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# A METRIC SPACE OF A.H. STONE AND AN EXAMPLE CONCERNING $\sigma$-MINIMAL BASES

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ABSTRACT. In this paper we use a metric space $Y$ due to A. H. Stone and one of its completions $X$ to construct a linearly ordered topological space $E = E(Y, X)$ that is Čech complete, has a $\sigma$-closed-discrete dense subset, is perfect, hereditarily paracompact, first-countable, and has the property that each of its subspaces has a $\sigma$-minimal base for its relative topology. However, $E$ is not metrizable and is not quasi-developable. The construction of $E(Y, X)$ is a point-splitting process that is familiar in ordered spaces, and an orderability theorem of Herrlich is the link between Stone's metric space and our construction.

## 1. INTRODUCTION

A collection $\mathcal{A}$ of subsets of a space $X$ is minimal (or irreducible) if each member of $\mathcal{A}$ contains a point that belongs to no other member of $\mathcal{A}$. Spaces with $\sigma$-minimal bases were introduced and studied by Aull in [A1], [A2], and the basic examples clarifying the relation of $\sigma$-minimal bases to other topological properties were constructed in [BB]. However, there remains one context in which the role of Aull's $\sigma$-minimal bases is not yet clear, and that is the theory of linearly ordered and generalized ordered spaces.

Among generalized ordered spaces, quasi-developability, the existence of a $\sigma$-disjoint base, and the existence of a $\sigma$-point-finite base are mutually equivalent properties ([B], [L]). Among generalized ordered spaces, each implies the existence of a $\sigma$-minimal base, but the converse is obviously false – the lexicographic square is a compact linearly ordered space that has a $\sigma$-minimal base but is not quasi-developable. Indeed, there are linearly ordered spaces that have $\sigma$-minimal bases that are not even first countable. However, all such spaces are hereditarily paracompact [BL1], even though the existence of a $\sigma$-minimal base is not itself a hereditary property.

Familiar examples suggest that generalized ordered spaces with $\sigma$-minimal bases are either quasi-developable or else contain a particular type of pathological subspace that does not have a $\sigma$-minimal base for its relative topology, and that suggests asking ([BL2], [L2]) whether a compact linearly ordered space must be metrizable if each of its subspaces has a $\sigma$-minimal base. That question is still

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open (see "Added in proof"). A weaker version of the question (originally misposed in [BL2]) asks whether a linearly ordered, or generalized ordered, space $X$ must be quasi-developable if every subspace of $X$ has a $\sigma$-minimal base for its relative topology. That is the question that we resolve negatively in this paper by constructing a non-metrizable, perfect linearly ordered space $X$ such that every subspace of $X$ has a $\sigma$-minimal base for its topology. The space $X$ has many other valuable features: it is hereditarily paracompact, first-countable, Čech-complete, and has a dense subspace that is $\sigma$-closed-discrete in $X$. But the space $X$ is not quasi-developable, as can be seen from the fact that $X$ is perfect but not metrizable.

The two key components of our construction are an elegant metric space constructed by A.H. Stone in [St] (and used as the basis for many other topological examples, e.g., [P1], [P2]) together with an orderability theorem of H. Herrlich [H].

Recall that a linearly ordered topological space (or LOTS) is a triple $(X, <, \mathcal{T})$, where $<$ is a linear ordering of $X$ and where $\mathcal{T}$ is the usual open interval topology of $<$. Unfortunately, a subspace of a LOTS may fail to be a LOTS, and that leads to the study of generalized ordered spaces (or GO-spaces), i.e., spaces that can be embedded in some LOTS. An internal characterization of GO-spaces is that they are triples $(X, <, \mathcal{T})$, where $<$ is a linear ordering of $X$ and where $\mathcal{T}$ is a Hausdorff topology on $X$ that has an open base consisting of order-convex sets.

In our paper we must carefully distinguish between subsets of a space $X$ that are the union of a countable collection of closed, discrete subspaces of $X$ (such subspaces will be called $\sigma$-closed-discrete) and subspaces that are countable unions of discrete, but not necessarily closed, subspaces (such spaces are said to be $\sigma$-discrete-in-themselves). However, as Stone pointed out in [St], the two notions are equivalent in any metric space, and indeed in any perfect space, where a space $X$ is perfect if every closed subset of $X$ is a $G_\delta$ in $X$.

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## 2. Stone's metric space and preliminary properties of $E(Y, X)$

In [St], A.H. Stone constructed a special metric space $Y$ that has been used as the starting point of many important examples in topology; see, for example, [P1], [P2]. The crucial properties of Stone's space $Y$ are:

**S-1):** $Y$ is a subspace of $D^\omega$, where $D$ is a discrete space of cardinality $\omega_1$;
**S-2):** the cardinality of $Y$ is $\omega_1$;
**S-3):** $Y$ is not the union of countably many subspaces, each discrete-in-itself;
**S-4):** if $C$ is a countable subset of $Y$, then the closure of $C$ in $Y$ is also countable.

Throughout this paper, $X$ will be the closure of $Y$ in the metric space $D^\omega$ and $\mathcal{M}$ will denote the topology that $X$ inherits from $D^\omega$. Let $d$ be a metric on $X$ that induces the topology $\mathcal{M}$, and for $x \in X$ and $\epsilon > 0$ let $B(d, x, \epsilon)$ denote the open $d$-ball of radius $\epsilon$ centered at $x$.

**2.1 Proposition.** *With $X$ as above, $(X, \mathcal{M})$ is a completely metrizable space and there is a linear ordering of the set $X$ that induces $\mathcal{M}$ as its open interval topology.*

*Proof.* Being closed in $D^\omega$, $(X, \mathcal{M})$ is completely metrizable. That some linear ordering of $X$ makes $(X, \mathcal{M})$ into a LOTS follows from a result of Herrlich ([H]; see [E, Problem 6.3.2] for another proof) because $X$ is strongly zero-dimensional. □

**2.2 Proposition.** *Let $Z \subset X$. Define $J(1) = \{z \in Z : \text{for some } x \in X,\ z < x \text{ and } ]z,x[\ \cap Z = \emptyset\}$ and $J(-1) = \{z \in Z : \text{for some } x \in X,\ z > x \text{ and } ]x,z[\ \cap Z = \emptyset\}$. Then $J = J(-1) \cup J(1)$ is a $\sigma$-closed-discrete subset of $(X, \mathcal{M})$.*

*Proof.* Because $X$ is metrizable, it will be enough to show that $J$ is the union of countably many subsets, each of which is discrete-in-itself. Consider $J(1)$, the argument for $J(-1)$ being analogous. For each $z \in J(1)$ let $z^+$ denote an element of $X$ such that $z < z^+$ and $]z, z^+[\ \cap Z = \emptyset$. Then there is an integer $n = n(z) \geq 1$ such that $B(d, z, \frac{1}{n}) \subset\ ]\leftarrow, z^+[$ . Let $J(1,m) = \{z \in J(1) : n(z) = m\}$. If $p < q$ are points of $J(1,m)$, then $p^+ \leq q$, so that we have $B(d, p, \frac{1}{m}) \subset\ ]\leftarrow, p^+[$ , showing that $q \notin B(d, p, \frac{1}{m})$, so that $d(p,q) \geq \frac{1}{m}$, as required. □

For any $Z \subset X$, there is a second natural topology on $Z$, namely the Sorgenfrey topology $\mathcal{S}_Z$ that has sets of the form $Z \cap [z, b[$ as a base, where $z \in Z$ and $b \in X$ with $z < b$. Clearly $\mathcal{M}_Z \subset \mathcal{S}_Z$, where $\mathcal{M}_Z$ is the topology that $Z$ inherits from the metrizable space $(X, \mathcal{M})$.

**2.3 Corollary.** *For any $Z \subset X$, the space $(Z, \mathcal{S}_Z)$ has a dense set $D$ that is the union of countably many closed, discrete subsets of $(Z, \mathcal{S}_Z)$.*

*Proof.* The metrizable space $(Z, \mathcal{M}_Z)$ has a dense set $E$ that is $\sigma$-closed-discrete in $(Z, \mathcal{M}_Z)$ and hence is also $\sigma$-closed-discrete in $(Z, \mathcal{S}_Z)$. By (2.2) the set $J(1) = \{z \in Z : \text{for some } x \in X \text{ with } x > z,\ [z, x[\ \cap Z = \{z\}\}$ is also $\sigma$-closed-discrete in $(Z, \mathcal{M}_Z)$ and therefore also in $(Z, \mathcal{S}_Z)$. Because $J(1) \cup E$ is dense in $(Z, \mathcal{S}_Z)$, the lemma is proved. □

**2.4 Construction of $E(Y, X)$.** Let $Y$ be Stone's metric space and $X$ its closure in $D^\omega$. Let $\mathcal{M}$ be the metrizable topology that $X$ inherits from the product space. Use (2.1) to choose a linear ordering of $X$ that induces $\mathcal{M}$ as its open interval topology. Let

$$E(Y, X) = (Y \times \{-1, 1\}) \cup ((X - Y) \times \{0\}).$$

Order $E(Y, X)$ lexicographically and let $E(Y, X)$ carry the open interval topology of that ordering. Let $\pi : E(Y, X) \to X$ be the function $\pi(x, i) = x$. One can think of $E(Y, X)$ as being the result of splitting each point of $Y$ into two consecutive points.

**2.5 Proposition.** *The mapping $\pi$ is a perfect mapping from $E(Y, X)$ onto $(X, \mathcal{M})$. Hence $E(Y, X)$:*
- *a) is Čech-complete;*
- *b) has a dense subset that is $\sigma$-closed-discrete;*
- *c) is perfect, hereditarily paracompact, and first countable.*

*Proof.* It is easy to see that $\pi$ is continuous and closed, so that, because $|\pi^{-1}(x)| \leq 2$ for each $x \in X$, $\pi$ is perfect. Hence a) holds.

Let $D$ be a dense $\sigma$-closed-discrete subset of the metric space $(X, \mathcal{M})$. Then $E_1 = \pi^{-1}[D]$ is a $\sigma$-closed-discrete subset of $E(Y, X)$. Consider the sets $J(-1) = \{z \in X : \text{for some } x < z,\ ]x, z[ = \emptyset\}$ and $J(1) = \{z \in X : \text{for some } x > z,\ ]z, x[ = \emptyset\}$. According to (2.2), each is $\sigma$-closed-discrete in $(X, \mathcal{M})$. Hence the subset $E_2 = \pi^{-1}[J(-1) \cup J(1)]$ is $\sigma$-closed-discrete in $E(Y, X)$. It is easy to verify that $E_1 \cup E_2$ is dense in $E(Y, X)$, so that (b) holds. Now (c) follows from the general theory of ordered spaces (see [L], [BLP], [Fa], [vW]). □

**2.6 Proposition.** *$E(Y, X)$ is not metrizable.*

*Proof.* For a contradiction, suppose $E(Y, X)$ is metrizable. Then so is its subspace $Y \times \{1\}$, which is homeomorphic to $(Y, \mathcal{S}_Y)$, where $\mathcal{S}_Y$ is the Sorgenfrey topology on $Y$ described just before (2.3). Let $d$ be a metric on $Y$ that induces the topology $\mathcal{M}_Y$, and suppose $\rho$ is a metric on $Y$ that induces the topology $\mathcal{S}_Y$. We will denote the open balls with respect to $d$ and $\rho$ by $B(d, x, \epsilon)$ and $B(\rho, x, \epsilon)$, respectively.

For each $y \in Y$ there is an integer $n = n(y) \geq 1$ such that $B(\rho, y, \frac{1}{n}) \subset [y, \rightarrow [$ . Let $Y(m) = \{y \in Y : n(y) = m\}$ and observe that

$$(*)\qquad \text{if } p < q \text{ are points of } Y(m), \text{ then } B(\rho, q, \frac{1}{m}) \subset [q, \rightarrow [, \quad \text{so that } \rho(p, q) \geq \frac{1}{m}.$$

Thus each $Y(m)$ is closed and discrete in $(Y, \mathcal{S}_Y)$.

For each $y \in Y(m)$ there is a $b > y$ such that $Y \cap [y, b[ \subset B(\rho, y, \frac{1}{m})$. Because $] \leftarrow, b[$ is a neighborhood of $y$ in $(Y, \mathcal{M}_Y)$, there is a $k = k(y, m) \geq 1$ such that $B(d, y, \frac{1}{k}) \subset ] \leftarrow, b[$ . Hence $B(d, y, \frac{1}{k}) \cap [y, \rightarrow [ \subset [y, b[ \subset B(\rho, y, \frac{1}{m})$. Let $Y(m, j) = \{y \in Y(m) : k(y, m) = j\}$ for each $m, j \geq 1$. Because $Y = \bigcup\{Y(m, j) : m, j \geq 1\}$, it follows from property (S-3) of $Y$ that for some $m, j \geq 1$, the set $Y(m, j)$ is not discrete-in-itself when $Y$ is topologized using $\mathcal{M}_Y$. Hence we may choose a strictly monotonic sequence $\{y_k : k \geq 1\}$ in $Y(m, j)$ that converges to a point $y_0 \in Y(m, j)$. Because $Y(m, j) \subset Y(m)$ and $Y(m)$ is closed and discrete in $(Y, \mathcal{S}_Y)$, it must be the case that $y_1 < y_2 < y_3 < ....$ Choose integers $k < i$ such that $\{y_k, y_i\} \subset B(d, y_0, \frac{1}{3j})$. The triangle inequality yields $d(y_k, y_i) < \frac{1}{j}$, so that $y_i \in B(d, y_k, \frac{1}{j}) \cap [y_k, \rightarrow [$ . Because $y_k \in Y(m, j)$ we have $B(d, y_k, \frac{1}{j}) \cap [y_k, \rightarrow [ \subset B(\rho, y_k, \frac{1}{m})$, so that $\rho(y_k, y_i) < \frac{1}{m}$, which contradicts (*) above, because $y_k, y_i \in Y(m, j) \subset Y(m)$. That contradiction shows that $(Y, \mathcal{S}_Y)$ cannot be metrizable, and completes the proof. □

## 3. σ-minimal bases

**3.1 Lemma.** *Let $Z$ be a subspace of a first-countable GO-space $Y$. Suppose that $Z$, in its relative topology, has a $\sigma$-minimal base. Then there is a $\sigma$-minimal collection of open subsets of $Y$ that contains a neighborhood base at each point of $Z$.*

*Proof.* Let $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$ be a $\sigma$-minimal base of relatively open subsets of $Z$. For each $B \in \mathcal{B}$, let $B^*$ be the convex hull of $B$ in $Y$. The set $B^*$ might not be open, but because $Y$ is first countable, there is a sequence $B^*(k)$ of convex open subsets of $Y$ such that if $G$ is a convex open subset of $Y$ that contains $B^*$, then for some $k$, $B^*(k) \subset G$.

For each $B \in \mathcal{B}(n)$, choose an open set $C(B)$ in $Y$ such that $B = C(B) \cap Z$, and define $D(B, n, k) = B^*(k) \cap C(B)$. Each $D(B, n, k)$ is open in $Y$, and the collection $\mathcal{D}(n, k) = \{D(B, n, k) : B \in \mathcal{B}(n)\}$ is irreducible. Finally, $\mathcal{D} = \bigcup\{\mathcal{D}(n, k) : n, k \geq 1\}$ contains a local base at each $y \in Y$. For suppose $G$ is a convex open subset of $Y$ and that $z \in G \cap Z$. Then there are an index $n$ and a set $B \in \mathcal{B}(n)$ such that $z \in B \subset G \cap Z$. Because $G$ is convex, $B^* \subset G$. As noted above, there is an index $k \geq 1$ such that $B^*(k) \subset G$. Then $z \in D(B, n, k) = C(B) \cap B^*(k) \subset G$, as required. □

**3.2 Corollary.** *Suppose $Z$ is a first-countable GO space and $Z = \bigcup\{Z(n) : n \geq 1\}$, where each subspace $Z(n)$ has a $\sigma$-minimal base for its relative topology. Then $Z$ has a $\sigma$-minimal base.*

**3.3 Lemma.** *With $(Y, \mathcal{S}_Y)$ as in Section 2, each subspace of $(Y, \mathcal{S}_Y)$ has a $\sigma$-minimal base for its relative topology.*

*Proof.* Let $Z \subset Y$. According to (2.3) there is a dense subset $D = \bigcup\{D(n) : n \geq 1\}$ of $(Z, \mathcal{S}_Z)$ where each $D(n)$ is a closed, discrete subspace of $(Z, \mathcal{S}_Z)$. Because $(Z, \mathcal{S}_Z)$ is first countable and hereditarily collectionwise normal, there is a $\sigma$-disjoint collection $\mathcal{D}$ of open subsets of $Z$ that contains a neighborhood base at each point of $D$.

Let $Z_0 = \{z \in Z :$ some $\mathcal{S}_Z$-neighborhood of $z$ is countable$\}$. Then $Z_0$ is an open subspace of $Z$ and, each countable GO-space being metrizable, $Z_0$ is locally metrizable. The existence of the $\sigma$-closed-discrete, dense set $D$ makes $Z$ perfect and hence hereditarily paracompact, so that $Z_0$ is a metrizable subspace of $Z$. Hence there is a $\sigma$-disjoint collection $\mathcal{E}$ of $\mathcal{S}_Z$-open sets that contains a neighborhood base at each point of $Z_0$.

Suppose $z \in Z - (D \cup Z_0)$. Then each $\mathcal{S}_Z$ neighborhood of $z$ is uncountable. We claim that for each $b \in X$ with $b > z$, the set $C = [z, b[\, \cap D$ is uncountable. For if not, then property (S-4) of Stone's space $(Y, \mathcal{M}_Y)$ forces the $\mathcal{M}_Y$-closure of $C$ to be countable. But then $[z, b[\, \cap Z$ is a subset of the $\mathcal{S}_Y$-closure of $C$, which is a subset of the $\mathcal{M}_Y$-closure of $C$, which is countable, so that $z \in Z_0$, contradicting $z \in Z - (D \cup Z_0)$.

Provided $z \in Z - (D \cup Z_0)$, each $B(d, z, \frac{1}{k}) \cap [z, \rightarrow[$ is an $\mathcal{S}_Z$-neighborhood of $z$. Hence for some $n \geq 1$, $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\, \cap D(n)| > \omega$. Let $A(n, k)$ be the set of all $z \in Z - (D \cup Z_0)$ such that $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\, \cap D(n)| > \omega$. Because $|Y| = \omega_1$ according to property (S-2), there is a 1-1 function $\phi_{n,k} : A(n, k) \rightarrow D(n)$ such that $\phi_{n,k}(z) \in B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\, \cap D(n)$ for each $z \in A(n, k)$. For $z \in A(n, k)$ define

$$C(z, n, k) = (Z \cap [z, \rightarrow[\, \cap B(d, z, \frac{1}{k})) - (D(n) - \{\phi_{n,k}(z)\})$$

and let $\mathcal{C}(n, k) = \{C(z, n, k) : z \in A(n, k)\}$. Each collection $\mathcal{C}(n, k)$ consists of $\mathcal{S}_Z$-open sets and is a minimal collection, because if $z, w$ are distinct points of $A(n, k)$, then $\phi_{n,k}(z) \in C(z, n, k) - C(w, n, k)$.

We claim that $\mathcal{C} = \bigcup\{\mathcal{C}(n, k) : n, k \geq 1\}$ contains an $\mathcal{S}_Z$-neighborhood base at each point $z \in Z - (D \cup Z_0)$. For suppose $z < b$ with $b \in X$, and consider $Z \cap [z, b[$ . Because $]\leftarrow, b[$ is an $\mathcal{M}_Y$-neighborhood of $z$, there is a $k \geq 1$ with $B(d, z, \frac{1}{k}) \subset\, ]\leftarrow, b[$. Then

$$B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\, \subset [z, b[\, .$$

Choose $n$ so that $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\, \cap D(n)| > \omega$. Then $C(z, n, k) \in \mathcal{C}(n, k)$ and $z \in C(z, n, k) \subset [z, b[\, \cap Z$, as required. □

**3.4 Corollary.** *Every subspace of $E(Y, X)$ has a $\sigma$-minimal base for its topology.*

*Proof.* Let $Z \subset E(Y, X)$, and define $Z(i) = (Y \times \{i\}) \cap Z$ if $i \in \{-1, 1\}$ and $Z(0) = Z \cap ((X - Y) \times \{0\}$. In the light of (3.2) it will be enough to show that each $Z(i)$ has a $\sigma$-minimal base for its relative topology.

The subspace $Z(0)$ is metrizable and therefore has a base of the required kind. The subspace $Z(1)$ is homeomorphic to a subspace of $(Y, \mathcal{S}_Y)$, so that it has a

$\sigma$-minimal base in the light of (3.3). The argument to show that $Z(-1)$ has a $\sigma$-minimal base for its relative topology is entirely analogous to the argument for $Z(1)$, and that completes the proof. □

## Added in proof

In a paper that will appear in the *Proceedings of the American Mathematical Society*, Wei-Xue Shi has constructed a non-metrizable compact LOTS $X$, every subspace of which has a $\sigma$-minimal base. Shi's result completely settles the question posed in [BL2] and [L2].

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