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A METRIC SPACE OF A.H. STONE AND AN EXAMPLE CONCERNING σ-MINIMAL BASES

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ABSTRACT. In this paper we use a metric space Y due to A. H. Stone and one of its completions X to construct a linearly ordered topological space $E = E(Y, X)$ that is Čech complete, has a σ-closed-discrete dense subset, is perfect, hereditarily paracompact, first-countable, and has the property that each of its subspaces has a σ-minimal base for its relative topology. However, E is not metrizable and is not quasi-developable. The construction of $E(Y, X)$ is a point-splitting process that is familiar in ordered spaces, and an orderability theorem of Herrlich is the link between Stone's metric space and our construction.

1. Introduction

A collection $\mathcal{A}$ of subsets of a space X is minimal (or irreducible) if each member of $\mathcal{A}$ contains a point that belongs to no other member of $\mathcal{A}$. Spaces with σ-minimal bases were introduced and studied by Aull in [A1], [A2], and the basic examples clarifying the relation of σ-minimal bases to other topological properties were constructed in [BB]. However, there remains one context in which the role of Aull's σ-minimal bases is not yet clear, and that is the theory of linearly ordered and generalized ordered spaces.

Among generalized ordered spaces, quasi-developability, the existence of a σ-disjoint base, and the existence of a σ-point-finite base are mutually equivalent properties ([B], [L]). Among generalized ordered spaces, each implies the existence of a σ-minimal base, but the converse is obviously false – the lexicographic square is a compact linearly ordered space that has a σ-minimal base but is not quasi-developable. Indeed, there are linearly ordered spaces that have σ-minimal bases that are not even first countable. However, all such spaces are hereditarily paracompact [BL1], even though the existence of a σ-minimal base is not itself a hereditary property.

Familiar examples suggest that generalized ordered spaces with σ-minimal bases are either quasi-developable or else contain a particular type of pathological subspace that does not have a σ-minimal base for its relative topology, and that suggests asking ([BL2], [L2]) whether a compact linearly ordered space must be metrizable if each of its subspaces has a σ-minimal base. That question is still

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open (see "Added in proof"). A weaker version of the question (originally misposed in [BL2]) asks whether a linearly ordered, or generalized ordered, space X must be quasi-developable if every subspace of X has a σ-minimal base for its relative topology. That is the question that we resolve negatively in this paper by constructing a non-metrizable, perfect linearly ordered space X such that every subspace of X has a σ-minimal base for its topology. The space X has many other valuable features: it is hereditarily paracompact, first-countable, Čech-complete, and has a dense subspace that is σ-closed-discrete in X. But the space X is not quasi-developable, as can be seen from the fact that X is perfect but not metrizable.

The two key components of our construction are an elegant metric space constructed by A.H. Stone in [St] (and used as the basis for many other topological examples, e.g., [P1], [P2]) together with an orderability theorem of H. Herrlich [H].

Recall that a linearly ordered topological space (or LOTS) is a triple $(X, <, \mathcal{T})$, where $<$ is a linear ordering of X and where $\mathcal{T}$ is the usual open interval topology of $<$. Unfortunately, a subspace of a LOTS may fail to be a LOTS, and that leads to the study of generalized ordered spaces (or GO-spaces), i.e., spaces that can be embedded in some LOTS. An internal characterization of GO-spaces is that they are triples $(X, <, \mathcal{T})$, where $<$ is a linear ordering of X and where $\mathcal{T}$ is a Hausdorff topology on X that has an open base consisting of order-convex sets.

In our paper we must carefully distinguish between subsets of a space X that are the union of a countable collection of closed, discrete subspaces of X (such subspaces will be called σ-closed-discrete) and subspaces that are countable unions of discrete, but not necessarily closed, subspaces (such spaces are said to be σ-discrete-in-themselves). However, as Stone pointed out in [St], the two notions are equivalent in any metric space, and indeed in any perfect space, where a space X is perfect if every closed subset of X is a G_δ in X.

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2. Stone's metric space and preliminary properties of $E(Y, X)$

In [St], A.H. Stone constructed a special metric space Y that has been used as the starting point of many important examples in topology; see, for example, [P1], [P2]. The crucial properties of Stone's space Y are:

S-1): Y is a subspace of D^ω, where D is a discrete space of cardinality ω_1;
S-2): the cardinality of Y is ω_1;
S-3): Y is not the union of countably many subspaces, each discrete-in-itself;
S-4): if C is a countable subset of Y, then the closure of C in Y is also countable.

Throughout this paper, X will be the closure of Y in the metric space D^ω and $\mathcal{M}$ will denote the topology that X inherits from D^ω. Let d be a metric on X that induces the topology $\mathcal{M}$, and for $x \in X$ and $\epsilon > 0$ let $B(d, x, \epsilon)$ denote the open d-ball of radius ϵ centered at x.

2.1 Proposition. *With X as above, $(X, \mathcal{M})$ is a completely metrizable space and there is a linear ordering of the set X that induces $\mathcal{M}$ as its open interval topology.*

Proof. Being closed in D^ω, $(X, \mathcal{M})$ is completely metrizable. That some linear ordering of X makes $(X, \mathcal{M})$ into a LOTS follows from a result of Herrlich ([H]; see [E, Problem 6.3.2] for another proof) because X is strongly zero-dimensional. □

2.2 Proposition. *Let $Z \subset X$. Define $J(1) = \{z \in Z : \text{for some } x \in X,\ z < x \text{ and }]z, x[\ \cap Z = \emptyset\}$ and $J(-1) = \{z \in Z : \text{for some } x \in X,\ z > x \text{ and }]x, z[\ \cap Z = \emptyset\}$. Then $J = J(-1) \cup J(1)$ is a σ-closed-discrete subset of $(X, \mathcal{M})$.*

Proof. Because X is metrizable, it will be enough to show that J is the union of countably many subsets, each of which is discrete-in-itself. Consider $J(1)$, the argument for $J(-1)$ being analogous. For each $z \in J(1)$ let z^+ denote an element of X such that $z < z^+$ and $]z, z^+[\ \cap Z = \emptyset$. Then there is an integer $n = n(z) \geq 1$ such that $B(d, z, \frac{1}{n}) \subset\ ]\leftarrow, z^+[$. Let $J(1, m) = \{z \in J(1) : n(z) = m\}$. If $p < q$ are points of $J(1, m)$, then $p^+ \leq q$, so that we have $B(d, p, \frac{1}{m}) \subset\ ]\leftarrow, p^+[$, showing that $q \notin B(d, p, \frac{1}{m})$, so that $d(p, q) \geq \frac{1}{m}$, as required. □

For any $Z \subset X$, there is a second natural topology on Z, namely the Sorgenfrey topology $\mathcal{S}_Z$ that has sets of the form $Z \cap [z, b[$ as a base, where $z \in Z$ and $b \in X$ with $z < b$. Clearly $\mathcal{M}_Z \subset \mathcal{S}_Z$, where $\mathcal{M}_Z$ is the topology that Z inherits from the metrizable space $(X, \mathcal{M})$.

2.3 Corollary. *For any $Z \subset X$, the space $(Z, \mathcal{S}_Z)$ has a dense set D that is the union of countably many closed, discrete subsets of $(Z, \mathcal{S}_Z)$.*

Proof. The metrizable space $(Z, \mathcal{M}_Z)$ has a dense set E that is σ-closed-discrete in $(Z, \mathcal{M}_Z)$ and hence is also σ-closed-discrete in $(Z, \mathcal{S}_Z)$. By (2.2) the set $J(1) = \{z \in Z : \text{for some } x \in X \text{ with } x > z,\ [z, x[\ \cap Z = \{z\}\}$ is also σ-closed-discrete in $(Z, \mathcal{M}_Z)$ and therefore also in $(Z, \mathcal{S}_Z)$. Because $J(1) \cup E$ is dense in $(Z, \mathcal{S}_Z)$, the lemma is proved. □

2.4 Construction of $E(Y, X)$. Let Y be Stone's metric space and X its closure in D^ω. Let $\mathcal{M}$ be the metrizable topology that X inherits from the product space. Use (2.1) to choose a linear ordering of X that induces $\mathcal{M}$ as its open interval topology. Let

$$E(Y, X) = (Y \times \{-1, 1\}) \cup ((X - Y) \times \{0\}).$$

Order $E(Y, X)$ lexicographically and let $E(Y, X)$ carry the open interval topology of that ordering. Let $\pi : E(Y, X) \to X$ be the function $\pi(x, i) = x$. One can think of $E(Y, X)$ as being the result of splitting each point of Y into two consecutive points.

2.5 Proposition. *The mapping π is a perfect mapping from $E(Y, X)$ onto $(X, \mathcal{M})$. Hence $E(Y, X)$:*
a) is Čech-complete;
b) has a dense subset that is σ-closed-discrete;
c) is perfect, hereditarily paracompact, and first countable.

Proof. It is easy to see that π is continuous and closed, so that, because $|\pi^{-1}(x)| \leq 2$ for each $x \in X$, π is perfect. Hence a) holds.

Let D be a dense σ-closed-discrete subset of the metric space $(X, \mathcal{M})$. Then $E_1 = \pi^{-1}[D]$ is a σ-closed-discrete subset of $E(Y, X)$. Consider the sets $J(-1) = \{z \in X : \text{for some } x < z,\ ]x, z[= \emptyset\}$ and $J(1) = \{z \in X : \text{for some } x > z,\ ]z, x[= \emptyset\}$. According to (2.2), each is σ-closed-discrete in $(X, \mathcal{M})$. Hence the subset $E_2 = \pi^{-1}[J(-1) \cup J(1)]$ is σ-closed-discrete in $E(Y, X)$. It is easy to verify that $E_1 \cup E_2$ is dense in $E(Y, X)$, so that (b) holds. Now (c) follows from the general theory of ordered spaces (see [L], [BLP], [Fa], [vW]). □

2.6 Proposition. *$E(Y,X)$ is not metrizable.*

Proof. For a contradiction, suppose $E(Y,X)$ is metrizable. Then so is its subspace $Y \times \{1\}$, which is homeomorphic to $(Y, \mathcal{S}_Y)$, where $\mathcal{S}_Y$ is the Sorgenfrey topology on Y described just before (2.3). Let d be a metric on Y that induces the topology $\mathcal{M}_Y$, and suppose ρ is a metric on Y that induces the topology $\mathcal{S}_Y$. We will denote the open balls with respect to d and ρ by $B(d,x,\epsilon)$ and $B(\rho,x,\epsilon)$, respectively.

For each $y \in Y$ there is an integer $n = n(y) \geq 1$ such that $B(\rho, y, \frac{1}{n}) \subset [y, \rightarrow[$. Let $Y(m) = \{y \in Y : n(y) = m\}$ and observe that

$$(*)\qquad \text{if } p < q \text{ are points of } Y(m), \text{ then } B(\rho, q, \frac{1}{m}) \subset [q, \rightarrow[, \quad \text{so that } \rho(p,q) \geq \frac{1}{m}.$$

Thus each $Y(m)$ is closed and discrete in $(Y, \mathcal{S}_Y)$.

For each $y \in Y(m)$ there is a $b > y$ such that $Y \cap [y, b[\subset B(\rho, y, \frac{1}{m})$. Because $]\leftarrow, b[$ is a neighborhood of y in $(Y, \mathcal{M}_Y)$, there is a $k = k(y,m) \geq 1$ such that $B(d, y, \frac{1}{k}) \subset]\leftarrow, b[$. Hence $B(d, y, \frac{1}{k}) \cap [y, \rightarrow[\subset [y, b[\subset B(\rho, y, \frac{1}{m})$. Let $Y(m,j) = \{y \in Y(m) : k(y,m) = j\}$ for each $m, j \geq 1$. Because $Y = \bigcup\{Y(m,j) : m, j \geq 1\}$, it follows from property (S-3) of Y that for some $m, j \geq 1$, the set $Y(m,j)$ is not discrete-in-itself when Y is topologized using $\mathcal{M}_Y$. Hence we may choose a strictly monotonic sequence $\{y_k : k \geq 1\}$ in $Y(m,j)$ that converges to a point $y_0 \in Y(m,j)$. Because $Y(m,j) \subset Y(m)$ and $Y(m)$ is closed and discrete in $(Y, \mathcal{S}_Y)$, it must be the case that $y_1 < y_2 < y_3 < \dots$. Choose integers $k < i$ such that $\{y_k, y_i\} \subset B(d, y_0, \frac{1}{3j})$. The triangle inequality yields $d(y_k, y_i) < \frac{1}{j}$, so that $y_i \in B(d, y_k, \frac{1}{j}) \cap [y_k, \rightarrow[$. Because $y_k \in Y(m,j)$ we have $B(d, y_k, \frac{1}{j}) \cap [y_k, \rightarrow[\subset B(\rho, y_k, \frac{1}{m})$, so that $\rho(y_k, y_i) < \frac{1}{m}$, which contradicts (*) above, because $y_k, y_i \in Y(m,j) \subset Y(m)$. That contradiction shows that $(Y, \mathcal{S}_Y)$ cannot be metrizable, and completes the proof. □

3. σ-MINIMAL BASES

3.1 Lemma. *Let Z be a subspace of a first-countable GO-space Y. Suppose that Z, in its relative topology, has a σ-minimal base. Then there is a σ-minimal collection of open subsets of Y that contains a neighborhood base at each point of Z.*

Proof. Let $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$ be a σ-minimal base of relatively open subsets of Z. For each $B \in \mathcal{B}$, let B^* be the convex hull of B in Y. The set B^* might not be open, but because Y is first countable, there is a sequence $B^*(k)$ of convex open subsets of Y such that if G is a convex open subset of Y that contains B^*, then for some k, $B^*(k) \subset G$.

For each $B \in \mathcal{B}(n)$, choose an open set $C(B)$ in Y such that $B = C(B) \cap Z$, and define $D(B,n,k) = B^*(k) \cap C(B)$. Each $D(B,n,k)$ is open in Y, and the collection $\mathcal{D}(n,k) = \{D(B,n,k) : B \in \mathcal{B}(n)\}$ is irreducible. Finally, $\mathcal{D} = \bigcup\{\mathcal{D}(n,k) : n, k \geq 1\}$ contains a local base at each $y \in Y$. For suppose G is a convex open subset of Y and that $z \in G \cap Z$. Then there are an index n and a set $B \in \mathcal{B}(n)$ such that $z \in B \subset G \cap Z$. Because G is convex, $B^* \subset G$. As noted above, there is an index $k \geq 1$ such that $B^*(k) \subset G$. Then $z \in D(B,n,k) = C(B) \cap B^*(k) \subset G$, as required. □

3.2 Corollary. *Suppose Z is a first-countable GO space and $Z = \bigcup\{Z(n) : n \geq 1\}$, where each subspace $Z(n)$ has a σ-minimal base for its relative topology. Then Z has a σ-minimal base.*

3.3 Lemma. *With $(Y, \mathcal{S}_Y)$ as in Section 2, each subspace of $(Y, \mathcal{S}_Y)$ has a σ-minimal base for its relative topology.*

Proof. Let $Z \subset Y$. According to (2.3) there is a dense subset $D = \bigcup\{D(n) : n \geq 1\}$ of $(Z, \mathcal{S}_Z)$ where each $D(n)$ is a closed, discrete subspace of $(Z, \mathcal{S}_Z)$. Because $(Z, \mathcal{S}_Z)$ is first countable and hereditarily collectionwise normal, there is a σ-disjoint collection $\mathcal{D}$ of open subsets of Z that contains a neighborhood base at each point of D.

Let $Z_0 = \{z \in Z :$ some $\mathcal{S}_Z$-neighborhood of z is countable$\}$. Then Z_0 is an open subspace of Z and, each countable GO-space being metrizable, Z_0 is locally metrizable. The existence of the σ-closed-discrete, dense set D makes Z perfect and hence hereditarily paracompact, so that Z_0 is a metrizable subspace of Z. Hence there is a σ-disjoint collection $\mathcal{E}$ of $\mathcal{S}_Z$-open sets that contains a neighborhood base at each point of Z_0.

Suppose $z \in Z - (D \cup Z_0)$. Then each $\mathcal{S}_Z$ neighborhood of z is uncountable. We claim that for each $b \in X$ with $b > z$, the set $C = [z, b[\cap D$ is uncountable. For if not, then property (S-4) of Stone's space $(Y, \mathcal{M}_Y)$ forces the $\mathcal{M}_Y$-closure of C to be countable. But then $[z, b[\cap Z$ is a subset of the $\mathcal{S}_Y$-closure of C, which is a subset of the $\mathcal{M}_Y$-closure of C, which is countable, so that $z \in Z_0$, contradicting $z \in Z - (D \cup Z_0)$.

Provided $z \in Z - (D \cup Z_0)$, each $B(d, z, \frac{1}{k}) \cap [z, \rightarrow [$ is an $\mathcal{S}_Z$-neighborhood of z. Hence for some $n \geq 1$, $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow [\cap D(n)| > \omega$. Let $A(n,k)$ be the set of all $z \in Z - (D \cup Z_0)$ such that $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow [\cap D(n)| > \omega$. Because $|Y| = \omega_1$ according to property (S-2), there is a 1-1 function $\phi_{n,k} : A(n,k) \to D(n)$ such that $\phi_{n,k}(z) \in B(d, z, \frac{1}{k}) \cap [z, \rightarrow [\cap D(n)$ for each $z \in A(n,k)$. For $z \in A(n,k)$ define

$$C(z, n, k) = (Z \cap [z, \rightarrow [\cap B(d, z, \frac{1}{k})) - (D(n) - \{\phi_{n,k}(z)\})$$

and let $\mathcal{C}(n,k) = \{C(z,n,k) : z \in A(n,k)\}$. Each collection $\mathcal{C}(n,k)$ consists of $\mathcal{S}_Z$-open sets and is a minimal collection, because if z, w are distinct points of $A(n,k)$, then $\phi_{n,k}(z) \in C(z,n,k) - C(w,n,k)$.

We claim that $\mathcal{C} = \bigcup\{\mathcal{C}(n,k) : n, k \geq 1\}$ contains an $\mathcal{S}_Z$-neighborhood base at each point $z \in Z - (D \cup Z_0)$. For suppose $z < b$ with $b \in X$, and consider $Z \cap [z, b[$. Because $]\leftarrow, b[$ is an $\mathcal{M}_Y$-neighborhood of z, there is a $k \geq 1$ with $B(d, z, \frac{1}{k}) \subset]\leftarrow, b[$. Then

$$B(d, z, \frac{1}{k}) \cap [z, \rightarrow [\subset [z, b[\; .$$

Choose n so that $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow [\cap D(n)| > \omega$. Then $C(z,n,k) \in \mathcal{C}(n,k)$ and $z \in C(z,n,k) \subset [z, b[\cap Z$, as required. □

3.4 Corollary. *Every subspace of $E(Y, X)$ has a σ-minimal base for its topology.*

Proof. Let $Z \subset E(Y, X)$, and define $Z(i) = (Y \times \{i\}) \cap Z$ if $i \in \{-1, 1\}$ and $Z(0) = Z \cap ((X - Y) \times \{0\}$. In the light of (3.2) it will be enough to show that each $Z(i)$ has a σ-minimal base for its relative topology.

The subspace $Z(0)$ is metrizable and therefore has a base of the required kind. The subspace $Z(1)$ is homeomorphic to a subspace of $(Y, \mathcal{S}_Y)$, so that it has a

σ-minimal base in the light of (3.3). The argument to show that $Z(-1)$ has a σ-minimal base for its relative topology is entirely analogous to the argument for $Z(1)$, and that completes the proof. ☐

Added in proof

In a paper that will appear in the *Proceedings of the American Mathematical Society*, Wei-Xue Shi has constructed a non-metrizable compact LOTS X, every subspace of which has a σ-minimal base. Shi's result completely settles the question posed in [BL2] and [L2].

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