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THE MODULI OF SUBSTRUCTURES OF A COMPACT COMPLEX SPACE

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ABSTRACT. We construct a space W_X of fine moduli for the substructures of an arbitrary compact complex space X. A substructure $(X, \mathcal{A})$ of X is given by a subalgebra $\mathcal{A}$ of the structure sheaf $\mathcal{O}_X$ with the additional feature that $(X, \mathcal{A})$ is also a complex space; $(X, \mathcal{A})$ and $(X, \mathcal{A}')$ are called equivalent if and only if $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic as subalgebras of $\mathcal{O}_X$.

Since substructures are quotients, it is only natural to start with the fine moduli space Q_X of all complex-analytic quotients of X. In order to obtain a representable moduli functor of substructures, we are forced to concentrate on families of quotients which satisfy some flatness condition for relative differential modules of higher order. Considering the corresponding flatification of Q_X, we realize that its open subset W_X consisting of all substructures turns out to be a complex space which has the required universal property.

1. FAMILIES OF QUOTIENTS AND SUBSTRUCTURES

Let X be a fixed complex space. A holomorphic map $f : X \to Y$ is called a *surjection* if f is surjective and $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective. By a *family of surjections* over an arbitrary complex space S we understand an S-morphism $g : X \times S \to Y$ with the property that $g_s : X \to Y_s$ is a surjection for every $s \in S$. Note that families of surjections over the reduced point are just surjections.

Two families of surjections $g_i : X \times S \to Y_i$ $(i = 1, 2)$ over S are called *equivalent* if and only if an S-isomorphism $h : Y_1 \to Y_2$ exists with $hg_1 = g_2$. By a *quotient (family of quotients)* we understand the equivalence class of a surjection (family of surjections). Henceforth, we use 'quotient' also as a synonym for 'surjection'.

The examples of quotients most important in this article are substructures. Let $\mathcal{A}$ be a subalgebra of the structure sheaf $\mathcal{O}_X$ of X. If $(X, \mathcal{A})$ is also a complex space, then we call the quotient $(X, \mathcal{O}_X) \to (X, \mathcal{A})$ a *substructure* of X. By a *family of substructures* of X over a complex space S we understand a family of quotients $g : X \times S \to Y$ over S such that $g_s : X \to Y_s$ is a substructure of X for every $s \in S$. Of course, the notion of equivalence given above is transferred from (families of) quotients also to (families of) substructures. A particular application of [3], 2.2 is

Lemma 1.1. *Let X be compact, and let $g : X \times S \to Y$ be a holomorphic map over an arbitrary complex space S. Then the following conditions are equivalent:*

- *g is a family of substructures over S.*

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- *g is a family of quotients over S and a homeomorphism.*
- *g is a substructure of $X \times S$, and $g_*\mathcal{O}_{X\times S}/\mathcal{O}_Y$ is S-flat.*

In order to give a quite useful criterion for a quotient to be even a substructure, we have to recall the notion of *relative infinitesimal neighbourhoods*. Let $X \to Y$ be a holomorphic mapping and $n \geq 0$ an integer. We denote by $X(n) = X/Y(n)$ the n^{th} infinitesimal neighbourhood of X in $X \times_Y X$. In other words, $X(n) = (X, \mathcal{O}_{X\times_Y X}/\mathcal{I}^{n+1})$, where $\mathcal{I}$ denotes the ideal sheaf of the diagonal embedding $\Delta_{X/Y} : X \to X \times_Y X$. According to [2], **IV** 16.4.5, the formation of infinitesimal neighbourhoods commutes with arbitrary base change:

Lemma 1.2. *If*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is a cartesian diagram of holomorphic maps, then the canonical mapping $u^*\mathcal{O}_{X/Y(n)} \to \mathcal{O}_{X'/Y'(n)}$ is an isomorphism for every $n \geq 0$.

A quotient is a substructure if and only if the underlying topological space is not affected at all:

Lemma 1.3. *Let X be compact, and let $f : X \to Y$ be a quotient. Then the following conditions are equivalent:*

- *f is a substructure of X.*
- *$X(n) = X \times_Y X$ for $n \gg 0$.*

Proof. The map f is injective if and only if $\Delta_{X/Y}$ is surjective. By Rückert's Nullstellensatz, this is equivalent to $\mathcal{I}^n = 0$ for $n \gg 0$. $\square$

Under certain circumstances, being a substructure turns out to be an open condition for the members of a family of quotients. In this context we use 1.3 together with

Lemma 1.4. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{\quad j \quad} & X \\ & \searrow \quad \swarrow_{p} & \\ & S & \end{array}$$

be a commutative triangle of holomorphic mappings such that j is a closed embedding with ideal sheaf $\mathcal{J}$. If Y is S-flat, then the closed embedding $j_s : Y_s \to X_s$ has ideal sheaf $(\mathcal{J}/m_{S,s}\mathcal{J})|X_s$ for each $s \in S$. Furthermore, we have

$$p(\operatorname{Supp}\mathcal{J}) = \{s \in S : j_s \textit{ is not biholomorphic}\},$$

which is an analytic subset of S provided that p is proper.

Proof. Let $s \in S$. The first statement is obvious. Furthermore, j_s is biholomorphic if and only if the sheaf $\mathcal{J}/m_s\mathcal{J}$ vanishes on X_s. By Nakayama's lemma, this is equivalent to $X_s \cap \operatorname{Supp}\mathcal{J} = \emptyset$, which is to say $s \notin p(\operatorname{Supp}\mathcal{J})$. The latter assertion of the lemma is an application of Remmert's mapping theorem. $\square$

2. Relative differential modules of higher order

This section prepares the definition of an appropriate moduli functor for substructures. Let $X \to Y$ be a holomorphic map, and let $\mathcal{I}$ be the ideal sheaf of the diagonal embedding $\Delta_{X/Y} : X \to X \times_Y X$. Then

$$\Omega_{X/Y}(n) = \Delta^* \mathcal{I}^n = (\mathcal{I}^n / \mathcal{I}^{n+1}) \mid X$$

is a coherent sheaf of $\mathcal{O}_X$-modules for every $n \geq 0$. In particular, we have $\Omega_{X/Y}(0) = \mathcal{O}_X$, and $\Omega_{X/Y}(1)$ is the sheaf $\Omega^1_{X/Y}$ of germs of relative holomorphic differential 1-forms with respect to $X \to Y$. According to 1.3, we have

Remark 2.1. If $X \to Y$ is a substructure, then $\Omega_{X/Y}(n) = 0$ for $n \gg 0$.

It is well-known that, without additional hypotheses, Ω^1 behaves perfectly under arbitrary base extension. What about $\Omega(n)$ for $n > 1$?

Lemma 2.2. *Let*

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & S' \\ {\scriptstyle u}\downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & S \end{array}$$

be a commutative diagram with cartesian squares. Provided that $\Omega_{X/Y}(n)$ is S-flat whenever $0 \leq n \leq N$, we have, for $0 \leq n \leq N$:

1. *$X(n)$ is S-flat.*
2. *The canonical morphism $u^*\Omega_{X/Y}(n) \to \Omega_{X'/Y'}(n)$ is bijective.*
3. *$\Omega_{X'/Y'}(n)$ is S'-flat.*

Proof. We consider the exact sequences

$$0 \to \Omega_{X/Y}(n) \to \mathcal{O}_{X(n)} \to \mathcal{O}_{X(n-1)} \to 0$$

and the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & u^*\Omega_{X/Y}(n) & \longrightarrow & u^*\mathcal{O}_{X(n)} & \longrightarrow & u^*\mathcal{O}_{X(n-1)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{X'/Y'}(n) & \longrightarrow & \mathcal{O}_{X'(n)} & \longrightarrow & \mathcal{O}_{X'(n-1)} & \longrightarrow & 0 \end{array}$$

with exact bottom rows. Having proven part 1 by means of iteration, we see that also the top rows are exact. By 1.2 we obtain part 2, and finally part 3, the latter representing an analog with [2], **IV** 16.4.6. □

3. Moduli spaces of quotients and substructures

Let X be a fixed complex space. Denoting by $Q_X(S)$ the set of all families of quotients of X over S, we get by means of obvious base change techniques the contravariant functor $Q = Q_X : S \mapsto Q_X(S)$ from the category of complex spaces to the category of sets. Provided that X is compact, there exists a space $Q = Q_X$ of fine moduli for quotients of X:

Theorem 3.1 (H. W. Schuster and A. Vogt [3]). *If X is compact, then Q_X is representable by a complex space.*

Encouraged by that success, one may ask whether also the set comprising all substructures of X carries a natural complex structure. The seminal question is: What is an appropriate moduli functor for substructures? At first sight, one would like to think about the subfunctor U of Q, where $U(S)$ consists of *all* families of substructures of X over S. But, unfortunately, U is in general *not* representable by a complex space even in the rather simple case where $X = \mathbb{P}^1_{\mathbb{C}}$, a fact which was already observed in [3], 4.4.

Fortunately, using the concepts introduced in section 2, we are able to define a subfunctor W of U in a way that the moduli problem for substructures can be settled: *For each complex space S, let $W_X(S)$ be the set consisting of all families of substructures $X \times S \to Y$ over S with the property that $\Omega_{X\times S/Y}(n)$ is S-flat for every $n \geq 0$.* According to 2.2.3, $W = W_X : S \mapsto W_X(S)$ actually is a functor. The following statement is the highlight of this article:

Theorem 3.2. *If X is a compact complex space, then W_X is representable.*

Proof. By $q : X \times Q \to Z$ we denote the universal family of quotients over Q. According to [1], there is the flatification $h_n : Q_n \to Q$ of the coherent $\mathcal{O}_{X\times Q}$-module $\bigoplus_{l\leq n} \Omega_{X\times Q/Z}(l)$ for each $n \geq 0$, and there is a sequence

$$(*) \qquad \ldots \longrightarrow Q_2 \xrightarrow{j_2} Q_1 \xrightarrow{j_1} Q_0 \xrightarrow{j_0} Q$$

consisting of bijective immersions with $h_0 = j_0$ and $h_n j_{n+1} = h_{n+1}$ for every $n \geq 0$. Let $Q_\infty = \varprojlim Q_n$ be the inverse limit of the Q_n (as a ringed space), $h : Q_\infty \to Q$ the morphism induced by the h_n, and $W_\infty = h^{-1}(|W|)$, where $|W|$ denotes the set of all substructures of X.

Claim. Given $f \in |W|$, there are $n \geq 0$ and an open neighbourhood U of f in Q_n such that $q_U : X \times U \to Z_U$ is a family of substructures, $\Omega_{X\times U/Z_U}(l) = 0$ if $l > n$, and $\Omega_{X\times U/Z_U}(l)$ is U-flat otherwise. In particular, q_U is in $W(U)$.

As a consequence we obtain that $(*)$ stabilizes locally at every point of $|W|$, and that W_∞ is open in Q_∞. Hence $W = W_X = (W_\infty, \mathcal{O}_{Q_\infty}|_{W_\infty})$ is a complex space. By its very construction W represents the functor W.

Finally, we prove the claim by means of a diagonal argument. Let $f : X \to Y$ be a substructure. According to 1.3, there is $n \geq 0$ such that $X(n) = X \times_Y X$. Consider the family of quotients $q_{Q_n} : X_n = X \times Q_n \to Z_n = Z_{Q_n}$ over Q_n. Since $X_n(n)$ is Q_n-flat by 2.2.1, we can apply 1.4 to the closed embedding $j : X_n(n) \to X_n \times_{Z_n} X_n$ over Q_n. Note that j_f is nothing else but the identity $X(n) = X \times_Y X$. Since the canonical projection $X_n \times_{Z_n} X_n \to Q_n$ is proper, we obtain an open $U \subset Q_n$ with $f \in U$ and such that j is biholomorphic over the whole of U. By 1.3, q_U is a family of substructures, and, by construction, $\Omega_{X\times U/Z_U}(l) = 0$ whenever $l > n$. Furthermore, $\Omega_{X\times U/Z_U}(l)$ is U-flat for $l \leq n$ because $U \subset Q_n$; see 2.2.3. □

It is worth mentioning a by-product of the above proof:

Remark 3.3. The natural mapping $W \to Q$ is an injective immersion.

As a complement we state that a compact Riemann surface X of arbitrary genus can be embedded as a connected component in its moduli space W_X. For each $x \in X$, let Y_x be the cuspidal cubic curve with normalization X and $f_x(x)$ as its only singularity, where $f_x : X \to Y_x$ denotes the normalization mapping.

Remark 3.4. The mapping $X \to W_X, x \mapsto f_x$ is holomorphic. Moreover, it is an open and closed embedding.

Proof. (sketch) We start with the family of quotients over the symmetric product $S^2(X)$ which is given by identifying two variable points on X. As restriction to the diagonal in $S^2(X)$ we obtain an element of $W_X(X)$ having the required properties. For more details we refer to [4], 3.2; see also [5]. □

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