PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 8, August 1998, Pages 2323–2331
S 0002-9939(98)04272-5

CONVEX LINEAR COMBINATIONS OF SEQUENCES OF MONIC ORTHOGONAL POLYNOMIALS

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(Communicated by J. Marshall Ash)

ABSTRACT. For a sequence $\{\Phi_n\}_0^\infty$ of monic orthogonal polynomials (SMOP), with respect to a positive measure supported on the unit circle, we obtain necessary and sufficient conditions on a SMOP $\{Q_n\}_0^\infty$ in order that a convex linear combination $\{R_n\}_0^\infty$ with $R_n = \beta\Phi_n + (1-\beta)Q_n$ be a SMOP with respect to a positive measure supported on the unit circle.

1. INTRODUCTION

Let μ be a finite positive Borel measure supported on $[0, 2\pi]$, and let $\{\varphi_n\}_0^\infty$ be the corresponding sequence of orthonormal polynomials, i.e.,

$$\int_0^{2\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}d\mu(\theta) = \delta_{n,m}, \qquad n, m \geq 0,$$

where $\varphi_n(z) = k_n z^n +$ lower degree terms, $k_n > 0$.

We denote the sequence of monic orthogonal polynomials (SMOP) associated with μ by $\{\Phi_n\}_0^\infty$, where $\Phi_n = {k_n}^{-1}\varphi_n$. It is well known that $\{\Phi_n\}_0^\infty$ satisfies for $n \geq 1$ the following recurrence relations:

$$\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z), \tag{1.1}$$

$$\Phi_n^*(z) = \Phi_{n-1}^*(z) + \overline{\Phi_n(0)}z\Phi_{n-1}(z), \tag{1.2}$$

$$\Phi_n(z) = (1 - |\Phi_n(0)|^2)z\Phi_{n-1}(z) + \Phi_n(0)\Phi_n^*(z), \tag{1.3}$$

$$\Phi_n^*(z) = (1 - |\Phi_n(0)|^2)\Phi_{n-1}^*(z) + \overline{\Phi_n(0)}\Phi_n(z), \tag{1.4}$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n}(z^{-1})$ is the reversed polynomial of $\Phi_n(z)$. For details about such recurrence relations, see [4], [5], and [9].

Furthermore, $\{\Phi_n\}_0^\infty$ satisfies the three-term recurrence relation

$$\Phi_n(0)\Phi_{n+1}(z) = (z\Phi_n(0) + \Phi_{n+1}(0))\,\Phi_n(z) - z(1 - |\Phi_n(0)|^2)\Phi_{n+1}(0)\Phi_{n-1}(z),$$

$n \geq 0$, with initial conditions $\Phi_{-1}(z) = 0$ and $\Phi_0(z) = 1$.

The values $\Phi_n(0)$ are called reflection parameters, and they satisfy $|\Phi_n(0)| < 1$

Received by the editors March 4, 1996 and, in revised form, January 13, 1997.
1991 *Mathematics Subject Classification.* Primary 42C05.
Key words and phrases. Orthogonal polynomials, C-functions, measures on the unit circle.
The work of the first author was supported by the DGICYT under grant number PB93-1169.
The work of the second author was supported by an Acción Integrada Hispano-Austriaca 4B/1995.

for $n \geq 1$. Conversely, given a sequence of complex numbers a_n such that $|a_n| < 1$, there exists a unique positive measure μ such that $a_n = \Phi_n(0)$, where $\{\Phi_n\}_0^\infty$ denotes the SMOP with respect to μ. In a certain sense, this is the analogue of Favard's theorem on the unit circle (see [3]).

On the other hand, if we consider the second order linear difference equation

$$a_n y_{n+1} = (za_n + a_{n+1})y_n - z(1 - |a_n|^2)a_{n+1}y_{n-1}, \quad n \geq 1,$$

the linear space of the solutions is two dimensional. Of course, one of the solutions is our SMOP $\{\Phi_n\}_0^\infty$, which corresponds to the initial conditions $y_0 = 1$ and $y_1 = z + a_1$.

If $a_N = 0$ for some $N \geq 1$, then

$$a_{N-1}y_N = za_{N-1}y_{N-1} \quad \text{and} \quad a_{N+1}y_N = za_{N+1}y_{N-1}.$$

By convention $y_N = zy_{N-1}$ and $y_N(0) = a_N$. Furthermore, if $a_{N+1} \neq 0$, then y_{N+2} can be given explicitly in terms of y_N and y_{N+1} by

$$a_{N+1}y_{N+2} = (za_{N+1} + a_{N+2})y_{N+1} - z(1 - |a_{N+1}|^2)a_{N+2}y_N.$$

In such a case, we can choose $y_{N+1} = zy_N + a_{N+1}y_N = z^2y_{N-1} + a_{N+1}y_{N-1}$, because $a_{N+1}y_{N+2}(0) = a_{N+2}y_{N+1}(0)$. If $a_{N+1} = 0$, again $y_{N+1} = zy_N$ and y_{N+2} cannot be explicitly defined from the recurrence relation. If $a_1 \neq 0$, a second linearly independent solution $\{\Psi_n\}_0^\infty$ of the above difference equation appears when $y_0 = 1$ and $y_1 = z - a_1$. This means that $\Psi_n(0) = -a_n$ and, as a consequence, $\{\Psi_n\}_0^\infty$ is a SMOP explicitly given in terms of $\{\Phi_n\}_0^\infty$ in the following way:

(1.5)
$$\Psi_n(z) = \frac{1}{c_0}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\left(\Phi_n(e^{i\theta}) - \Phi_n(z)\right)d\mu(\theta), \quad n \geq 1, \quad \Psi_0(z) = 1,$$

where $c_0 = \int_0^{2\pi} d\mu(\theta)$. These polynomials $\{\Psi_n\}_0^\infty$ are called polynomials of the second kind with respect to μ.

Since $\{\Phi_n\}_0^\infty$ and $\{\Psi_n\}_0^\infty$ constitute a basis in the above linear space, it is straightforward to deduce that for $|\lambda| = 1$ the SMOP $\{\Phi_n^\lambda\}_0^\infty$, whose reflection parameters are $\tilde{a}_n = \lambda a_n$ for $n \geq 1$, can be expressed in terms of $\{\Phi_n\}_0^\infty$ and $\{\Psi_n\}_0^\infty$ in the following way:

$$\Phi_n^\lambda(z) = \frac{1+\lambda}{2}\Phi_n(z) + \frac{1-\lambda}{2}\Psi_n(z), \tag{1.6}$$

i.e., a linear convex combination of the elements of the above basis (see [5, Section 7]).

The aim of this paper is to extend this property of convexity in order to characterize sequences of monic orthogonal polynomials $\{\Phi_n\}_0^\infty$ such that there exist $\beta \in \mathbb{C}$ and a SMOP $\{Q_n\}_0^\infty$ for which the sequence $\{R_n\}_0^\infty$ defined by

$$R_n = \beta\Phi_n + (1-\beta)Q_n, \quad n \geq 0,$$

is a SMOP. We will use the Carathéodory function (or C-function) associated with the measure μ defined by

$$F(z) = \frac{1}{c_0}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}d\mu(\theta)$$

in order to obtain the C-function associated with $\{R_n\}_0^\infty$ in terms of F. Then the measure associated with $\{R_n\}_0^\infty$ can be easily deduced.

In [2] the authors considered the problem of finding necessary and sufficient conditions on a SMOP $\{\Phi_n\}_0^\infty$ and a sequence of complex numbers $\{\alpha_n\}_0^\infty$ so that $\{\Phi_n + \alpha_n \Phi_{n-1}\}_0^\infty$ would be a SMOP. This result was extended in [6], where for a fixed k a finite linear combination

$$\Omega_n = \Phi_n + \sum_{j=n-k}^{n-1} \lambda_{n,j} \Phi_j$$

is considered. $\{\Omega_n\}_0^\infty$ belongs to the Bernstein-Szegő class (see [9, Theorem 11.2, p. 289]), and $\{\Phi_n\}_0^\infty$ is a SMOP relative to a positive trigonometric rational weight function.

2. Main results

Lemma 1. *Let* $\{\Phi_n\}_0^\infty$ *and* $\{\Psi_n\}_0^\infty$ *be the* SMOPs *defined in* (1.1) *and* (1.5) *respectively and* $\beta \in \mathbb{C}$.

i) If $|2\beta - 1| = 1$, *then the sequence* $\{R_n\}_0^\infty$ *given by*

$$R_n = \beta \Phi_n + (1-\beta)\Psi_n \tag{2.1}$$

is always a SMOP.

ii) If $|2\beta - 1| \neq 1$, (2.1) *is a* SMOP *if and only if*

$$\Phi_n(z) = z^{n-1}(z+a) \text{ for } n \geq 1, \text{ with } \quad |a| < \min\left\{\frac{1}{|2\beta - 1|}, 1\right\},$$

or

$$\Phi_n(z) = z^n \quad 1 \leq n \leq N \quad \text{and} \quad \Phi_n(z) = z^{n-(N+1)}(z^{N+1} + b) \quad n \geq N+1,$$

for some $N \geq 1$, *with* $0 < |b| < \min\{1/|2\beta - 1|, 1\}$, *i.e.,* $\{\Phi_n\}_0^\infty$ *belongs to the Bernstein-Szegő class.*

Proof. $\Rightarrow$) Because of (1.1), for $n \geq 0$

$$R_{n+1}(z) = zR_n(z) + R_{n+1}(0)R_n^*(z).$$

Taking into account the definition of $R_n(z)$, we have for $n \geq 0$

$$\begin{aligned}&\beta\Phi_{n+1}(z) + (1-\beta)\Psi_{n+1}(z)\\ &\quad = \beta z \Phi_n(z) + (1-\beta) z \Psi_n(z) + R_{n+1}(0)\left(\overline{\beta}\Phi_n^*(z) + (1-\overline{\beta})\Psi_n^*(z)\right).\end{aligned}$$

Thus,

$$\beta\Phi_{n+1}(0)\Phi_n^* - (1-\beta)\Phi_{n+1}(0)\Psi_n^* = R_{n+1}(0)\left(\overline{\beta}\Phi_n^* + (1-\overline{\beta})\Psi_n^*\right),$$

or, equivalently,

$$\left(\beta\Phi_{n+1}(0) - \overline{\beta}R_{n+1}(0)\right)\Phi_n^* = \left((1-\overline{\beta})R_{n+1}(0) + (1-\beta)\Phi_{n+1}(0)\right)\Psi_n^*.$$

Using

$$R_n(0) = \beta\Phi_n(0) - (1-\beta)\Phi_n(0) = (2\beta - 1)\Phi_n(0)$$

in the above relation, we obtain for $n \geq 0$

$$\left(\beta - \overline{\beta}(2\beta - 1)\right)\Phi_{n+1}(0)\Phi_n^* = \left((1-\overline{\beta})(2\beta-1) + (1-\beta)\right)\Phi_{n+1}(0)\Psi_n^*.$$

That is,

$$\left(\beta + \overline{\beta} - 2|\beta|^2\right)\Phi_{n+1}(0)\Phi_n^* = \left(\beta + \overline{\beta} - 2|\beta|^2\right)\Phi_{n+1}(0)\Psi_n^*,$$

from which it follows that

$$(1-|1-2\beta|^2)\Phi_{n+1}(0)\,(\Phi_n^*-\Psi_n^*)=0,$$

or, equivalently,

$$(1-|1-2\beta|^2)\overline{\Phi_{n+1}(0)}\,(\Phi_n-\Psi_n)=0,\ \ n\geq 0.$$

Therefore, either $|1-2\beta|=1$ or, if $|1-2\beta|\neq 1$, then $\overline{\Phi_{n+1}(0)}\,(\Phi_n-\Psi_n)=0$ for $n\geq 0$. In this second case, either $\Phi_{n+1}(0)=0$ for every $n\geq 0$, or there exists at most one Φ_N with $N\geq 1$ such that $\Phi_N(0)\neq 0$. To prove this, suppose $\Phi_N(0)\neq 0$ and $\Phi_M(0)\neq 0$ with $M>N\geq 1$. Then $\Phi_{M-1}=\Psi_{M-1}$, i.e., $\Phi_{M-1}(0)=0$. Thus, if we use the recurrence relation (1.1), then $\Phi_{M-2}=\Psi_{M-2}$. After a finite number of steps, we get $\Phi_N=\Psi_N$, i.e., $\Phi_N(0)=0$, and this yields a contradiction.

We conclude that
either $\Phi_{n+1}(0)=0$ for $n\geq 1$, i.e., $\Phi_n(z)=z^{n-1}(z+a)$ for $n\geq 1$, where $|a|<1$,
or there exists a unique $N\geq 1$ such that $\Phi_{N+1}(0)\neq 0$ and $\Phi_n(0)=0$ for $n\neq N+1$. Then $\Phi_n(z)=z^{n-(N+1)}\Phi_{N+1}(z)$ for $n\geq N+1$, and $\Phi_n(z)=z^n$ for $1\leq n\leq N$.

$\Leftarrow$) If $|2\beta-1|=1$, then $2\beta=1+\lambda$ with $|\lambda|=1$. Thus $\beta=\dfrac{1+\lambda}{2}$ and $R_n(z)=\Phi_n^\lambda(z)$ follows from (1.6).

If $|2\beta-1|\neq 1$ and $\Phi_n(z)=z^{n-(N+1)}\Phi_{N+1}(z)$ for $n\geq N+1$ and $\Phi_n(z)=z^n$ for $1\leq n\leq N$, it is easy to check that

$$R_{n+1}(z)=\beta\Phi_{n+1}(z)+(1-\beta)\Psi_{n+1}(z)=z\,(\beta\Phi_n(z)+(1-\beta)\Psi_n(z))=zR_n(z)$$

when $n\geq N+1$ or $n<N$. Furthermore, $R_{N+1}(0)=(2\beta-1)\Phi_{N+1}(0)$, and since $|\Phi_{N+1}(0)|<\min\left\{\frac{1}{|2\beta-1|},1\right\}$, then $|R_{N+1}(0)|<1$. If $\Phi_n(z)=z^{n-1}\Phi_1(z)$ for $n\geq 1$, then $R_{n+1}(z)=zR_n(z)$, $n\geq 1$, and since $|\Phi_1(0)|<\min\left\{\frac{1}{|2\beta-1|},1\right\}$, then $|R_1(0)|<1$. Therefore $\{R_n\}_0^\infty$ is a SMOP. □

Remark 1. Notice that from ii) in the above lemma, we can obtain the sequence $\{R_n\}_0^\infty$ as follows:

- If $\Phi_n(z)=z^{n-1}\Phi_1(z)=z^{n-1}(z+a)$, $n\geq 1$, then

$$R_n(z)=\beta z^{n-1}(z+a)+(1-\beta)z^{n-1}(z-a)=z^{n-1}\left(z+(2\beta-1)a\right),\quad n\geq 1.$$

- If $\Phi_1(0)=0$ and $\Phi_{N+1}(0)\neq 0$ for some $N\geq 1$, then

$$R_n(z)=z^n,\quad n\leq N,$$

and

$$\begin{aligned}R_n(z)&=\beta z^{n-1}\left(z+\Phi_{N+1}(0)z^{-N}\right)+(1-\beta)z^{n-1}\left(z-\Phi_{N+1}(0)z^{-N}\right)\\&=z^{n-1}\left(z+(2\beta-1)\Phi_{N+1}(0)z^{-N}\right),\qquad n\geq N+1.\end{aligned}$$

Lemma 2. *Let* $\{\Phi_n\}_0^\infty$ *be a* SMOP. *For* $\lambda\neq 1$, *let* $\{\Phi_n^\lambda\}_0^\infty$ *be as in* (1.6). *Then the sequence* $\{R_n\}_0^\infty$ *given by*

$$R_n=\beta\Phi_n+(1-\beta)\Phi_n^\lambda \tag{2.2}$$

is a SMOP *if and only if either*

i) $\left|\beta+\dfrac{\lambda}{1-\lambda}\right|=\dfrac{1}{|1-\lambda|}$, *or* *ii)* $\left|\beta + \frac{\lambda}{1-\lambda}\right| \neq \frac{1}{|1-\lambda|}$, *and* $\{\Phi_n\}_0^\infty$ *is either of the form*

$$\Phi_n(z) = z^{n-1}(z+a) \quad \text{for } n \geq 1, \text{ with } \quad |a| < \min\left\{\frac{1}{|(1-\lambda)\beta+\lambda|}, 1\right\},$$

or of the form

$$\Phi_n(z) = z^n, \quad 1 \leq n \leq N, \quad \text{and} \quad \Phi_n(z) = z^{n-(N+1)}(z^{N+1}+b), \quad n \geq N+1,$$

for some $N \geq 1$, *with* $0 < |b| < \min\{1/|(1-\lambda)\beta+\lambda|, 1\}$.

Proof. By (1.6), (2.2) can be reduced to

$$\begin{aligned} R_n &= \beta\Phi_n + (1-\beta)\left(\frac{1+\lambda}{2}\Phi_n + \frac{1-\lambda}{2}\Psi_n\right) \\ &= \left(\beta + (1-\beta)\frac{1+\lambda}{2}\right)\Phi_n + \frac{(1-\beta)(1-\lambda)}{2}\Psi_n. \end{aligned}$$

According to Lemma 1, $\{R_n\}_0^\infty$ is a SMOP if and only if either

i) $|2\beta + (1-\beta)(1+\lambda) - 1| = |(1-\lambda)\beta + \lambda| = 1$,

or

ii) $|(1-\lambda)\beta+\lambda| \neq 1$, in which case $\{\Phi_n\}_0^\infty$ is the corresponding SMOP defined as in Lemma 1, *ii*). □

Theorem 1. *Let* $\{\Phi_n\}_0^\infty$ *and* $\{Q_n\}_0^\infty$ *be two* SMOPs. *Let*

$$R_n = \beta\Phi_n + (1-\beta)Q_n, \quad n \geq 0,$$

where $\beta \in \mathbb{C}\backslash\{0,1\}$. *Then* $\{R_n\}_0^\infty$ *is a* SMOP *if and only if*

$$Q_n(0) = \Phi_n(0) \text{ for } n \leq N, \ Q_n(0) = \Phi_n^\lambda(0) \text{ for } n \geq N+2, \text{ with } N \geq 0$$

and either

i) $Q_{N+1}(0) = \Phi_{N+1}^\lambda(0)$

or

ii) $Q_{N+1}(0) \neq \Phi_{N+1}^\lambda(0)$, *and* $|\beta\Phi_{N+1}(0) + (1-\beta)Q_{N+1}(0)| < 1$, *with* $\lambda = \beta(1-\overline{\beta})/\overline{\beta}(1-\beta)$.

Proof. ⇒) Let us suppose that $\{R_n\}_0^\infty$ is a SMOP. Then

$$\begin{aligned} &\beta\Phi_{n+1}(z) + (1-\beta)Q_{n+1}(z) \\ &\qquad = z\left(\beta\Phi_n(z) + (1-\beta)Q_n(z)\right) + R_{n+1}(0)\left(\overline{\beta}\Phi_n^*(z) + (1-\overline{\beta})Q_n^*(z)\right). \end{aligned}$$

Using the recurrence relations for $\{\Phi_n\}_0^\infty$ and $\{Q_n\}_0^\infty$, we have

$$\left(\beta\Phi_{n+1}(0) - \overline{\beta}R_{n+1}(0)\right)\Phi_n^*(z) = \left((1-\overline{\beta})R_{n+1}(0) - (1-\beta)Q_{n+1}(0)\right)Q_n^*(z).$$

Thus

$$\begin{aligned} &\left(\beta(1-\overline{\beta})\Phi_{n+1}(0) - \overline{\beta}(1-\beta)Q_{n+1}(0)\right)\Phi_n^*(z) \\ &= \left(\beta(1-\overline{\beta})\Phi_{n+1}(0) - \overline{\beta}(1-\beta)Q_{n+1}(0)\right)Q_n^*(z) \end{aligned}$$

or, equivalently,

$$\left(\overline{\beta}(1-\beta)\overline{\Phi_{n+1}(0)} - \beta(1-\overline{\beta})\overline{Q_{n+1}(0)}\right)(\Phi_n(z) - Q_n(z)) = 0, \quad \text{for} \quad n \geq 0.$$

Let $A = \{n \geq 0 \ : \overline{\beta}(1-\beta)\overline{\Phi_{n+1}(0)} \neq \beta(1-\overline{\beta})\overline{Q_{n+1}(0)}\}$.

1) If A is a finite set, we will consider two situations:

1.i) $A = \varnothing$ leads to $Q_{n+1}(0) = \dfrac{\beta(1-\overline{\beta})}{\overline{\beta}(1-\beta)}\Phi_{n+1}(0)$, that is,

$$Q_{n+1}(0) = \lambda\Phi_{n+1}(0) \quad \text{for} \quad n \geq 0 \quad \text{and} \quad \lambda = \frac{\beta(1-\overline{\beta})}{\overline{\beta}(1-\beta)}.$$

1.ii) If $A \neq \varnothing$, let $M = \max A$. Then $\Phi_M = Q_M$ and as a consequence of the recurrence relation (1.3) we have $Q_n = \Phi_n$ for $n \leq M$.

On the other hand, since

$$\beta(1-\overline{\beta})\Phi_{n+1}(0) = \overline{\beta}(1-\beta)Q_{n+1}(0) \quad \text{for} \quad n > M,$$

then

$$Q_{n+1}(0) = \lambda\Phi_{n+1}(0), \quad \text{where} \quad \lambda = \frac{\beta(1-\overline{\beta})}{\overline{\beta}(1-\beta)}, \quad \text{for} \quad n \geq M+1.$$

2) If A is an infinite set, given $N \geq 0$ there exists $M' \in A$ such that $M' > N$. Then, as before, $\Phi_{M'} = Q_{M'}$ and $Q_n = \Phi_n$ for $n \leq M'$, i.e., $Q_n = \Phi_n$ for $n \geq 0$.

$\Leftarrow$) Straightforward calculations give

$$\begin{aligned}
&R_{n+1}(z) - zR_n(z) - R_{n+1}(0)R_n^*(z) \\
&\quad = \left(\beta\Phi_{n+1}(0) - \overline{\beta}R_{n+1}(0)\right)\Phi_n^*(z) \\
&\qquad + \left((1-\beta)Q_{n+1}(0) - (1-\overline{\beta})R_{n+1}(0)\right)Q_n^*(z) \\
&\quad = \left(\beta(1-\overline{\beta})\Phi_{n+1}(0) - \overline{\beta}(1-\beta)Q_{n+1}(0)\right)\Phi_n^*(z) \\
&\qquad - \left(\beta(1-\overline{\beta})\Phi_{n+1}(0) - \overline{\beta}(1-\beta)Q_{n+1}(0)\right)Q_n^*(z) \\
&\quad = \left(\beta(1-\overline{\beta})\Phi_{n+1}(0) - \overline{\beta}(1-\beta)Q_{n+1}(0)\right)\left(\Phi_n^*(z) - Q_n^*(z)\right).
\end{aligned}$$

If $Q_n(0) = \Phi_n(0)$ for $n \leq N$, then $Q_n = \Phi_n$ for $n \leq N$ and the above expression vanishes for $n \leq N$. If $Q_n(0) = \Phi_n^\lambda(0)$ for $n \geq N+2$, the above expression vanishes for $n \geq N+1$. □

Remark 2. 1) If $\beta \in \mathbb{R}\backslash\{0,1\}$, then $\lambda = 1$ and $Q_n(0) = \Phi_n(0)$ for $n \neq N+1$. This is a perturbation of the reflection parameter $\Phi_{N+1}(0)$, while the others remain invariant. If $Q_{N+1}(0) \neq \Phi_{N+1}(0)$, then $R_n(0) = \Phi_n(0)$ for $n \neq N+1$, and $R_{N+1}(0) = \Phi_{N+1}(0) + \alpha$.

On the other hand, $R_n(z) = \Phi_n(z)$ for $n \leq N$, and $R_{N+1}(z) = \Phi_{N+1}(z) + \alpha\Phi_n^*(z)$. For the other terms of the sequence $\{R_n\}_0^\infty$, notice that

$$R_n^{(N+1)}(z) = \Phi_n^{(N+1)}(z), \;\; n \geq 0,$$

where the superscript denotes the $(N+1)$th associated polynomial introduced in [7, Definition 3.1]. But according to [7, Theorem 3.1], we have for $n \geq 0$

$$\begin{aligned}
&\Phi_n^{(N+1)}(z) \\
&\quad = \frac{\left(\Psi_{N+1}(z) + \Psi_{N+1}^*(z)\right)\Phi_{n+N+1}(z) + \left(\Phi_{N+1}^*(z) - \Phi_{N+1}(z)\right)\Psi_{n+N+1}(z)}{d_{N+1}z^{N+1}}, \\
&\Psi_n^{(N+1)}(z) \\
&\quad = \frac{\left(\Phi_{N+1}(z) + \Phi_{N+1}^*(z)\right)\Psi_{n+N+1}(z) + \left(\Psi_{N+1}^*(z) - \Psi_{N+1}(z)\right)\Phi_{n+N+1}(z)}{d_{N+1}z^{N+1}},
\end{aligned}$$

where

$$d_{N+1} = 2c_0 \prod_{k=1}^{N+1} (1 - |\Phi_k(0)|^2).$$

Then, if $\{S_n\}_0^\infty$ denotes the SMOP of second kind associated with $\{R_n\}_0^\infty$,

(2.3)
$$\begin{aligned}\left(S_{N+1} + S^*_{N+1}\right) R_{n+N+1} &+ \left(R^*_{N+1} - R_{N+1}\right) S_{n+N+1} \\ &= \frac{(1 - |\Phi_{N+1}(0) + \alpha|^2)}{1 - |\Phi_{N+1}(0)|^2} \big((\Psi_{N+1} + \Psi^*_{N+1})\Phi_{n+N+1} \\ &\qquad + (\Phi^*_{N+1} - \Phi_{N+1})\Psi_{n+N+1}\big),\end{aligned}$$

and

(2.4)
$$\begin{aligned}\left(R_{N+1} + R^*_{N+1}\right) S_{n+N+1} &+ \left(S^*_{N+1} - S_{N+1}\right) R_{n+N+1} \\ &= \frac{(1 - |\Phi_{N+1}(0) + \alpha|^2)}{1 - |\Phi_{N+1}(0)|^2} \big((\Phi_{N+1} + \Phi^*_{N+1})\Psi_{n+N+1} \\ &\qquad + (\Psi^*_{N+1} - \Psi_{N+1})\Phi_{n+N+1}\big).\end{aligned}$$

Denoting

$$R_\pm = R_{N+1} \pm R^*_{N+1} = \left(\Phi_{N+1} \pm \Phi^*_{N+1}\right) + (\alpha\Phi^*_N \pm \overline{\alpha} z \Phi_N)$$

and

$$S_\pm = S_{N+1} \pm S^*_{N+1} = \left(\Psi_{N+1} \pm \Psi^*_{N+1}\right) - (\alpha\Psi^*_N \pm \overline{\alpha} z \Psi_N),$$

formulas (2.3) and (2.4) may be expressed in matrix form as follows:

$$\begin{pmatrix} S_+ & -R_- \\ -S_- & R_+ \end{pmatrix} \begin{pmatrix} R_{n+N+1} \\ S_{n+N+1} \end{pmatrix} = \alpha_N \begin{pmatrix} \Psi_+ & -\Phi_- \\ -\Psi_- & \Phi_+ \end{pmatrix} \begin{pmatrix} \Phi_{n+N+1} \\ \Psi_{n+N+1} \end{pmatrix}.$$

Then

$$\begin{aligned}\begin{pmatrix} R_{n+N+1} \\ S_{n+N+1} \end{pmatrix} &= \alpha_N \begin{pmatrix} S_+ & -R_- \\ -S_- & R_+ \end{pmatrix}^{-1} \begin{pmatrix} \Psi_+ & -\Phi_- \\ -\Psi_- & \Phi_+ \end{pmatrix} \begin{pmatrix} \Phi_{n+N+1} \\ \Psi_{n+N+1} \end{pmatrix} \\ &= \frac{\alpha_N}{S_+R_+ - S_-R_-} \begin{pmatrix} R_+ & R_- \\ S_- & S_+ \end{pmatrix} \begin{pmatrix} \Psi_+ & -\Phi_- \\ -\Psi_- & \Phi_+ \end{pmatrix} \begin{pmatrix} \Phi_{n+N+1} \\ \Psi_{n+N+1} \end{pmatrix}.\end{aligned}$$

Therefore,

$$\begin{aligned}R_{n+N+1} = \frac{\alpha_N}{S_+R_+ - S_-R_-} \big(&(R_+\Psi_+ - R_-\Psi_-)\Phi_{n+N+1} \\ &- (R_+\Phi_- - R_-\Phi_+)\Psi_{n+N+1}\big)\end{aligned}$$

and

$$\begin{aligned}S_{n+N+1} = \frac{\alpha_N}{S_+R_+ - S_-R_-} \big(&(S_-\Psi_+ - S_+\Psi_-)\Phi_{n+N+1} \\ &+ (S_+\Phi_+ - S_-\Phi_-)\Psi_{n+N+1}\big).\end{aligned}$$

Thus, the relation between the corresponding C-functions is

(2.5)
$$\tilde{F} = \frac{A + BF}{C + DF},$$

where $A = S_+\Psi_- - S_-\Psi_+$, $B = \Phi_+S_+ - S_-\Phi_-$, $C = R_+\Psi_+ - R_-\Psi_-$ and $D = \Phi_-R_+ - R_-\Phi_+$ are self-reciprocal polynomials.

Hence, as in [7, Theorem 2.3] we can obtain the measure $\tilde{\mu}$ associated with $\tilde{F}$.

2) If $\beta \in \mathbb{C}\backslash\mathbb{R}$, then $\{Q_n(0)\}_0^\infty$ is a perturbation of the sequence $\{\Phi_n^\lambda(0)\}_0^\infty$ given by

$$Q_n(0) = \Phi_n(0), \quad n \leq N, \quad \text{and} \quad Q_{N+1}(0) \neq \Phi_{N+1}^\lambda(0)$$

for some $N \geq 0$. Using arguments similar to those employed above (taking into account that $Q_n^{(N+1)}(z) = (\Phi_n^\lambda)^{(N+1)}(z)$), we obtain a relation between the corresponding C-functions analogous to (2.5).

3) For both cases

$$R_n(0) = \Phi_n(0), \quad \text{for} \quad n \leq N,$$

and

$$\begin{aligned} R_n(0) &= \beta\Phi_n(0) + (1-\beta)\Phi_n^\lambda(0) = (\beta + \lambda(1-\beta))\,\Phi_n(0) \\ &= \left(\beta + \frac{\beta(1-\overline{\beta})}{\overline{\beta}}\right)\Phi_n(0) = \frac{\beta}{\overline{\beta}}\Phi_n(0) = \mu\Phi_n(0), \end{aligned}$$

for $n \geq N+2$, with $|\mu| = 1$.

Thus, we have proved

Corollary 1. *Under the assumptions of Theorem 1, the sequence* $\{R_n\}_0^\infty$ *given by*

$$R_n = \beta\Phi_n + (1-\beta)Q_n$$

is a SMOP *if and only*

$$R_n = \Phi_n, \quad n \leq N, \ \textit{and}\ R_n(0) = \Phi_n^\mu(0) = \mu\Phi_n(0), \quad n \geq N+2,$$

with $N \geq 0$ *and* $\mu = \beta/\overline{\beta}$.

Thus, $\{R_n\}_0^\infty$ *is a finite perturbation of* $\{\Phi_n^\mu\}_0^\infty$*, and the first* $N+1$ *reflection parameters are given by* $\{\Phi_n(0)\}_0^N$ *with the convention that* $\Phi_0(0) = 1$.

Such perturbations were introduced in [7]. For more details, see [8]. Notice that the C-function associated with $\{\Phi_n^\mu\}_0^\infty$ (see [1]) is

$$F^\mu = \frac{(\mu-1) + (\mu+1)F}{(\mu+1) + (1-\mu)F}.$$

Acknowledgements

We are grateful to Professor Guillermo López Lagomasino for his very helpful comments on the revised version of this manuscript.

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