CONVEX LINEAR COMBINATIONS OF SEQUENCES
OF MONIC ORTHOGONAL POLYNOMIALS

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Abstract. For a sequence \( \{ \Phi_n \}_0^\infty \) of monic orthogonal polynomials (SMOP), with respect to a positive measure supported on the unit circle, we obtain necessary and sufficient conditions on a SMOP \( \{ Q_n \}_0^\infty \) in order that a convex linear combination \( \{ R_n \}_0^\infty \) with \( R_n = \beta \Phi_n + (1 - \beta)Q_n \) be a SMOP with respect to a positive measure supported on the unit circle.

1. Introduction

Let \( \mu \) be a finite positive Borel measure supported on \([0, 2\pi] \), and let \( \{ \varphi_n \}_0^\infty \) be the corresponding sequence of orthonormal polynomials, i.e.,

\[
\int_0^{2\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}d\mu(\theta) = \delta_{n,m}, \quad n, m \geq 0,
\]

where \( \varphi_n(z) = k_n z^n + \text{lower degree terms}, \ k_n > 0. \)

We denote the sequence of monic orthogonal polynomials (SMOP) associated with \( \mu \) by \( \{ \Phi_n \}_0^\infty \), where \( \Phi_n = k_n^{-1} \varphi_n. \) It is well known that \( \{ \Phi_n \}_0^\infty \) satisfies for \( n \geq 1 \) the following recurrence relations:

\[
(1.1) \quad \Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi^{*}_{n-1}(z),
\]

\[
(1.2) \quad \Phi^{*}_n(z) = \Phi^{*}_{n-1}(z) + \overline{\Phi_n(0)}z\Phi_{n-1}(z),
\]

\[
(1.3) \quad \Phi_n(z) = (1 - |\Phi_n(0)|^2)z\Phi_{n-1}(z) + \Phi_n(0)\Phi^{*}_n(z),
\]

\[
(1.4) \quad \Phi^{*}_n(z) = (1 - |\Phi_n(0)|^2)\Phi^{*}_{n-1}(z) + \overline{\Phi_n(0)}\Phi_n(z),
\]

where \( \Phi^{*}_n(z) = z^n\Phi_n(z^{-1}) \) is the reversed polynomial of \( \Phi_n(z). \) For details about such recurrence relations, see [4], [5], and [9].

Furthermore, \( \{ \Phi_n \}_0^\infty \) satisfies the three-term recurrence relation

\[
\Phi_n(0)\Phi_{n+1}(z) = (z\Phi_n(0) + \Phi_{n+1}(0))\Phi_n(z) - z(1 - |\Phi_n(0)|^2)\Phi_{n+1}(0)\Phi_{n-1}(z),
\]

\( n \geq 0, \) with initial conditions \( \Phi_{-1}(z) = 0 \) and \( \Phi_0(z) = 1. \)

The values \( \Phi_n(0) \) are called reflection parameters, and they satisfy \( |\Phi_n(0)| < 1 \)

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for \( n \geq 1 \). Conversely, given a sequence of complex numbers \( a_n \) such that \( |a_n| < 1 \), there exists a unique positive measure \( \mu \) such that \( a_n = \Phi_n(0) \), where \( \{\Phi_n\}_0^\infty \) denotes the SMOP with respect to \( \mu \). In a certain sense, this is the analogue of Favard's theorem on the unit circle (see [3]).

On the other hand, if we consider the second order linear difference equation

\[
a_{n+1}y_{n+1} = (za_n + a_{n+1})y_n - z(1 - |a_n|^2)a_{n+1}y_{n-1}, \quad n \geq 1,
\]

the linear space of the solutions is two-dimensional. Of course, one of the solutions is our SMOP \( \{\Phi_n\}^\infty_0 \), which corresponds to the initial conditions \( y_0 = 1 \) and \( y_1 = z + a_1 \).

If \( a_N = 0 \) for some \( N \geq 1 \), then

\[
a_{N-1}y_N = za_{N-1}y_{N-1} \quad \text{and} \quad a_{N+1}y_N = za_{N+1}y_{N-1}.
\]

By convention \( y_N = zy_{N-1} \) and \( y_N(0) = a_N \). Furthermore, if \( a_{N+1} \neq 0 \), then \( y_{N+2} \)

\[
\text{can be given explicitly in terms of } y_N \text{ and } y_{N+1} \text{ by}
\]

\[
a_{N+1}y_{N+2} = (za_{N+1} + a_{N+2})y_{N+1} - z(1 - |a_{N+1}|^2)a_{N+2}y_N.
\]

In such a case, we can choose \( y_{N+1} = zy_N + a_{N+1}y_N = z^2y_{N-1} + a_{N+1}y_{N-1} \), because \( a_{N+1}y_{N+2}(0) = a_{N+2}y_{N+1}(0) \). If \( a_{N+1} = 0 \), again \( y_{N+1} = zy_N \) and \( y_{N+2} \)

\[
cannot be explicitly defined from the recurrence relation. If \( a_{N} \neq 0 \), a second linearly independent solution \( \{\Psi_n\}^\infty_0 \) of the above difference equation appears when \( y_0 = 1 \)

\[
\text{and } y_1 = z - a_1. \text{ This means that } \Psi_n(0) = -a_n \text{ and, as a consequence, } \{\Psi_n\}^\infty_0 \text{ is a SMOP explicitly given in terms of } \{\Phi_n\}^\infty_0 \text{ in the following way:}
\]

\[
(1.5) \quad \Psi_n(z) = \frac{1}{c_0} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \left( \Phi_n(e^{i\theta}) - \Phi_n(z) \right) d\mu(\theta), \quad n \geq 1, \quad \Psi_0(z) = 1,
\]

where \( c_0 = \int_0^{2\pi} d\mu(\theta) \). These polynomials \( \{\Psi_n\}^\infty_0 \) are called polynomials of the second kind with respect to \( \mu \).

Since \( \{\Phi_n\}^\infty_0 \) and \( \{\Psi_n\}^\infty_0 \) constitute a basis in the above linear space, it is straightforward to deduce that for \( |\lambda| = 1 \) the SMOP \( \{\Phi_n(\lambda)\}^\infty_0 \), whose reflection parameters are \( \tilde{a}_n = \lambda a_n \) for \( n \geq 1 \), can be expressed in terms of \( \{\Phi_n\}^\infty_0 \) and \( \{\Psi_n\}^\infty_0 \) in the following way:

\[
(1.6) \quad \Phi^\lambda_n(z) = \frac{1 + \lambda}{2} \Phi_n(z) + \frac{1 - \lambda}{2} \Psi_n(z),
\]

i.e., a linear convex combination of the elements of the above basis (see [5, Section 7]).

The aim of this paper is to extend this property of convexity in order to characterize sequences of monic orthogonal polynomials \( \{\Phi_n\}^\infty_0 \) such that there exist \( \beta \in \mathbb{C} \) and a SMOP \( \{Q_n\}^\infty_0 \) for which the sequence \( \{R_n\}^\infty_0 \) defined by

\[
R_n = \beta \Phi_n + (1 - \beta)Q_n, \quad n \geq 0,
\]

is a SMOP. We will use the Carathéodory function (or C-function) associated with the measure \( \mu \) defined by

\[
F(z) = \frac{1}{c_0} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)
\]

in order to obtain the C-function associated with \( \{R_n\}^\infty_0 \) in terms of \( F \). Then the measure associated with \( \{R_n\}^\infty_0 \) can be easily deduced.
In [2] the authors considered the problem of finding necessary and sufficient conditions on a SMOP \( \{\Phi_n\}_0^\infty \) and a sequence of complex numbers \( \{\alpha_n\}_0^\infty \) so that \( \{\Phi_n + \alpha_n \Phi_{n-1}\}_0^\infty \) would be a SMOP. This result was extended in [6], where for a fixed \( k \) a finite linear combination
\[
\Omega_n = \Phi_n + \sum_{j=n-k}^{n-1} \lambda_{n,j} \Phi_j
\]
is considered. \( \{\Omega_n\}_0^\infty \) belongs to the Bernstein-Szegő class (see [9, Theorem 11.2, p. 289]), and \( \{\Phi_n\}_0^\infty \) is a SMOP relative to a positive trigonometric rational weight function.

2. Main results

**Lemma 1.** Let \( \{\Phi_n\}_0^\infty \) and \( \{\Psi_n\}_0^\infty \) be the SMOPs defined in (1.1) and (1.5) respectively and \( \beta \in \mathbb{C} \).

i) If \( |2\beta - 1| = 1 \), then the sequence \( \{R_n\}_0^\infty \) given by
\[
(2.1) \quad R_n = \beta \Phi_n + (1 - \beta) \Psi_n
\]
is always a SMOP.

ii) If \( |2\beta - 1| \neq 1 \), (2.1) is a SMOP if and only if
\[
\Phi_n(z) = z^{n-1}(z + a) \quad \text{for} \quad n \geq 1, \quad \text{with} \quad |a| < \min\left\{\frac{1}{|2\beta - 1|}, 1\right\},
\]
or
\[
\Phi_n(z) = z^n \quad 1 \leq n \leq N \quad \text{and} \quad \Phi_n(z) = z^{n-(N+1)}(z^{N+1} + b) \quad n \geq N + 1,
\]
for some \( N \geq 1 \), with \( 0 < |b| < \min\{1/|2\beta - 1|, 1\} \), i.e., \( \{\Phi_n\}_0^\infty \) belongs to the Bernstein-Szegő class.

**Proof.** \( \Rightarrow \) Because of (1.1), for \( n \geq 0 \)
\[
R_{n+1}(z) = z R_n(z) + R_{n+1}(0) R_n^*(z).
\]
Taking into account the definition of \( R_n(z) \), we have for \( n \geq 0 \)
\[
\beta \Phi_{n+1}(z) + (1 - \beta) \Psi_{n+1}(z) = \beta z \Phi_n(z) + (1 - \beta) z \Psi_n(z) + R_{n+1}(0) \left( \overline{\beta} \Phi^*_n(z) + (1 - \overline{\beta}) \Psi^*_n(z) \right).
\]
Thus,
\[
\beta \Phi_{n+1}(0) \Phi^*_n - (1 - \beta) \Phi_{n+1}(0) \Psi^*_n = R_{n+1}(0) \left( \overline{\beta} \Phi^*_n + (1 - \overline{\beta}) \Psi^*_n \right),
\]
or, equivalently,
\[
(\beta \Phi_{n+1}(0) - \overline{\beta} R_{n+1}(0)) \Phi^*_n = ((1 - \overline{\beta}) R_{n+1}(0) + (1 - \beta) \Phi_{n+1}(0)) \Psi^*_n.
\]
Using
\[
R_n(0) = \beta \Phi_n(0) - (1 - \beta) \Phi_n(0) = (2\beta - 1) \Phi_n(0)
\]
in the above relation, we obtain for \( n \geq 0 \)
\[
(\beta - \overline{\beta}(2\beta - 1)) \Phi_{n+1}(0) \Phi^*_n = ((1 - \overline{\beta})(2\beta - 1) + (1 - \beta)) \Phi_{n+1}(0) \Psi^*_n.
\]
That is,
\[
\left(\beta + \overline{\beta} - 2|\beta|^2\right) \Phi_{n+1}(0) \Phi^*_n = \left(\beta + \overline{\beta} - 2|\beta|^2\right) \Phi_{n+1}(0) \Psi^*_n,
\]
from which it follows that
\[ (1 - |1 - 2\beta|^2)\Phi_{n+1}(0) (\Phi_n^* - \Psi_n^*) = 0, \]
or, equivalently,
\[ (1 - |1 - 2\beta|^2)\Phi_{n+1}(0) (\Phi_n - \Psi_n) = 0, \quad n \geq 0. \]

Therefore, either \(|1 - 2\beta| = 1\) or, if \(|1 - 2\beta| \neq 1\), then \(\Phi_{n+1}(0) (\Phi_n - \Psi_n) = 0\) for \(n \geq 0\). In this second case, either \(\Phi_N(0) = 0\) for every \(n \geq 0\), or there exists at most one \(\Phi_N\) with \(N \geq 1\) such that \(\Phi_N(0) \neq 0\). To prove this, suppose \(\Phi_N(0) \neq 0\) and \(\Phi_M(0) \neq 0\) with \(M > N \geq 1\). Then \(\Phi_{M-1} = \Psi_{M-1}\), i.e., \(\Phi_{M-1}(0) = 0\). Thus, if we use the recurrence relation (1.1), then \(\Phi_{M-2} = \Psi_{M-2}\). After a finite number of steps, we get \(\Phi_N = \Psi_N\), i.e., \(\Phi_N(0) = 0\), and this yields a contradiction.

We conclude that either \(\Phi_{n+1}(0) = 0\) for \(n \geq 1\), i.e., \(\Phi_n(z) = z^{n-1}(z+a)\) for \(n \geq 1\), where \(|a| < 1\), or there exists a unique \(N \geq 1\) such that \(\Phi_{N+1}(0) \neq 0\) and \(\Phi_n(0) = 0\) for \(n \neq N+1\). Then \(\Phi_n(z) = z^n\Phi_{N+1}(z)\) for \(n \geq N+1\), and \(\Phi_n(z) = z^n\) for \(1 \leq n \leq N\).

\(\Leftarrow\) If \(|2\beta - 1| = 1\), then \(2\beta = 1 + \lambda\) with \(|\lambda| = 1\). Thus \(\beta = \frac{1 + \lambda}{2}\) and \(R_n(z) = \Phi_n^*(z)\) follows from (1.6).

If \(|2\beta - 1| \neq 1\) and \(\Phi_n(z) = z^{n-(N+1)}\Phi_{N+1}(z)\) for \(n \geq N + 1\) and \(\Phi_n(z) = z^n\) for \(1 \leq n \leq N\), it is easy to check that
\[ R_{n+1}(z) = \beta \Phi_{n+1}(+1) + (1 - \beta)\Psi_{n+1}(z) = z (\beta \Phi_n(z) + (1 - \beta)\Psi_n(z)) = z R_n(z) \]
when \(n \geq N + 1\) or \(n < N\). Furthermore, \(R_{N+1}(0) = (2\beta - 1)\Phi_{N+1}(0)\), and since \(|\Phi_{N+1}(0)| < \min \left\{ \frac{1}{|2\beta - 1|}, 1 \right\}\), then \(|R_{N+1}(0)| < 1\). If \(\Phi_n(z) = z^{n-1}\Phi_1(z)\) for \(n \geq 1\), then \(R_{n+1}(z) = z R_n(z)\), \(n \geq 1\), and since \(|\Phi_1(0)| < \min \left\{ \frac{1}{|2\beta - 1|}, 1 \right\}\), then \(|R_1(0)| < 1\). Therefore \(\{R_n\}^\infty\) is a SMOP. \(\square\)

**Remark 1.** Notice that from ii) in the above lemma, we can obtain the sequence \(\{R_n\}^\infty\) as follows:

- If \(\Phi_1(z) = z^{n-1}\Phi_1(z) = z^{n-1}(z+a), \ n \geq 1\), then
  \[ R_n(z) = \beta z^{n-1}(z+a) + (1 - \beta)z^{n-1}(z-a) = z^{n-1}(z + (2\beta - 1)a), \ n \geq 1. \]
- If \(\Phi_1(0) = 0\) and \(\Phi_{N+1}(0) \neq 0\) for some \(N \geq 1\), then
  \[ R_n(z) = z^n, \quad n \leq N, \]
and
  \[ R_n(z) = \beta z^{n-1} \left( z + \Phi_{N+1}(0)z^{-N} \right) + (1 - \beta)z^{n-1} \left( z - \Phi_{N+1}(0)z^{-N} \right) = z^{n-1} \left( z + (2\beta - 1)\Phi_{N+1}(0)z^{-N} \right), \quad n \geq N + 1. \]

**Lemma 2.** Let \(\{\Phi_n\}^\infty_{0}\) be a SMOP. For \(\lambda \neq 1\), let \(\Phi_n^\lambda\) be as in (1.6). Then the sequence \(\{R_n\}^\infty\) given by
\[ R_n = \beta \Phi_n + (1 - \beta)\Phi_n^\lambda \]
is a SMOP if and only if either
\[ \left| \beta + \frac{\lambda}{1 - \lambda} \right| = \frac{1}{|1 - \lambda|}, \]

\[ i) \left| \beta + \frac{\lambda}{1 - \lambda} \right| = \frac{1}{|1 - \lambda|}, \]

or
ii) \( |\beta + \frac{\lambda}{1 - \lambda}| \neq \frac{1}{|1 - \lambda|} \), and \( \{\Phi_n\}_0^\infty \) is either of the form

\[ \Phi_n(z) = z^{n-1}(z + a) \quad \text{for } n \geq 1, \text{ with } |a| < \min \left\{ \frac{1}{|1 - \lambda|\beta + \lambda}, 1 \right\}, \]

or of the form

\[ \Phi_n(z) = z^n, \quad 1 \leq n \leq N, \quad \text{and} \quad \Phi_n(z) = z^{n-(N+1)}(z^{N+1} + b), \quad n \geq N + 1, \]

for some \( N \geq 1, \) with \( 0 < |b| < \min \{1/|1 - \lambda|\beta + \lambda, 1\}. \)

**Proof.** By (1.6), (2.2) can be reduced to

\[ R_n = \beta\Phi_n + (1 - \beta) \left( \frac{1 + \lambda}{2\Phi_n} + \frac{1 - \lambda}{2\Psi_n} \right) \]

\[ = \left( \beta + (1 - \beta) \frac{1 + \lambda}{2} \right) \Phi_n + \frac{(1 - \beta)(1 - \lambda)}{2\Psi_n}. \]

According to Lemma 1, \( \{R_n\}_0^\infty \) is a SMOP if and only if either

i) \( |2\beta + (1 - \beta)(1 + \lambda) - 1| = |(1 - \lambda)\beta + \lambda| = 1, \)

or

ii) \( |(1 - \lambda)\beta + \lambda| \neq 1, \) in which case \( \{\Phi_n\}_0^\infty \) is the corresponding SMOP defined as in Lemma 1, ii).

\[ \square \]

**Theorem 1.** Let \( \{\Phi_n\}_0^\infty \) and \( \{Q_n\}_0^\infty \) be two SMOPs. Let

\[ R_n = \beta\Phi_n + (1 - \beta)Q_n, \quad n \geq 0, \]

where \( \beta \in \mathbb{C}\setminus\{0,1\}. \) Then \( \{R_n\}_0^\infty \) is a SMOP if and only if

\( Q_n(0) = \Phi_n(0) \) for \( n \leq N, \) \( Q_n(0) = \Phi_n^\lambda(0) \) for \( n \geq N + 2, \) with \( N \geq 0 \)

and either

i) \( Q_{N+1}(0) = \Phi_{N+1}^\lambda(0) \)

or

ii) \( Q_{N+1}(0) \neq \Phi_{N+1}^\lambda(0), \) and \( |\beta\Phi_{N+1}(0) + (1 - \beta)Q_{N+1}(0)| < 1, \) with \( \lambda = \beta(1 - \beta)/\beta(1 - \beta). \)

**Proof.** \( \Rightarrow \) Let us suppose that \( \{R_n\}_0^\infty \) is a SMOP. Then

\[ \beta\Phi_{n+1}(z) + (1 - \beta)Q_{n+1}(z) \]

\[ = z(\beta\Phi_n(z) + (1 - \beta)Q_n(z)) + R_{n+1}(0) \left( \beta\Phi_n^\lambda(z) + (1 - \beta)Q_n^\lambda(z) \right). \]

Using the recurrence relations for \( \{\Phi_n\}_0^\infty \) and \( \{Q_n\}_0^\infty, \) we have

\( (\beta\Phi_{n+1}(0) - \beta R_{n+1}(0)) \Phi_n^\lambda(z) = ((1 - \beta)R_{n+1}(0) - (1 - \beta)Q_{n+1}(0)) Q_n^\lambda(z). \)

Thus

\[ (\beta(1 - \beta)\Phi_{n+1}(0) - \beta(1 - \beta)Q_{n+1}(0)) \Phi_n^\lambda(z) \]

\[ = (\beta(1 - \beta)\Phi_{n+1}(0) - \beta(1 - \beta)Q_{n+1}(0)) Q_n^\lambda(z), \]

or, equivalently,

\[ (\beta(1 - \beta)\Phi_{n+1}(0) - \beta(1 - \beta)Q_{n+1}(0)) (\Phi_n(z) - Q_n(z)) = 0, \quad \text{for } n \geq 0. \]

Let \( A = \{n \geq 0 : \beta(1 - \beta)\Phi_{n+1}(0) \neq \beta(1 - \beta)Q_{n+1}(0)\}. \)

1) If \( A \) is a finite set, we will consider two situations:
1.i) $A = \emptyset$ leads to $Q_{n+1}(0) = \frac{\beta(1 - \overline{\beta})}{\overline{\beta}(1 - \beta)} \Phi_{n+1}(0)$, that is,

$$Q_{n+1}(0) = \lambda \Phi_{n+1}(0) \quad \text{for} \quad n \geq 0 \quad \text{and} \quad \lambda = \frac{\beta(1 - \overline{\beta})}{\overline{\beta}(1 - \beta)}.$$

1.ii) If $A \neq \emptyset$, let $M = \max A$. Then $\Phi_M = Q_M$ and as a consequence of the recurrence relation (1.3) we have $Q_n = \Phi_n$ for $n \leq M$.

On the other hand, since $Q_{n+1}(0) = \lambda \Phi_{n+1}(0)$, the above expression vanishes for $n \geq M + 1$.

2) If $A$ is an infinite set, given $N \geq 0$ there exists $M' \in A$ such that $M' > N$. Then, as before, $\Phi_{M'} = Q_{M'}$ and $Q_n = \Phi_n$ for $n \leq M'$, i.e., $Q_n = \Phi_n$ for $n \geq 0$.

$\Leftarrow$) Straightforward calculations give

$$R_{n+1}(z) - zR_n(z) - R_{n+1}(0)R_n^*(z)$$

$$= (\beta \Phi_{n+1}(0) - \overline{\beta}R_{n+1}(0)) \Phi_n^*(z)$$

$$+ ((1 - \beta)Q_{n+1}(0) - (1 - \overline{\beta})R_{n+1}(0)) Q_n^*(z)$$

$$= (\beta(1 - \overline{\beta})\Phi_{n+1}(0) - \overline{\beta}(1 - \beta)Q_{n+1}(0)) \Phi_n^*(z)$$

$$- (\beta(1 - \overline{\beta})\Phi_{n+1}(0) - \overline{\beta}(1 - \beta)Q_{n+1}(0)) Q_n^*(z)$$

$$= (\beta(1 - \overline{\beta})\Phi_{n+1}(0) - \overline{\beta}(1 - \beta)Q_{n+1}(0)) (\Phi_n^*(z) - Q_n^*(z)).$$

If $Q_n(0) = \Phi_n(0)$ for $n \leq N$, then $Q_n = \Phi_n$ for $n \leq N$ and the above expression vanishes for $n \leq N$. If $Q_n(0) = \Phi_n^*(0)$ for $n \geq N + 2$, the above expression vanishes for $n \geq N + 1$.

Remark 2. 1) If $\beta \in \mathbb{R} \setminus \{0, 1\}$, then $\lambda = 1$ and $Q_n(0) = \Phi_n(0)$ for $n \neq N + 1$. This is a perturbation of the reflection parameter $\Phi_{N+1}(0)$, while the others remain invariant. If $Q_{N+1}(0) \neq \Phi_{N+1}(0)$, then $R_n(0) = \Phi_n(0)$ for $n \neq N + 1$, and $R_{N+1}(0) = \Phi_{N+1}(0) + \alpha$.

On the other hand, $R_n(z) = \Phi_n(z)$ for $n \leq N$, and $R_{N+1}(z) = \Phi_{N+1}(z) + \alpha \Phi_n^*(z)$. For the other terms of the sequence $\{R_n\}$, notice that

$$R_{n+1}^{(N+1)}(z) = \Phi_n^{(N+1)}(z), \quad n \geq 0,$$

where the superscript denotes the $(N + 1)$th associated polynomial introduced in [7, Definition 3.1]. But according to [7, Theorem 3.1], we have for $n \geq 0$

$$\Phi_n^{(N+1)}(z)$$

$$= (\Psi_{N+1}(z) + \Phi_n^*(z)) \Phi_{n+N+1}(z) + (\Phi_n^*(z) - \Phi_{N+1}(z)) \Psi_{n+N+1}(z),$$

$$\Psi_n^{(N+1)}(z)$$

$$= (\Phi_{N+1}(z) + \Phi_n^*(z)) \Psi_{n+N+1}(z) + (\Psi_{N+1}(z) - \Psi_{N+1}(z)) \Phi_{n+N+1}(z).$$
Then, if \( \{S_n\}_0^\infty \) denotes the SMOP of second kind associated with \( \{R_n\}_0^\infty \),
\[
(2.3) \quad (S_{n+1} + S^*_N) R_{n+N+1} + (R^*_N - R_{N+1}) S_{n+N+1}
\]
\[
= \frac{(1 - |\Phi_{N+1}(0) + \alpha|^2)}{1 - |\Phi_{N+1}(0)|^2} ((\Psi_{N+1} + \Psi^*_N) \Phi_{n+N+1}
\]
\[
+ (\Phi^*_{N+1} - \Phi_{N+1}) \Psi_{n+N+1}),
\]
and
\[
(2.4) \quad (R_{N+1} + R^*_N) S_{n+N+1} + (S^*_{N+1} - S_{N+1}) R_{n+N+1}
\]
\[
= \frac{(1 - |\Phi_{N+1}(0) + \alpha|^2)}{1 - |\Phi_{N+1}(0)|^2} ((\Phi_{N+1} + \Phi^*_N) \Psi_{n+N+1}
\]
\[
+ (\Psi^*_{N+1} - \Psi_{N+1}) \Phi_{n+N+1}).
\]

Denoting
\[
R_\pm = R_{N+1} \pm R^*_N = (\Phi_{N+1} \pm \Phi^*_N) + (\alpha \Phi^*_N \pm \alpha \Psi_N),
\]
and
\[
S_\pm = S_{N+1} \pm S^*_N = (\Psi_{N+1} \pm \Psi^*_N) - (\alpha \Psi^*_N \pm \alpha \Phi_N),
\]
formulas (2.3) and (2.4) may be expressed in matrix form as follows:
\[
\begin{pmatrix}
S_+ & -R_- \\
-S_- & R_+
\end{pmatrix}
\begin{pmatrix}
R_{n+N+1} \\
S_{n+N+1}
\end{pmatrix}
= \alpha_N \begin{pmatrix}
\Psi_+ & -\Phi_- \\
-\Psi_- & \Phi_+
\end{pmatrix}
\begin{pmatrix}
\Phi_{n+N+1} \\
\Psi_{n+N+1}
\end{pmatrix}.
\]

Then
\[
\begin{pmatrix}
R_{n+N+1} \\
S_{n+N+1}
\end{pmatrix}
= \alpha_N \begin{pmatrix}
S_+ & -R_- \\
-S_- & R_+
\end{pmatrix}^{-1}
\begin{pmatrix}
\Psi_+ & -\Phi_- \\
-\Psi_- & \Phi_+
\end{pmatrix}
\begin{pmatrix}
\Phi_{n+N+1} \\
\Psi_{n+N+1}
\end{pmatrix}
\]
\[
= \frac{\alpha_N}{S_+R_+ - S_-R_-}
\begin{pmatrix}
R_+ & R_- \\
S_+ & S_-
\end{pmatrix}
\begin{pmatrix}
\Psi_+ & -\Phi_- \\
-\Psi_- & \Phi_+
\end{pmatrix}
\begin{pmatrix}
\Phi_{n+N+1} \\
\Psi_{n+N+1}
\end{pmatrix}.
\]

Therefore,
\[
R_{n+N+1} = \frac{\alpha_N}{S_+R_+ - S_-R_-}
((\Psi_+ - \Psi-) \Phi_{n+N+1}
\]
\[
- (R_+ \Phi_+ - R_- \Phi_-) \Psi_{n+N+1})
\]
and
\[
S_{n+N+1} = \frac{\alpha_N}{S_+R_+ - S_-R_-}
((\Psi_+ - \Psi_-) \Psi_{n+N+1}
\]
\[
+ (S_+ \Phi_+ - S_- \Phi_-) \Psi_{n+N+1}).
\]

Thus, the relation between the corresponding C-functions is
\[
(2.5) \quad \tilde{F} = \frac{A + BF}{C + DF},
\]
where \( A = S_\Psi \Psi_\Psi - S_\Psi \Psi_+ \), \( B = \Phi_\Psi S_\Psi - S_\Psi \Phi_- \), \( C = R_+ \Psi_+ - R_- \Psi_- \) and \( D = \Phi_- R_+ - R_- \Phi_+ \) are self-reciprocal polynomials.
Hence, as in [7, Theorem 2.3] we can obtain the measure \( \tilde{\mu} \) associated with \( \tilde{F} \).

2) If \( \beta \in \mathbb{C} \setminus \mathbb{R} \), then \( \{Q_n(0)\}_{0}^{\infty} \) is a perturbation of the sequence \( \{\Phi_n^\lambda(0)\}_{0}^{N+1} \) given by

\[
Q_n(0) = \Phi_n(0), \quad n \leq N, \quad \text{and} \quad Q_{N+1}(0) \neq \Phi_{N+1}^\lambda(0)
\]

for some \( N \geq 0 \). Using arguments similar to those employed above (taking into account that \( Q_n^{(N+1)}(z) = (\Phi_n^\lambda)^{(N+1)}(z) \)), we obtain a relation between the corresponding C-functions analogous to (2.5).

3) For both cases

\[
R_n(0) = \Phi_n(0), \quad \text{for} \quad n \leq N,
\]

and

\[
R_n(0) = \beta \Phi_n(0) + (1 - \beta) \Phi_n^\lambda(0) = (\beta + \lambda(1 - \beta)) \Phi_n(0)
\]

\[
= \left( \beta + \frac{\beta(1 - \beta)}{\beta} \right) \Phi_n(0) = \frac{\beta}{\beta} \Phi_n(0) = \mu \Phi_n(0),
\]

for \( n \geq N + 2 \), with \( |\mu| = 1 \).

Thus, we have proved

**Corollary 1.** Under the assumptions of Theorem 1, the sequence \( \{R_n\}_{0}^{\infty} \) given by

\[
R_n = \beta \Phi_n + (1 - \beta)Q_n
\]

is a SMOP if and only

\[
R_n = \Phi_n, \quad n \leq N, \quad \text{and} \quad R_n(0) = \Phi_n^\mu(0) = \mu \Phi_n(0), \quad n \geq N + 2,
\]

with \( N \geq 0 \) and \( \mu = \beta/\beta \).

Thus, \( \{R_n\}_{0}^{\infty} \) is a finite perturbation of \( \{\Phi_n^\mu\}_{0}^{N} \), and the first \( N + 1 \) reflection parameters are given by \( \{\Phi_n(0)\}_{0}^{N} \) with the convention that \( \Phi_0(0) = 1 \).

Such perturbations were introduced in [7]. For more details, see [8]. Notice that the C-function associated with \( \{\Phi_n^\mu\}_{0}^{\infty} \) (see [1]) is

\[
F^\mu = \frac{(\mu - 1) + (\mu + 1)F}{(\mu + 1) + (1 - \mu)F}.
\]

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**References**


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