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CONJUGATE HARDY'S INEQUALITIES WITH DECREASING WEIGHTS

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ABSTRACT. We prove that for a decreasing weight ω on $\mathbf{R}^+$, the conjugate Hardy transform is bounded on $L_p(\omega)$ $(1 \le p < \infty)$ if and only if it is bounded on the cone of all decreasing functions of $L_p(\omega)$. This property does not depend on p.

1. INTRODUCTION

All functions are assumed to be measurable on $\mathbf{R}^+ = (0,\infty)$, which is endowed with Lebesgue measure. A decreasing function will be a non–negative and non–increasing function, and ω will be a weight, i.e., a non–negative function.

If $1 \le p < \infty$, we denote

$$L_p(\omega) = \Big\{ f;\ \|f\|_{L_p(\omega)} = \Big(\int_0^\infty |f(t)|^p \omega(t)\, dt \Big)^{1/p} < \infty \Big\}$$

and

$$L_p(\omega)^d = \{ f \in L_p(\omega);\ f \text{ decreasing} \}.$$

Results by Muckenhoupt [2] state that the Hardy operator

$$Pf(x) = \frac{1}{x} \int_0^x f(t)\, dt$$

is bounded on $L_p(\omega)$ if and only if

$$\sup_{r>0} \Big(\int_r^\infty \frac{\omega(s)}{s^p} ds \Big)^{1/p} \Big(\int_0^r \omega(s)^{-p'/p} ds \Big)^{1/p'} < \infty \tag{1}$$

in the case $1 < p < \infty$, and, for $p = 1$, if and only if

$$\int_t^\infty \frac{\omega(s)}{s} ds \le C\omega(t). \tag{2}$$

The corresponding condition for the conjugate Hardy operator

$$Qf(x) = \int_x^\infty f(t) \frac{dt}{t}$$

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is, if $1 < p < \infty$,

$$\sup_{r>0} \Big(\int_0^r \omega(s)\,ds \Big)^{1/p} \Big(\int_r^\infty \frac{\omega(s)^{-p'/p}}{s^{p'}} ds \Big)^{1/p'} < \infty \tag{3}$$

and, in the case $p = 1$,

$$\frac{1}{r} \int_0^r \omega(s)\,ds \le C\omega(r). \tag{4}$$

It is a result by Ariño and Muckenhoupt [1] that the operator

$$P : L_p(\omega)^d \longrightarrow L_p(\omega)$$

is bounded on $L_p(\omega)^d$ if and only if ω satisfies the so–called B_p–condition:

$$t^p \int_t^\infty \frac{\omega(s)}{s^p} ds \le C \int_0^t \omega(s) ds. \tag{5}$$

More recently, Neugebauer [3] has shown that Q is bounded on $L_p(\omega)^d$ if and only if

$$\int_0^t P\omega(s)\,ds \le C \int_0^t \omega(s)\,ds. \tag{6}$$

Thus, the boundedness of Q on $L_p(\omega)^d$ does not depend on $p \ge 1$.

If ω is decreasing, we prove that conditions (3), (4) and (6) are equivalent. Hence, Q is bounded on $L_p(\omega)$ if and only if the decreasing weight ω satisfies (6). For decreasing ω, (3), (4) and (6) are essentially a rate of decrease condition; this is proved in Proposition 2.

By $h_1(t) \simeq h_2(t)$ we mean that $h_1(t) \le C_1 h_2(t)$ and $h_2(t) \le C_2 h_1(t)$.

2. The boundedness theorem for the conjugate Hardy operator

Theorem 1. *Let ω be a decreasing weight. If Q is bounded on $L_p(\omega)^d$ ($1 \le p < \infty$), it is also bounded on $L_p(\omega)$.*

Proof. We start by showing that (3) and (4) are equivalent.

Obviously, (3) implies (4), since ω is decreasing and then

$$\int_0^r \omega(s)\,ds \le C_1 \omega(r) \Big(\int_r^\infty \frac{ds}{s^{p'}} \Big)^{-p/p'} = Cr\omega(r).$$

Assume now that ω has property (4). Since ω is decreasing,

$$\int_r^\infty \frac{\omega(s)^{-p'/p}}{s^{p'}} ds = \int_r^\infty \frac{\omega(s)}{(s\omega(s))^{p'}} ds \le C \int_r^\infty \frac{\omega(s)}{(\int_0^\infty \omega(t)dt)^{p'}} ds,$$

and on performing the integration in the last expression we obtain

$$\int_r^\infty \frac{\omega(s)^{-p'/p}}{s^{p'}} ds \le \frac{C}{p'-1} \Big(\int_0^r \omega(t)\,dt \Big)^{-p'/p},$$

which is condition (3).

Obviously, (4) implies (6). To prove that (6) implies (4) we first observe that, for $a > 1$,

$$\frac{1}{r} \int_0^r \omega(s)\,ds = \frac{1}{r \log a} \int_r^{ar} \frac{\int_0^r \omega(s)\,ds}{t}\,dt \le \frac{1}{r \log a} \int_r^{ar} P\omega(t)\,dt.$$

Now, (6) and the monotonicity of ω show that

$$\frac{1}{r}\int_0^r \omega(s)\,ds \le \frac{C}{r\log a}\int_0^{ar} \omega(s)\,ds \le \frac{C}{r\log a}\int_0^r \omega(s)\,ds + \frac{C(a-1)}{\log a}\omega(r),$$

whence

$$\frac{1}{r}\int_0^r \omega(s)\,ds\Big(1-\frac{C}{\log a}\Big) \le \frac{C(a-1)}{\log a}\omega(r),$$

and (4) follows taking $a = e^{2C}$. ☐

Property (4) is a decrease condition on ω:

Proposition 1. *For a decreasing weight w, the conditions*

$$\frac{1}{s}\int_0^s \omega(t)\,dt \simeq \omega(s) \qquad \Big(i.e.,\ \frac{1}{s}\int_0^s \omega(t)\,dt \le C\omega(s)\Big) \tag{7}$$

and

$$\inf_{x>0}\frac{\omega(rx)}{\omega(x)} > \frac{1}{r} \text{ for some constant } r>1 \tag{8}$$

are equivalent.

Proof. Assume that (8) holds. Then $\omega(rx) \ge a\omega(x)$ with $ar > 1$, and $\omega(r^{-n}x)a^n \le \omega(x)$. Hence, from

$$\int_0^s \omega(t)\,dt = \sum_{n=0}^{\infty}\int_{s/r^{n+1}}^{s/r^n} \omega(t)\,dt \le \sum_{n=0}^{\infty}\omega(sr^{-(n+1)})sr^{-n}$$

it follows that

$$\int_0^s \omega(t)\,dt \le s\omega(s)\sum_{n=0}^{\infty}\frac{1}{r^n a^{n+1}}$$

with $C = (1/a)\sum_{n=0}^{\infty}(ra)^{-n} < \infty$.

Assume now that (7) holds and observe that, if $\tau \ge 1$,

$$\omega(\tau x) \ge \frac{1}{C\tau x}\int_0^{\tau x}\omega \ge \frac{1}{C\tau x}\int_0^{x}\omega \ge \frac{\omega(x)}{C\tau}.$$

Let us see that (7) holds if $r > \exp(C^2)$. From the above remark we have

$$\int_1^r \omega(\tau x)\,d\tau \ge \omega(x)\frac{\log r}{C}$$

and

$$\int_1^r \omega(\tau x)\,d\tau \le \int_0^r \omega(\tau x)\,d\tau = r\Big(\frac{1}{rx}\int_0^{rx}\omega(s)\,ds\Big) \le Cr\omega(rx).$$

Thus, putting together both estimates,

$$\frac{\omega(rx)}{\omega(x)} \ge \frac{1}{r}\frac{\log r}{C^2} > \frac{1}{r}. \quad \square$$

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References

[1] M. Ariño and B. Muckenhoupt, Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions, Trans. Amer. Math. Soc. **320**(1990), 727–735 MR **90k:**42034

[2] B. Muckenhoupt, Hardy's inequality with weights, Studia Math. **44**(1972), 31–38 MR **47:**418

[3] C.J. Neugebauer, Some classical operators on Lorentz space, Forum Math. **4**(1992), 135–146. MR **93i:**42013

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