SOME EXTREMAL PROBLEMS IN $L^p(w)$

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Abstract. Fix a positive integer $n$ and $1 < p < \infty$. We provide expressions for the weighted $L^p$ distance
$$\inf_{f} \int_0^{2\pi} |1-f|^p w \, d\lambda,$$
where $d\lambda$ is normalized Lebesgue measure on the unit circle, $w$ is a nonnegative integrable function, and $f$ ranges over the trigonometric polynomials with frequencies in

$S_1 = \{\ldots, -3, -2, -1\} \cup \{1, 2, 3, \ldots, n\}$,

$S_2 = \{\ldots, -3, -2, -1\} \setminus \{-n\}$,

or

$S_3 = \{\ldots, -3, -2, -1\} \cup \{n\}$.

These distances are related to other extremal problems, and are shown to be positive if and only if $\log w$ is integrable. In some cases they are expressed in terms of the series coefficients of the outer functions associated with $w$.

Let $w$ be a nonnegative integrable function on the unit circle in the complex plane, and consider the Banach space $L^p(w)$ for $1 < p < \infty$. A natural subspace of $L^p(w)$ is associated with each subset $S$ of the integers $\mathbb{Z}$, namely

$$\mathcal{M}(S) = \bigvee \{e^{ik\theta} : k \in S\}.$$

Writing $d\lambda$ for normalized Lebesgue measure on the unit circle, we denote the distance from the constant function 1 to the subspace $\mathcal{M}(S)$ by

$$\sigma_p(w, S) = \inf \left\{ \left( \int |1 - f|^p w \, d\lambda \right)^{1/p} : f \in \mathcal{M}(S) \right\}.$$

This notion has been of considerable interest in the theory of stationary processes and harmonic analysis (see [4, 6, 7, 9]). This paper is concerned with evaluating $\sigma_p(w, S)$, and exploring its relationship to other constructions. It is a sequel to [6], to which the reader is referred for further history and background material.

In particular, for a fixed integer $n \geq 1$, we are interested in examining $\sigma_p(w, S)$ for

$$S_1 = \{\ldots, -3, -2, -1\} \cup \{1, 2, 3, \ldots, n\},$$

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\[ S_2 = \{ \ldots, -3, -2, -1 \} \setminus \{ -n \}, \]

and

\[ S_3 = \{ \ldots, -3, -2, -1 \} \cup \{ n \}. \]

All of these frequency sets are natural departures from the classical case of the halfline, \( S_0 = \{ \ldots, -3, -2, -1 \}. \)

Let \( S_0 = \mathbb{Z} \setminus \{ 0 \} \), and for any subset \( S \subseteq \mathbb{Z}_0 \), let \( S^c = \mathbb{Z}_0 \setminus S \) be the complement of \( S \) in \( \mathbb{Z}_0 \). The following result provides a way to calculate \( \sigma_p(w, S) \) when the prediction problem for the complementary frequency set \( S^c \) is understood. It is a special case of Theorem 3.1 in [6] with a shorter and more direct proof. In what follows, we fix the parameter \( p \) with \( 1 < p < \infty \), we define \( q \) by

\[
\frac{1}{q} + \frac{1}{p} = 1,
\]

and we put

\[
s = \frac{-1}{p - 1}.
\]

**Theorem 1.** Suppose that \( w \) is a nonnegative integrable function on the circle, and \( w^s \) is integrable. For any subset \( S \subseteq \mathbb{Z}_0 \), we have

\[
\sigma_p(w, S) = \sigma_q(w^s, S^c)^{-1},
\]

provided that \( \sigma_p(w, S) \) or \( \sigma_q(w^s, S^c) \) is positive.

**Proof.** By logical symmetry it suffices to assume that \( \sigma_p(w, S) \) is positive. Indeed, this is because \( p \) and \( q \) are conjugate indices to each other, \( (S^c)^c = S \), and

\[
(w^s)^{-1/(q-1)} = w^{1/(p-1)(q-1)} = w.
\]

Let us write \( H(S) \) for the collection of finite trigonometric sums with frequencies in \( S \). Then

\[
\sigma_p(w, S)^p = \inf \left\{ \int |1 - \phi|^p w \, d\lambda : \phi \in H(S) \right\}.
\]

Replace each \( 1 - \phi \) with a member of \( H(S \cup \{ 0 \}) \), divided by its constant term. Then we get

\[
\sigma_p(w, S)^p = \inf \left\{ \int \frac{|\phi|^p w \, d\lambda}{\int |\phi|^p \, d\lambda} : \phi \in H(S \cup \{ 0 \}), \hat{\phi}_0 \neq 0 \right\}.
\]

Note that we used \( \hat{\phi}_0 = \int \phi \, d\lambda \). Now

\[
\sigma_p(w, S)^p = \left[ \sup \left\{ \int \frac{|\phi|^p \, d\lambda}{\int |\phi|^p \, d\lambda} : \phi \in H(S \cup \{ 0 \}), \hat{\phi}_0 \neq 0 \right\} \right]^{-1}
\]

\[
= \left[ \sup \left\{ \int \frac{|\phi w^{-\phi} \cdot 1| w^s \, d\lambda}{\int |\phi w^{-\phi} \cdot w^s| \, d\lambda} : \phi \in H(S \cup \{ 0 \}), \hat{\phi}_0 \neq 0 \right\} \right]^{-1}.
\]

Note that \( s - sp = 1 \), so the denominator is indeed unchanged.

In the last expression, we may allow \( \phi \) to range over all of \( H(S \cup \{ 0 \}) \), excluding only the null function, without affecting the value of the supremum. This expresses \( \sigma_p(w, S) \) as the reciprocal of the norm of 1, viewed as a bounded linear functional on the span of \( \{ \phi w^{-\phi} : \phi \in H(S \cup \{ 0 \}) \} \) in \( L^p(w^s) \). But then this is just the distance from 1 to the annihilator of \( \{ \phi w^{-\phi} : \phi \in H(S \cup \{ 0 \}) \} \) in \( L^q(w^s) \). That annihilator consists exactly of those \( f \) in \( L^q(w^s) \) such that \( \int f \cdot \phi w^{-\phi} \cdot w^s \, d\lambda = 0 \) for
all $\phi \in H(S \cup \{0\})$. The collection of such $f$, in turn, is spanned by $H(\mathbb{Z} \setminus [S \cup \{0\}]) = H(S^c)$. Hence

$$\sigma_p(w, S) = \left[ \inf \left\{ \int |1 - f|^q w^q \, d\lambda : f \in H(S^c) \right\} \right]^{-1} = [\sigma_q(w^q, S^c)]^{-1}.$$  

The next proposition asserts that if the index set $S$ is a halfline with finitely many points of $\mathbb{Z}$ added or deleted, then $\sigma_p(w, S)$ is positive exactly when $\log w \in L^1 = L^1(\lambda)$. In the classical case $S = S_0$ and $p > 0$, the result is well-known [3, p. 136]. Here and henceforth, the exponential function $e^{ik\theta}$, $k \in \mathbb{Z}$, is denoted by $e_k$.

**Theorem 2.** Suppose that $w$ is a nonnegative integrable function on the circle. Let

$$S = (\{.,.,-3,-2,-1\} \cup \{J_1, J_2, \ldots, J_M\}) \setminus \{K_1, K_2, \ldots, K_N\},$$

where

$$0 < J_1 < J_2 < \ldots < J_M,$$

and

$$0 > K_1 > K_2 > \ldots > K_N.$$

Then $\sigma_p(w, S)$ is positive if and only if $\log w \in L^1$.

**Proof.** If $\log w$ is not integrable, then $\sigma_p(w, T) = 0$ for $T = \{.,.,K_N - 3, K_N - 2, K_N - 1\}$. Since $T \subseteq S$, we have

$$\sigma_p(w, T) \geq \sigma_p(w, S) \geq 0.$$  

It follows that $\sigma_p(w, S) = 0$.

Conversely, suppose that $\sigma_p(w, S) = 0$, so that $e_0$ belongs to the subspace $\mathcal{M}(S)$. Then $e_0$ certainly belongs to $\mathcal{M}(U)$, where $U$ is an index set of the form

$$U = (\{.,.,-3,-2,-1\} \cup \{1, 2, \ldots, r\}).$$  

Indeed, the inclusion $S \subseteq U$ holds when $r = J_M$. But let us choose $r$ to be the smallest positive integer for which $\sigma_p(w, U) = 0$.

There exist coefficients $c_k^{(j)}$ and $V_j \in \mathcal{M}(S_0 \cup \{1, 2, \ldots, r - 1\})$ such that

$$e_0 = \lim_{j \to \infty} (c_r^{(j)} e_r + V_j)$$

in the norm of $L^p(w)$. The sequence $c_r^{(1)}, c_r^{(2)}, c_r^{(3)}, \ldots$ must be bounded away from zero by some positive distance $\rho$. If not, then $c_r^{(jm)} \to 0$ for some subsequence. This would imply that

$$e_0 = \lim_{m \to \infty} c_r^{(jm)} e_r + \lim_{m \to \infty} V_{jm}.$$  

The first limit is zero, and the resulting equation violates the minimality of $r$.

Now we have

$$0 \leq \rho \left\| \frac{1}{c_r^{(j)}} e_0 - e_r - \frac{1}{c_r^{(j)}} V_j \right\|_p \leq \left\| e_0 - c_r^{(j)} e_r - V_j \right\|_p \to 0.$$  

This shows that $e_r \in \mathcal{M}(\{.,.,r - 3, r - 2, r - 1\})$, giving $\log w \notin L^1$. \qed
Thus the condition \( w^s \in L^1 \) in Theorem 1 is not natural to the present applications, and we seek to replace it with the weaker condition \( \log w \in L^1 \). Unfortunately, the related quantity \( \sigma_q(w^s, S^c) \) might not be defined under this weaker condition. Thus we must bring in yet another dual extremal problem, one tied to the metric projection of \( L^p \) onto the Hardy space \( H^p \) of the unit circle.

**Theorem 3.** Suppose that \( w \) is nonnegative and integrable. If \( \log w \in L^1 \), then

\[
\sigma_p(w, S_1) = \text{dist}_{L^q}(\phi(n), e_{n+1}H^q)^{-1},
\]

where \( \phi \) is the outer function satisfying

\[
w^s = |\phi|^q,
\]

and \( \phi(n) \) is the truncated series

\[
\phi(n)(e^{i\theta}) = \sum_{k=0}^{n} \hat{\phi}_k e^{ik\theta}.
\]

**Proof.** First assume that \( w^s \in L^1 \). Then Theorem 1 applies, yielding

\[
\sigma_p(w, S_1) = \sigma_q(w^s, S^c)_1^{-1}
\]

\[
= \inf \left\{ \left( \int |1 - f|^q w^s \, d\lambda \right)^{1/q} : f \in H(S^c_1) \right\}^{-1}
\]

\[
= \inf \left\{ \left( \int |\phi - f|^q \, d\lambda \right)^{1/q} : f \in H(S^c_1) \right\}^{-1}
\]

\[
= \inf \left\{ \left( \int |\phi(n) - f|^q \, d\lambda \right)^{1/q} : f \in H(S^c_1) \right\}^{-1},
\]

where in the last step we used the fact that \( \phi \) is outer in \( H^q \). This confirms (1) when \( w^s \in L^1 \). More generally, for any positive integer \( m \) define \( w_m = \max \{ w, 1/m \} \), and let \( \phi_m \) be the outer function satisfying

\[
w^s_m = |\phi_m|^q.
\]

Since \( \log w_m \in L^1 \), the preliminary result applies, and we get

\[
\sigma_p(w_m, S_1) = \text{dist}_{L^q}(\phi_m(n), e_{n+1}H^q)^{-1}.
\]

Next, we argue that \( \sigma_p(w_m, S_1) \to \sigma_p(w, S_1) \) as \( m \to \infty \). To see this, note that for any \( \epsilon > 0 \), there exists an \( f_0 \in H(S_1) \) such that

\[
\sigma_p(w, S_1)^p \leq \int |1 - f_0|^p w \, d\lambda < \sigma_p(w, S_1)^p + \frac{\epsilon}{2}.
\]

Since \( w_m \) is a decreasing sequence of functions converging to \( w \), the dominated convergence theorem (with dominating function \( w + 1 \)) provides that

\[
\int |1 - f_0|^p w_m \, d\lambda \to \int |1 - f_0|^p w \, d\lambda.
\]

Hence there exists an \( M > 0 \) such that for all \( m > M \) we have

\[
\int |1 - f_0|^p w \, d\lambda \leq \int |1 - f_0|^p w_m \, d\lambda \leq \int |1 - f_0|^p w \, d\lambda + \frac{\epsilon}{2}.
\]

(2)
Combining (2) and (3), we get
\[ \sigma_p(w, S_1)^p \leq \int |1 - f_0|^p w_m \, d\lambda \leq \sigma_p(w, S_1)^p + \epsilon, \]
whenever \( m > M \). This implies that
\[ \sigma_p(w, S_1)^p \leq \sigma_p(w_m, S_1)^p \leq \sigma_p(w, S_1)^p + \epsilon, \]
and hence
\[ \lim_{m \to \infty} \sigma_p(w_m, S_1) = \sigma_p(w, S_1). \]

Finally we confirm that \( \phi_m^{(n)} \to \phi^{(n)} \) uniformly on the circle, hence in \( L^q \) norm. Since
\[ | \log w_m | \leq | \log w | \]
for all positive integers \( m \), it follows that for each \( z \) with \( | z | < 1 \) we have
\[ \left| \frac{e_1 + z}{e_1 - z} \log w_m \right| \leq g(|z|) | \log w_m | \leq g(|z|) | \log w | \]
for some positive function \( g \). Applying the dominated convergence theorem to
\[ \phi_m = \exp \left[ \frac{s}{q} \int \frac{e_1 + z}{e_1 - z} \log w_m \, d\lambda \right], \]
we get
\[ \lim_{m \to \infty} \phi_m(z) = \phi(z) \]
for all \( |z| < 1 \). In fact, the convergence is uniform on any closed disc \( |z| \leq R < 1 \). Thus for the corresponding power series coefficients we have
\[ \lim_{m \to \infty} \hat{\phi}_{m,k} = \hat{\phi}_k, \]
\( k = 0, 1, 2, \ldots \). In particular, the truncated series \( \phi_m^{(n)} \) converges uniformly to \( \phi^{(n)} \). Now the continuity of the metric projection in \( L^q \) yields
\[ \lim_{m \to \infty} \text{dist}_{L^q}(\psi_m^{(n)}, e_{n+1}H^q) = \text{dist}_{L^q}(\phi^{(n)}, e_{n+1}H^q). \]
The claim is proved. \( \square \)

This second formulation turns out to be a fruitful one, since it reduces the computation of \( \sigma_p(w, S_1) \) to the well established dual extremal problem of computing \( \text{dist}_{L^q}(\phi^{(n)}, e_{n+1}H^q) \) (see [2, pp. 136–146]).

With these preliminaries it is possible to solve the prediction problems for the case \( p = 2 \). Here we suppose that \( \log w \in L^1 \), so that \( w = |\psi|^2 \) for some outer function \( \psi \) in \( H^2 \). In this case \( s = -1 \), and the function \( w^s \) factors into \( |1/\psi|^2 \). We write the associated power series expansions
\[ \psi(z) = \sum_{k=0}^{\infty} c_k z^k, \]
\[ \frac{1}{\psi(z)} = \sum_{k=0}^{\infty} d_k z^k, \]
\(|z| < 1\).
Theorem 4. Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$, then
\[
\sigma_2(w, S_1) = \left( \sum_{k=0}^{n} |d_k|^2 \right)^{-1/2}.
\]
Otherwise, $\sigma_2(w, S_1) = 0$.

Proof. The quantity $\sigma_2(w, S_1)$ is nonzero if and only if $\log w \in L^1$, by Theorem 2. Now Theorem 3 shows that
\[
\sigma_2(w, S_1) = \text{dist}_{L^2}(\phi^{-1}(n), e_{n+1}H^2)^{-1}.
\]
The right side is easily seen to be
\[
\left( \sum_{k=0}^{n} |d_k|^2 \right)^{-1/2}.
\]
This improves upon the corresponding results of [6, 8], in which the stronger hypothesis $w^{-1} \in L^1$ is needed.

Next, an elementary argument settles the case $p = 2$ and $S = S_3$.

Theorem 5. Let $w$ be nonnegative and integrable. If $\log w \in L^1$, then
\[
\sigma_2(w, S_3) = |c_0| \left( \sum_{k=0}^{n-1} |c_k|^2 \right)^{1/2} \left( \sum_{k=0}^{n} |c_k|^2 \right)^{-1/2}.
\]
Otherwise, $\sigma_2(w, S_3) = 0$.

Proof. Again, Theorem 2 asserts that $\sigma_2(w, S_3)$ is nonzero precisely when $\log w$ is integrable. In that case, let $\hat{e}_n$ be the orthogonal projection of $e_n$ onto $\mathcal{M}(S_0)$. Since
\[
S_3 = \mathcal{M}(S_0) \oplus \{e_n - \hat{e}_n\},
\]
the projection of $e_0$ onto $\mathcal{M}(S_3)$ is given by
\[
P_{S_3}e_0 = \hat{e}_0 + a(e_n - \hat{e}_n),
\]
where
\[
a = \frac{\langle e_0, e_n - \hat{e}_n \rangle_{L^2(w)}}{\|e_n - \hat{e}_n\|_{L^2(w)}}.
\]
Thus, using the orthogonality of $\hat{e}_0$ and $e_n - \hat{e}_n$ we get
\[
\sigma_2(w, S_3)^2 = \|e_0 - P_{S_3}e_0\|_{L^2(w)}^2 = \|e_0 - \hat{e}_0\|_{L^2(w)}^2 - \frac{\|e_0 - \hat{e}_0, e_n - \hat{e}_n \|_{L^2(w)}^2}{\|e_n - \hat{e}_n\|_{L^2(w)}^2}.
\]
But it is straightforward to check that
\[
e_n - \hat{e}_n = \sum_{k=0}^{n} c_k e_{n-k} \tilde{\phi}^{-1}.
\]
The desired result now follows from substituting this in (4).

With this done, the case with $p = 2$ and $S = S_2$ is immediate.
Theorem 6. Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$, then

$$\sigma_2(w, S_2) = |c_0| \left( \sum_{k=0}^{n} |d_k|^2 \right)^{1/2} \left( \sum_{k=0}^{n-1} |d_k|^2 \right)^{-1/2}.$$

Otherwise, $\sigma_2(w, S_2) = 0$.

Proof. By Theorem 2 the quantity $\sigma_2(w, S_2)$ is nonzero precisely when $\log w \in L^1$. When that holds, Theorem 1 gives

$$\sigma_2(w, S_2) = \sigma_2(w^{-1}, S_2)^{-1}.$$

But $S_2^c = \{-n\} \cup \{1, 2, 3, \ldots\}$ is simply the reflection of $S_3$ about the origin. Now an application of Theorem 5 completes the proof.

It is interesting and instructive to compare these least-squares error formulas with the classical $n$-step prediction error variance, where $S = \{\ldots, -n - 3, -n - 2, -n - 1\}$; for then

$$\sigma_2(w, S) = \left( \sum_{k=0}^{n} |c_k|^2 \right)^{1/2}.$$

In particular, we can explicitly observe the changes in $\sigma_2(w, S)$ as indices are added to or deleted from the frequency set $S$.

When $p \neq 2$ these Hilbert space techniques do not apply. However, in this situation the work of Rajput and Sundberg [10, Theorem 2 and Remark 1(b)] makes the following approach possible. As before, let $w$ be nonnegative and integrable with $\log w \in L^1$, and let $\phi$ be the outer function for which $w^s = |\phi|^q$. Assume for the sake of argument that $w^s \in L^1$, so that $\phi \in H^q$. Let $Q(h)$ denote the metric projection of $h \in L^q$ onto $H^q$; by Theorem 3, we need to consider $h = e^{-\{n+1\}}\phi^{(n)}$.

Note that $\phi^{(n)}(0) = \hat{\phi}_0 > 0$, and let

$$[\phi^{(n)}(z)]^{q/2} = \sum_{k=0}^{\infty} c_k z^k = P_n(z) + \sum_{k=n+1}^{\infty} c_k z^k \quad (5)$$

be an analytic root of $\phi^{(n)}(z)$ defined in a neighborhood of zero. Assuming that $P_n(z) \neq 0$ for all $|z| < 1$, we get

$$Q(h)(z) = z^{-(n+1)}[\phi^{(n)}(z) - P_n(z)^{2/q}], \quad (6)$$

$|z| < 1$, where $P_n^{2/q}$ is the analytic root of $P_n$ which satisfies and is uniquely determined by the condition

$$P_n^{2/q}(0) = \hat{\phi}_0.$$

Now the desired distance is given by

$$\sigma_p(w, S_1) = \text{dist}_{L^q}(h, H^q)^{-1}$$

$$= \left( \int |P_n|^2 \, d\lambda \right)^{-1/q}$$

$$= \left( \sum_{k=0}^{n} |c_k|^2 \right)^{-1/q}.$$

For more general $w$, we apply the above argument with $w$ replaced by $w_m = \max\{w, 1/m\}$, and extract limits as $m \to \infty$, as in the proof of Theorem 3. Note
that if the polynomial $P_n$ has no roots in the closed disk $|z| \leq 1$, then its approximants also do not vanish in the open disk $|z| < 1$, for sufficiently large $m$. This yields the following.

**Theorem 7.** Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$ and $P_n(z)$ has no roots in the closed disk $|z| \leq 1$, then

$$\sigma_p(w, S_1) = \left( \sum_{k=0}^{n} |c_k|^2 \right)^{1/q},$$

where $P_n$ and the coefficients $c_k$ are given in (5).

The explicit formula (6) and [6, Theorem 3.1] can be used to compute the function in $M(S_1)$ for which $\sigma_p(w, S_1)$ is attained.

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