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# SOME EXTREMAL PROBLEMS IN $L^p(w)$

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Abstract. Fix a positive integer $n$ and $1 < p < \infty$. We provide expressions for the weighted $L^p$ distance

$$\inf_f \int_0^{2\pi} |1 - f|^p w \, d\lambda,$$

where $d\lambda$ is normalized Lebesgue measure on the unit circle, $w$ is a nonnegative integrable function, and $f$ ranges over the trigonometric polynomials with frequencies in

$$S_1 = \{\dots, -3, -2, -1\} \cup \{1, 2, 3, \dots, n\},$$
$$S_2 = \{\dots, -3, -2, -1\} \setminus \{-n\},$$

or

$$S_3 = \{\dots, -3, -2, -1\} \cup \{n\}.$$

These distances are related to other extremal problems, and are shown to be positive if and only if $\log w$ is integrable. In some cases they are expressed in terms of the series coefficients of the outer functions associated with $w$.

Let $w$ be a nonnegative integrable function on the unit circle in the complex plane, and consider the Banach space $L^p(w)$ for $1 < p < \infty$. A natural subspace of $L^p(w)$ is associated with each subset $S$ of the integers $\mathbb{Z}$, namely

$$\mathcal{M}(S) = \bigvee \{e^{ik\theta} : k \in S\}.$$

Writing $d\lambda$ for normalized Lebesgue measure on the unit circle, we denote the distance from the constant function 1 to the subspace $\mathcal{M}(S)$ by

$$\sigma_p(w, S) = \inf \left\{ \left( \int |1 - f|^p w \, d\lambda \right)^{1/p} : \ f \in \mathcal{M}(S) \right\}.$$

This notion has been of considerable interest in the theory of stationary processes and harmonic analysis (see [4, 6, 7, 9]). This paper is concerned with evaluating $\sigma_p(w, S)$, and exploring its relationship to other constructions. It is a sequel to [6], to which the reader is referred for further history and background material.

In particular, for a fixed integer $n \geq 1$, we are interested in examining $\sigma_p(w, S)$ for

$$S_1 = \{\dots, -3, -2, -1\} \cup \{1, 2, 3, \dots, n\},$$

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$$S_2 = \{\dots, -3, -2, -1\} \setminus \{-n\},$$

and

$$S_3 = \{\dots, -3, -2, -1\} \cup \{n\}.$$

All of these frequency sets are natural departures from the classical case of the halfline, $S_0 = \{\dots, -3, -2, -1\}$.

Let $\mathbb{Z}_0 = \mathbb{Z}\backslash\{0\}$, and for any subset $S \subseteq \mathbb{Z}_0$, let $S^c = \mathbb{Z}_0\backslash S$ be the complement of $S$ in $\mathbb{Z}_0$. The following result provides a way to calculate $\sigma_p(w, S)$ when the prediction problem for the complementary frequency set $S^c$ is understood. It is a special case of Theorem 3.1 in [6] with a shorter and more direct proof. In what follows, we fix the parameter $p$ with $1 < p < \infty$, we define $q$ by

$$\frac{1}{q} + \frac{1}{p} = 1,$$

and we put

$$s = \frac{-1}{p-1}.$$

**Theorem 1.** *Suppose that $w$ is a nonnegative integrable function on the circle, and $w^s$ is integrable. For any subset $S \subseteq \mathbb{Z}_0$, we have*

$$\sigma_p(w, S) = \sigma_q(w^s, S^c)^{-1},$$

*provided that $\sigma_p(w, S)$ or $\sigma_q(w^s, S^c)$ is positive.*

*Proof.* By logical symmetry it suffices to assume that $\sigma_p(w, S)$ is positive. Indeed, this is because $p$ and $q$ are conjugate indices to each other, $(S^c)^c = S$, and

$$(w^s)^{-1/(q-1)} = w^{1/(p-1)(q-1)} = w.$$

Let us write $H(S)$ for the collection of finite trigonometric sums with frequencies in $S$. Then

$$\sigma_p(w, S)^p = \inf\left\{\int |1 - \phi|^p w \, d\lambda : \phi \in H(S)\right\}.$$

Replace each $1 - \phi$ with a member of $H(S \cup \{0\})$, divided by its constant term. Then we get

$$\sigma_p(w, S)^p = \inf\left\{\frac{\int |\phi|^p w \, d\lambda}{|\int \phi \, d\lambda|^p} : \phi \in H(S \cup \{0\}),\ \hat{\phi}_0 \neq 0\right\}.$$

Note that we used $\hat{\phi}_0 = \int \phi \, d\lambda$. Now

$$\begin{aligned}
\sigma_p(w, S)^p &= \left[\sup\left\{\frac{|\int \phi \, d\lambda|^p}{\int |\phi|^p w \, d\lambda} : \phi \in H(S \cup \{0\}),\ \hat{\phi}_0 \neq 0\right\}\right]^{-1} \\
&= \left[\sup\left\{\frac{|\int (\phi w^{-s} \cdot 1) w^s \, d\lambda|^p}{\int |\phi w^{-s}|^p w^s \, d\lambda} : \phi \in H(S \cup \{0\}),\ \hat{\phi}_0 \neq 0\right\}\right]^{-1}.
\end{aligned}$$

Note that $s - sp = 1$, so the denominator is indeed unchanged.

In the last expression, we may allow $\phi$ to range over all of $H(S \cup \{0\})$, excluding only the null function, without affecting the value of the supremum. This expresses $\sigma_p(w, S)$ as the reciprocal of the norm of 1, viewed as a bounded linear functional on the span of $\{\phi w^{-s} : \phi \in H(S \cup \{0\})\}$ in $L^p(w^s)$. But then this is just the distance from 1 to the annihilator of $\{\phi w^{-s} : \phi \in H(S \cup \{0\})\}$ in $L^q(w^s)$. That annihilator consists exactly of those $f$ in $L^q(w^s)$ such that $\int f \cdot \phi w^{-s} \cdot w^s \, d\lambda = 0$ for all $\phi \in H(S \cup \{0\})$. The collection of such $f$, in turn, is spanned by $H(\mathbb{Z} \setminus [S \cup \{0\}]) = H(S^c)$. Hence

$$\begin{aligned} \sigma_p(w,S) &= \left[\inf \left\{ \int |1-f|^q w^s \, d\lambda : f \in H(S^c) \right\}\right]^{-1} \\ &= [\sigma_q(w^s, S^c)]^{-1}. \end{aligned}$$

□

The next proposition asserts that if the index set $S$ is a halfline with finitely many points of $\mathbb{Z}$ added or deleted, then $\sigma_p(w,S)$ is positive exactly when $\log w \in L^1 = L^1(\lambda)$. In the classical case $S = S_0$ and $p > 0$, the result is well-known [3, p. 136]. Here and henceforth, the exponential function $e^{ik\theta}$, $k \in \mathbb{Z}$, is denoted by $e_k$.

**Theorem 2.** *Suppose that $w$ is a nonnegative integrable function on the circle. Let*

$$S = (\{\dots,-3,-2,-1\} \cup \{J_1, J_2, \dots, J_M\}) \setminus \{K_1, K_2, \dots, K_N\},$$

*where*

$$0 < J_1 < J_2 < \dots < J_M,$$

*and*

$$0 > K_1 > K_2 > \dots > K_N.$$

*Then $\sigma_p(w,S)$ is positive if and only if $\log w \in L^1$.*

*Proof.* If $\log w$ is not integrable, then $\sigma_p(w,T) = 0$ for $T = \{\dots, K_N - 3, K_N - 2, K_N - 1\}$. Since $T \subseteq S$, we have

$$\sigma_p(w,T) \geq \sigma_p(w,S) \geq 0.$$

It follows that $\sigma_p(w,S) = 0$.

Conversely, suppose that $\sigma_p(w,S) = 0$, so that $e_0$ belongs to the subspace $\mathcal{M}(S)$. Then $e_0$ certainly belongs to $\mathcal{M}(U)$, where $U$ is an index set of the form

$$U = \{\dots,-3,-2,-1\} \cup \{1,2,\dots,r\}.$$

Indeed, the inclusion $S \subseteq U$ holds when $r = J_M$. But let us choose $r$ to be the smallest positive integer for which $\sigma_p(w,U) = 0$.

There exist coefficients $c_k^{(j)}$ and $V_j \in \mathcal{M}(S_0 \cup \{1,2,\dots,r-1\})$ such that

$$e_0 = \lim_{j\to\infty} (c_r^{(j)} e_r + V_j)$$

in the norm of $L^p(w)$. The sequence $c_r^{(1)}, c_r^{(2)}, c_r^{(3)}, \dots$ must be bounded away from zero by some positive distance $\rho$. If not, then $c_r^{(j_m)} \to 0$ for some subsequence. This would imply that

$$e_0 = \lim_{m\to\infty} c_r^{(j_m)} e_r + \lim_{m\to\infty} V_{j_m}.$$

The first limit is zero, and the resulting equation violates the minimality of $r$.

Now we have

$$\begin{aligned} 0 &\leq \rho \left\| \frac{1}{c_r^{(j)}} e_0 - e_r - \frac{1}{c_r^{(j)}} V_j \right\|_p \\ &\leq \|e_0 - c_r^{(j)} e_r - V_j\|_p \\ &\to 0. \end{aligned}$$

This shows that $e_r \in \mathcal{M}(\{\dots, r-3, r-2, r-1\})$, giving $\log w \notin L^1$. □

Thus the condition $w^s \in L^1$ in Theorem 1 is not natural to the present applications, and we seek to replace it with the weaker condition $\log w \in L^1$. Unfortunately, the related quantity $\sigma_q(w^s, S^c)$ might not be defined under this weaker condition. Thus we must bring in yet another dual extremal problem, one tied to the metric projection of $L^p$ onto the Hardy space $H^p$ of the unit circle.

**Theorem 3.** *Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$, then*

$$\sigma_p(w, S_1) = dist_{L^q}(\phi^{(n)}, e_{n+1}H^q)^{-1}, \tag{1}$$

*where $\phi$ is the outer function satisfying*

$$w^s = |\phi|^q,$$

*and $\phi^{(n)}$ is the truncated series*

$$\phi^{(n)}(e^{i\theta}) = \sum_{k=0}^{n} \hat{\phi}_k e^{ik\theta}.$$

*Proof.* First assume that $w^s \in L^1$. Then Theorem 1 applies, yielding

$$\begin{aligned}
\sigma_p(w, S_1) &= \sigma_q(w^s, S_1^c)^{-1} \\
&= \inf\left\{\left(\int |1 - f|^q\, w^s\, d\lambda\right)^{1/q} : f \in H(S_1^c)\right\}^{-1} \\
&= \inf\left\{\left(\int |\phi - \phi f|^q\, d\lambda\right)^{1/q} : f \in H(S_1^c)\right\}^{-1} \\
&= \inf\left\{\left(\int |\phi^{(n)} - f|^q\, d\lambda\right)^{1/q} : f \in H(S_1^c)\right\}^{-1},
\end{aligned}$$

where in the last step we used the fact that $\phi$ is outer in $H^q$. This confirms (1) when $w^s \in L^1$. More generally, for any positive integer $m$ define $w_m = \max\{w, 1/m\}$, and let $\phi_m$ be the outer function satisfying

$$w_m^s = |\phi_m|^q.$$

Since $\log w_m \in L^1$, the preliminary result applies, and we get

$$\sigma_p(w_m, S_1) = \mathrm{dist}_{L^q}(\phi_m^{(n)}, e_{n+1}H^q)^{-1}.$$

Next, we argue that $\sigma_p(w_m, S_1) \to \sigma_p(w, S_1)$ as $m \to \infty$. To see this, note that for any $\epsilon > 0$, there exists an $f_0 \in H(S_1)$ such that

$$\sigma_p(w, S_1)^p \le \int |1 - f_0|^p\, w\, d\lambda < \sigma_p(w, S_1)^p + \frac{\epsilon}{2}. \tag{2}$$

Since $w_m$ is a decreasing sequence of functions converging to $w$, the dominated convergence theorem (with dominating function $w+1$) provides that

$$\int |1 - f_0|^p\, w_m\, d\lambda \to \int |1 - f_0|^p\, w\, d\lambda.$$

Hence there exists an $M > 0$ such that for all $m > M$ we have

$$\int |1 - f_0|^p\, w\, d\lambda \le \int |1 - f_0|^p\, w_m\, d\lambda \le \int |1 - f_0|^p\, w\, d\lambda + \frac{\epsilon}{2}. \tag{3}$$

Combining (2) and (3), we get

$$\sigma_p(w, S_1)^p \leq \int |1 - f_0|^p \, w_m \, d\lambda \leq \sigma_p(w, S_1)^p + \epsilon,$$

whenever $m > M$. This implies that

$$\sigma_p(w, S_1)^p \leq \sigma_p(w_m, S_1)^p \leq \sigma_p(w, S_1)^p + \epsilon,$$

and hence

$$\lim_{m\to\infty} \sigma_p(w_m, S_1) = \sigma_p(w, S_1).$$

Finally we confirm that $\phi_m^{(n)} \to \phi^{(n)}$ uniformly on the circle, hence in $L^q$ norm. Since

$$|\log w_m| \leq |\log w|$$

for all positive integers $m$, it follows that for each $z$ with $|z| < 1$ we have

$$\left| \frac{e_1 + z}{e_1 - z} \log w_m \right| \leq g(|z|) |\log w_m| \leq g(|z|) |\log w|$$

for some positive function $g$. Applying the dominated convergence theorem to

$$\phi_m = \exp\left[ \frac{s}{q} \int \frac{e_1 + z}{e_1 - z} \log w_m \, d\lambda \right],$$

we get

$$\lim_{m\to\infty} \phi_m(z) = \phi(z)$$

for all $|z| < 1$. In fact, the convergence is uniform on any closed disc $|z| \leq R < 1$. Thus for the corresponding power series coefficients we have

$$\lim_{m\to\infty} \hat{\phi}_{m,k} = \hat{\phi}_k,$$

$k = 0, 1, 2, \ldots$. In particular, the truncated series $\phi_m^{(n)}$ converges uniformly to $\phi^{(n)}$. Now the continuity of the metric projection in $L^q$ yields

$$\lim_{m\to\infty} \mathrm{dist}_{L^q}(\phi_m^{(n)}, e_{n+1}H^q) = \mathrm{dist}_{L^q}(\phi^{(n)}, e_{n+1}H^q).$$

The claim is proved. □

This second formulation turns out to be a fruitful one, since it reduces the computation of $\sigma_p(w, S_1)$ to the well established dual extremal problem of computing $\mathrm{dist}_{L^q}(\phi^{(n)}, e_{n+1}H^q)$ (see [2, pp. 136–146]).

With these preliminaries it is possible to solve the prediction problems for the case $p = 2$. Here we suppose that $\log w \in L^1$, so that $w = |\psi|^2$ for some outer function $\psi$ in $H^2$. In this case $s = -1$, and the function $w^s$ factors into $|1/\psi|^2$. We write the associated power series expansions

$$\begin{aligned} \psi(z) &= \sum_{k=0}^{\infty} c_k z^k, \\ \frac{1}{\psi(z)} &= \sum_{k=0}^{\infty} d_k z^k, \end{aligned}$$

$|z| < 1$.

**Theorem 4.** *Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$, then*

$$\sigma_2(w, S_1) = \left(\sum_{k=0}^{n} |d_k|^2\right)^{-1/2}.$$

*Otherwise, $\sigma_2(w, S_1) = 0$.*

*Proof.* The quantity $\sigma_2(w, S_1)$ is nonzero if and only if $\log w \in L^1$, by Theorem 2. Now Theorem 3 shows that

$$\sigma_2(w, S_1) = \mathrm{dist}_{L^2}((\phi^{-1})^{(n)}, e_{n+1}H^2)^{-1}.$$

The right side is easily seen to be

$$\left(\sum_{k=0}^{n} |d_k|^2\right)^{-1/2}.$$

$\square$

This improves upon the corresponding results of [6, 8], in which the stronger hypothesis $w^{-1} \in L^1$ is needed.

Next, an elementary argument settles the case $p = 2$ and $S = S_3$.

**Theorem 5.** *Let $w$ be nonnegative and integrable. If $\log w \in L^1$, then*

$$\sigma_2(w, S_3) = |c_0| \left(\sum_{k=0}^{n-1} |c_k|^2\right)^{1/2} \left(\sum_{k=0}^{n} |c_k|^2\right)^{-1/2}.$$

*Otherwise, $\sigma_2(w, S_3) = 0$.*

*Proof.* Again, Theorem 2 asserts that $\sigma_2(w, S_3)$ is nonzero precisely when $\log w$ is integrable. In that case, let $\hat{e}_n$ be the orthogonal projection of $e_n$ onto $\mathcal{M}(S_0)$. Since

$$S_3 = \mathcal{M}(S_0) \oplus \bigvee\{e_n - \hat{e}_n\},$$

the projection of $e_0$ onto $\mathcal{M}(S_3)$ is given by

$$P_{S_3} e_0 = \hat{e}_0 + a(e_n - \hat{e}_n),$$

where

$$a = \frac{\langle e_0, e_n - \hat{e}_n \rangle_{L^2(w)}}{\|e_n - \hat{e}_n\|^2_{L^2(w)}}.$$

Thus, using the orthgonality of $\hat{e}_0$ and $e_n - \hat{e}_n$ we get

$$\begin{aligned} \sigma_2(w, S_3)^2 &= \|e_0 - P_{S_3} e_0\|^2_{L^2(w)} \\ &= \|e_0 - \hat{e}_0\|^2_{L^2(w)} - \frac{|\langle e_0 - \hat{e}_0, e_n - \hat{e}_n \rangle|^2_{L^2(w)}}{\|e_n - \hat{e}_n\|^2_{L^2(w)}}. \end{aligned} \tag{4}$$

But it is straightforward to check that

$$e_n - \hat{e}_n = (\sum_{k=0}^{n} c_k e_{n-k}) \overline{\phi^{-1}}.$$

The desired result now follows from substituting this in (4). $\square$

With this done, the case with $p = 2$ and $S = S_2$ is immediate.

**Theorem 6.** *Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$, then*

$$\sigma_2(w, S_2) = |c_0| \left( \sum_{k=0}^{n} |d_k|^2 \right)^{1/2} \left( \sum_{k=0}^{n-1} |d_k|^2 \right)^{-1/2}.$$

*Otherwise, $\sigma_2(w, S_2) = 0$.*

*Proof.* By Theorem 2 the quantity $\sigma_2(w, S_2)$ is nonzero precisely when $\log w \in L^1$. When that holds, Theorem 1 gives

$$\sigma_2(w, S_2) = \sigma_2(w^{-1}, S_2^c)^{-1}.$$

But $S_2^c = \{-n\} \cup \{1, 2, 3, \dots\}$ is simply the reflection of $S_3$ about the origin. Now an application of Theorem 5 completes the proof. $\square$

It is interesting and instructive to compare these least-squares error formulas with the classical $n$-step prediction error variance, where $S = \{\dots, -n-3, -n-2, -n-1\}$; for then

$$\sigma_2(w, S) = \left( \sum_{k=0}^{n} |c_k|^2 \right)^{1/2}.$$

In particular, we can explicitly observe the changes in $\sigma_2(w, S)$ as indices are added to or deleted from the frequency set $S$.

When $p \neq 2$ these Hilbert space techniques do not apply. However, in this situation the work of Rajput and Sundberg [10, Theorem 2 and Remark 1(b)] makes the following approach possible. As before, let $w$ be nonnegative and integrable with $\log w \in L^1$, and let $\phi$ be the outer function for which $w^s = |\phi|^q$. Assume for the sake of argument that $w^s \in L^1$, so that $\phi \in H^q$. Let $Q(h)$ denote the metric projection of $h \in L^q$ onto $H^q$; by Theorem 3, we need to consider $h = e_{-(n+1)}\phi^{(n)}$.

Note that $\phi^{(n)}(0) = \hat{\phi}_0 > 0$, and let

$$[\phi^{(n)}(z)]^{q/2} = \sum_{k=0}^{\infty} c_k z^k = P_n(z) + \sum_{k=n+1}^{\infty} c_k z^k \tag{5}$$

be an analytic root of $\phi^{(n)}(z)$ defined in a neighborhood of zero. Assuming that $P_n(z) \neq 0$ for all $|z| < 1$, we get

$$(Q(h))(z) = z^{-(n+1)}[\phi^{(n)}(z) - P_n(z)^{2/q}], \tag{6}$$

$|z| < 1$, where $P_n^{2/q}$ is the analytic root of $P_n$ which satisfies and is uniquely determined by the condition

$$P_n^{2/q}(0) = \hat{\phi}_0.$$

Now the desired distance is given by

$$\begin{aligned} \sigma_p(w, S_1) &= \operatorname{dist}_{L^q}(h, H^q)^{-1} \\ &= \left( \int |P_n|^2 \, d\lambda \right)^{-1/q} \\ &= \left( \sum_{k=0}^{n} |c_k|^2 \right)^{-1/q}. \end{aligned}$$

For more general $w$, we apply the above argument with $w$ replaced by $w_m = \max\{w, 1/m\}$, and extract limits as $m \to \infty$, as in the proof of Theorem 3. Note that if the polynomial $P_n$ has no roots in the closed disk $|z| \le 1$, then its approximants also do not vanish in the open disk $|z| < 1$, for sufficiently large $m$. This yields the following.

**Theorem 7.** *Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$ and $P_n(z)$ has no roots in the closed disk $|z| \le 1$, then*

$$\sigma_p(w, S_1) = \left( \sum_{k=0}^{n} |c_k|^2 \right)^{-1/q},$$

*where $P_n$ and the coefficients $c_k$ are given in* (5).

The explicit formula (6) and [6, Theorem 3.1] can be used to compute the function in $\mathcal{M}(S_1)$ for which $\sigma_p(w, S_1)$ is attained.

## References


1. S. Cambanis and A. R. Soltani (1984). Prediction of stable processes: Spectral and moving average representations, *Z. Wahrsch. Verw. Geb.* **66**, 593–612. MR **86g:**60054
2. P. L. Duren (1970). *Theory of $H^p$ spaces.* Academic Press, New York. MR **42:**3552
3. T. W. Gamelin (1967). *Uniform Algebras.* Prentice Hall, Englewood Cliffs, N.J. MR **53:**14137
4. H. Helson (1964). *Lectures On Invariant Subspaces.* Academic Press, New York. MR **30:**1409
5. A. G. Miamee (1993). On basicity of exponentials in $L^p(d\mu)$ and general prediction problems, *Period. Math. Hungar.* **26**, 115–124. MR **94k:**60066
6. A. G. Miamee and M. Pourahmadi (1988). Best approximation in $L^p(d\mu)$ and prediction problems of Szegö, Kolmogorov, Yaglom and Nakazi, *J. London Math. Soc.* **(2)** 38, 133–145. MR **90g:**60042
7. T. Nakazi (1984). Two problems in prediction theory, *Studia Math.* **78**, 7–14. MR **86i:**60122
8. T. Nakazi and K. Takahashi (1980). Prediction $n$ units of time ahead, *Proc. Amer. Math. Soc.* **80**, 658–659. MR **82b:**60041
9. M. Pourahmadi (1992). Alternating projections and interpolation of stationary processes. *J. Appl. Probab.* **29**, 921–931. MR **93m:**60077
10. B. S. Rajput and C. Sundberg (1994). On some extremal problems in $H^p$ and the prediction of $L^p$-harmonizable stochastic processes, *Probab. Theory Related Fields* **99**, 197–210. MR **95g:**30048



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