

## MEASURES WITH NATURAL SPECTRA ON LOCALLY COMPACT ABELIAN GROUPS

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**ABSTRACT.** Every bounded regular Borel measure on noncompact LCA groups is a sum of an absolutely continuous measure and a measure with natural spectrum. The set of bounded regular Borel measures with natural spectrum on a nondiscrete LCA group  $G$  whose Fourier-Stieltjes transforms vanish at infinity is closed under addition if and only if  $G$  is compact.

Let  $G$  be a locally compact abelian group and  $\Gamma$  its dual group. Let  $M(G)$  be the measure algebra on  $G$ , and  $M_0(G)$  the subalgebra of  $M(G)$  which consists of measures whose Fourier-Stieltjes transforms vanish at infinity. For every  $\mu \in M(G)$  we denote by  $\check{\mu}$  the Gelfand transform of  $\mu$ .  $X$  denotes the maximal ideal space of  $M(G)$ . We denote by  $M_{00}(G)$  the algebra of all  $\mu \in M(G)$  whose Gelfand transforms vanish off  $\Gamma$ .

Note that  $L^1(G) \subset M_{00}(G)$ . Note also that  $\Gamma \subset X$  and  $\hat{\mu} = \check{\mu}$  on  $\Gamma$  for every  $\mu \in M(G)$ , where  $\hat{\mu}$  is the Fourier-Stieltjes transform of  $\mu$ . Let  $\text{sp}(\mu)$  denote the spectrum of  $\mu$ . Then  $\text{sp}(\mu) = \check{\mu}(X)$ . We denote by  $\text{NS}(G)$  the set of all measures with natural spectra, that is,  $\text{NS}(G) = \{\mu \in M(G) : \text{sp}(\mu) = \overline{\check{\mu}(\Gamma)}\}$ . Williamson [6] proved that  $\text{NS}(G)$  is a proper subset of  $M(G)$  for every non-discrete  $G$ . Put  $\text{NS}_0(G) = \text{NS}(G) \cap M_0(G)$ . Then  $M_{00}(G) \subset \text{NS}_0(G)$  holds for every  $G$ . Neumann [3, Theorem 9] proved, as a generalization of a theorem of Zafran [7, Theorem 3.2], that  $\text{NS}_0(G) = M_{00}(G) = \text{Reg } M_0(G) = \text{Dec } M_0(G)$  if  $G$  is compact, where  $\text{Reg } M_0(G)$  is the greatest regular closed subalgebra of  $M_0(G)$  and  $\text{Dec } M_0(G)$  is the Apostol algebra of  $M_0(G)$ .

Rudin [4] for  $G = \mathbb{R}$  and Varopoulos [5] for an arbitrary non-discrete  $G$  proved that  $\text{NS}_0(G)$  is a proper subset of  $M_0(G)$ . Eschmeier, Laursen and Neumann [1] gave examples of measures in  $\text{NS}_0(G) \setminus M_{00}(G)$  for certain non-compact  $G$ . In this note we show that for every non-compact  $G$  and for every  $\mu \in M(G)$  (resp.  $M_0(G)$ ) there exists an  $f \in L^1(G)$  such that  $\mu - f \in \text{NS}(G)$  (resp.  $\text{NS}_0(G)$ ). It follows that  $\text{NS}_0(G) \setminus M_{00}(G) \neq \emptyset$  for every non-compact and non-discrete  $G$ , which is a solution to the question posed by Eschmeier, Laursen and Neumann [1, p.288].

**Theorem 1.** *Let  $G$  be a non-compact locally compact abelian group. Then we have*

$$\text{NS}(G) + L^1(G) = M(G).$$

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*Proof.* Let  $\Gamma$  be the dual group of  $G$  and  $X$  the maximal ideal space of  $M(G)$ . By [2, Theorem 41.5, Theorem 41.13] there exists a Helson set  $K \subset \Gamma$  which is homeomorphic with Cantor's ternary set  $H$ , since  $\Gamma$  is non-discrete. Let  $\pi$  be a homeomorphism from  $K$  onto  $H$ . Let  $c$  be the restriction to  $H$  of Cantor's function defined on the unit interval  $I$ . Then  $c(H) = I$ . Let  $p$  be a continuous function defined on  $I$  onto the square  $\Delta = \{z \in C : |\operatorname{Re} z| \leq 1, |\operatorname{Im} z| \leq 1\}$  (Peano curve!). Then  $p \circ c \circ \pi$  is continuous function defined on  $K$  onto  $\Delta$ . Since  $K$  is a Helson set, there exists a function  $f \in L^1(G)$  such that  $\hat{f} = p \circ c \circ \pi$  on  $K$ . Let  $h$  be a function in  $L^1(G)$  such that  $\hat{h} = 1$  on  $K$ .

Let  $\mu$  be a measure in  $M(G)$ . Put  $\nu = \mu - \mu * h$ . Let  $r$  be the spectral radius of  $\nu$ . Then the measure  $\nu_1 = \nu - r f$  satisfies  $\mu = \nu_1 + r f + \mu * h$ , where  $r f + \mu * h \in L^1(G)$ . We show that  $\nu_1 \in \operatorname{NS}(G)$ . Let  $p \in X \setminus \Gamma$ . Then  $\check{\nu}(p) = \check{\nu}_1(p)$  since  $r \hat{f}(p) = 0$ ; hence  $|\check{\nu}_1(p)| \leq r$ . Since  $r \hat{f}(K) = r \Delta$ , there is  $x \in K$  such that  $r \hat{f}(x) = -\check{\nu}_1(p)$ . Since  $\check{\nu} = 0$  on  $K$ , we have  $\check{\nu}_1(x) = \check{\nu}_1(p)$ . It follows that  $\check{\nu}_1(X) = \check{\nu}_1(\Gamma)$ . Thus we have  $\nu_1 \in \operatorname{NS}(G)$ .  $\square$

Zafran [7, Example 3.4] pointed out that  $\operatorname{NS}(G)$  is not closed under addition if  $G$  is an  $I$ -group. By Theorem 1 and a theorem of Williamson we see that  $\operatorname{NS}(G)$  is not closed under addition if  $G$  is non-discrete and non-compact, since  $L^1(G) \subset \operatorname{NS}(G)$ .

Next we consider that case of  $M_0(G)$ .

**Corollary 2.** *Let  $G$  be a non-compact locally compact abelian group. Then we have*

$$\operatorname{NS}_0(G) + L^1(G) = M_0(G).$$

*Proof.* Let  $\mu$  be a measure in  $M_0(G)$ . Then by Theorem 1 there are  $\nu \in \operatorname{NS}(G)$  and  $f \in L^1(G)$  such that  $\mu = \nu + f$ . Since the Fourier-Stieltjes transforms of  $\mu$  and  $f$  vanish at infinity, we see that  $\nu \in \operatorname{NS}_0(G)$ .  $\square$

Eschmeier, Laursen and Neumann [1, Proposition 14] proved for a locally compact abelian group  $G$  that  $\operatorname{NS}_0(G) = M_{00}(G)$  if and only if  $\operatorname{NS}_0(G)$  is closed under addition. For a discrete  $G$  the situation is simple, since  $M(G) = L^1(G)$ . In particular,  $\operatorname{NS}_0(G) = M_0(G)$  holds if  $G$  is discrete. For a non-discrete  $G$  we have the following.

**Corollary 3.** *Let  $G$  be a non-discrete locally compact abelian group. Then the following are equivalent.*

- (1)  $\operatorname{NS}_0(G)$  is closed under addition.
- (2)  $\operatorname{NS}_0(G) = M_{00}(G)$ .
- (3)  $G$  is compact.

*Proof.* The equivalence between (1) and (2) is proved by Eschmeier, Laursen and Neumann [1, Proposition 14]. Suppose that  $G$  is compact. Neumann [3, Theorem 9] proved that  $\operatorname{NS}_0(G) = M_{00}(G)$ .

We show that (3) follows from (1). Suppose that (1) holds. If  $G$  is not compact, then  $\operatorname{NS}_0(G) + \operatorname{NS}_0(G) = M_0(G)$  by Corollary 2. Hence we have  $\operatorname{NS}_0(G) = M_0(G)$ , which is a contradiction (cf. [4], [5]). Thus we see that  $G$  is compact.  $\square$

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