PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 8, August 1998, Pages 2377–2388
S 0002-9939(98)04300-7

ASYMPTOTICS FOR SOBOLEV ORTHOGONAL POLYNOMIALS WITH COHERENT PAIRS: THE JACOBI CASE, TYPE 1

K. PAN

(Communicated by J. Marshall Ash)

ABSTRACT. Define $P_n(x)$ and $Q_n(x)$ as the nth monic orthogonal polynomials with respect to $d\mu$ and $d\nu$ respectively. The pair $\{d\mu, d\nu\}$ is called a coherent pair if there exist non-zero constants D_n such that

$$Q_n(x) = \frac{P'_{n+1}(x)}{n+1} + D_n \frac{P'_n(x)}{n}, \qquad n \geq 1.$$

One can divide the coherent pairs into two cases: the Jacobi case and the Laguerre case. There are two types for each case: type 1 and 2. We investigate the asymptotic properties and zero distribution of orthogonal polynomials with respect to Sobolev inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)d\mu(x) + \lambda \int_a^b f'(x)g'(x)d\nu(x)$$

for the coherent pair $\{d\mu, d\nu\}$: the Jacobi case, type 1.

1. INTRODUCTION

Recently polynomials orthogonal with respect to Sobolev inner products of the form

$$\langle f, g \rangle_S = \int_a^b f(x)g(x)d\mu(x) + \lambda \int_a^b f'(x)g'(x)d\nu(x) \tag{1.1}$$

were introduced, where (a, b) denotes a finite or infinite interval on the real axis, μ and ν are distribution functions on (a, b) with infinitely many points of increase and $\lambda > 0$. Let S_n be the nth monic orthogonal polynomial with respect to (1.1). For the inner product (1.1), the results known are mostly connected with the formal theory: recurrence relation, location of zeros, differential formulas and so on (see [C], [IKNS] and [M1]–[M2]). Little is known concerning asymptotic properties. Let $\{P_n\}$ and $\{Q_n\}$ be the monic orthogonal polynomial sequences with respect to $d\mu$ and $d\nu$ respectively. The pair $\{d\mu, d\nu\}$ is called a coherent pair if there exist non-zero constants D_n such that

$$Q_n(x) = \frac{P'_{n+1}(x)}{n+1} + D_n \frac{P'_n(x)}{n}, \qquad n \geq 1.$$

The concept of coherent pair was introduced by Iserles et al. in [IKNS]. It was proved to be very important in the research of Sobolev orthogonal polynomials. Recently a complete classification of all coherent pairs has been given in [M2]. It

Received by the editors July 24, 1996 and, in revised form, January 22, 1997.
1991 *Mathematics Subject Classification.* Primary 42C05.

has been proved that they can be divided into two types depending on the regularity of $d\nu$.

Type 1. $d\nu$ is classical.

(i) The Laguerre case, $(a,b) = (0,\infty)$, $d\mu = (x-c)x^{\alpha-1}e^{-x}dx$, $d\nu = x^{\alpha}e^{-x}dx$, with $c \le 0$, $\alpha > 0$.

(ii) The Jacobi case, $(a,b) = (-1,1)$, $d\mu = (x-c)(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$, $d\nu = (1-x)^{\alpha}(1+x)^{\beta}\,dx$, with $c < -1$, $\alpha > 0$, $\beta > 0$.

Type 2. $(x-c)\,d\nu$ is classical, $d\mu$ is classical.

(iii) The Laguerre case, $(a,b) = (0,\infty)$, $d\mu = x^{\alpha}e^{-x}dx$, $d\nu = \frac{x^{\alpha+1}e^{-x}}{x-c}dx + M\delta(c)$, with $c \le 0$, $\alpha > -1$, $M \ge 0$.

(iv) The Jacobi case, $(a,b) = (-1,1)$, $d\mu = (1-x)^{\alpha}(1+x)^{\beta}\,dx$, $d\nu = \frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{x-c}dx + M\delta(c)$, with $c < -1$, $\alpha > -1$, $\beta > -1$, $M \ge 0$.

In [BdM], Bruinsma, de Bruin and Meijer investigated some properties of S_n. It was proved that coherent pairs of type 1 are more regular than those of type 2; they studied the zeros location and proved that the n zeros of S_n are separated by those of Q_{n-1} ($n \ge 2$) for the coherent pairs of type 1. This result does not hold for all coherent pairs of type 2. They also proved that for all coherent pairs the $n-1$ extremata of S_n interlace with the $n-1$ zeros of Q_{n-1} ($n \ge 2$). In [P], Pan studied the limits of $\lim_{n\to\infty} P_n(x)/Q_n(x)$ when $x \in \mathbf{C}\backslash[-1,1]$ and $\lim_{n\to\infty} D_n$ for the Jacobi case in both types 1 and 2.

In this paper, we focus on the Jacobi case, type 1. The structure of the paper is as follows. In section 2, we introduce the asymptotic results and the proofs are given in section 6. In section 3, we define a differential operator on the linear space of the real polynomials. This operator allows us to relate the Sobolev inner product with $\langle\ ,\ \rangle_{\mu}$. Some auxiliary lemmas will be discussed in section 4 and we study the zeros distribution in section 5.

2. Asymptotic results

From now on, we only consider the Jacobi case, type 1, i.e. $d\mu(x) = w_0(x)\,dx$ and $d\nu(x) = w_1(x)\,dx$, where $w_0(x) = (x-c)(1-x)^{\alpha-1}(1+x)^{\beta-1}$ and $w_1(x) = (1-x)^{\alpha}(1+x)^{\beta}$.

Let $\{S_n\}$ denote the sequence of monic orthogonal polynomials (SMOP) with respect to (1.1), and $\{p_n(x) = \kappa_n x^n + \cdots\}$, $\kappa_n > 0$, be the sequence of orthonormal polynomials with respect to w_0dx. Denote by $\{q_n(x) = p^{(\alpha,\beta)}(x) = \gamma_n^{(\alpha,\beta)}x^n + \cdots\}$, $\gamma_n^{(\alpha,\beta)} > 0$, the sequence of orthonormal polynomials with respect to w_1dx (i.e. the orthonormal Jacobi polynomials). First, we have the following ratio limit.

Theorem 2.1. *The following is valid:*

$$\lim_{n\to\infty} \frac{S_{n+1}(x)}{S_n(x)} = \phi(x)$$

locally uniformly in $\mathbf{C}\backslash[-1,1]$*, where* $\phi(x) = [x + \sqrt{x^2-1}]/2$ *with* $\sqrt{x^2-1} > 0$ *for* $x > 1$.

Next, we find the relative asymptotic between $S_n(x)$ and $P_n(x)$.

Theorem 2.2. *The following holds:*

$$\lim_{n\to\infty}\frac{S_n(x)}{P_n(x)}=\frac{x-c}{\phi(x)-\phi(c)}$$

locally uniformly in $\mathbf{C}\backslash[-1,1]$, *where* $P_n(x)=p_n(x)/\kappa_n$.

Finally, we compare $S_n(x)$ with $Q_n(x)=P_n^{(\alpha,\beta)}(x)=q_n(x)/\gamma_n^{(\alpha,\beta)}$ and have the following relative asymptotic.

Theorem 2.3. *The following is valid:*

$$\lim_{n\to\infty}\frac{S_n(x)}{P_n^{(\alpha,\beta)}(x)}=\frac{\sqrt{x^2-1}}{\phi(x)}$$

locally uniformly in $\mathbf{C}\backslash[-1,1]$.

By taking advantage of the classical results of $P_n^{(\alpha,\beta)}(x)$, one can derive the corresponding results for $S_n(x)$ from Theorem 2.3. The proofs will be given in section 6.

3. Differential operator

Let $\rho(x)=w_0(x)/(x-c)$. Integrating by parts on the second integral, the constant term vanishes and we obtain

$$\langle p,q\rangle_S=\int_{-1}^{1}q(x)\{p(x)w_0(x)-\lambda p'(x)w'(x)-\lambda p''(x)w_1(x)\}\,dx.$$

We define the linear operator $\mathbf{F}$ by

$$\mathbf{F}p=\frac{x-c}{w_0}\{pw_0-\lambda p'w_1'-\lambda p''w_1\}.$$

Then

$$\langle p,q\rangle_S=\int_{-1}^{1}q(x)\mathbf{F}(p)\rho(x)\,dx=\langle\mathbf{F}p,q\rangle_\rho. \tag{3.1}$$

For the investigation of the operator $\mathbf{F}$, it is convenient to make the change of variable $x-c=t$. Then, after some simple calculations, we obtain for $\mathbf{F}$

$$\mathbf{F}p=xp+\lambda\{x(\alpha+\beta)+c(\alpha+\beta)-\beta+\alpha\}p'+\lambda\{x^2+2xc+c^2-1\}p''.$$

Lemma 3.1. *The differential operator* $\mathbf{F}$ *is self-adjoint with respect to the inner product* $\langle\ ,\ \rangle_S$, *that is,*

$$\langle\mathbf{F}p,q\rangle_S=\langle p,\mathbf{F}q\rangle_S,$$

for polynomials p *and* q.

Proof. From (3.1), we obtain

$$\langle\mathbf{F}p,q\rangle_S=\langle q,\mathbf{F}p\rangle_S=\langle\mathbf{F}q,\mathbf{F}p\rangle_\rho=\langle\mathbf{F}p,\mathbf{F}q\rangle_\rho=\langle p,\mathbf{F}q\rangle_S. \quad\square$$

Next, by the Fourier expansion and the orthogonality with respect to $\langle\ ,\ \rangle_S$, we express $\mathbf{F}S_n(x)$ in terms of $P_n^{(\alpha-1,\beta-1)}(x)$.

Lemma 3.2. *For the differential operator* $\mathbf{F}$, *we have*

$$\mathbf{F}S_n(x)=P_{n+1}^{(\alpha-1,\beta-1)}(x)+b_nP_n^{(\alpha-1,\beta-1)}(x),\ \ \textit{where}\ b_n=\frac{\langle S_n,S_n\rangle_S}{\langle P_n^{(\alpha-1,\beta-1)},P_n^{(\alpha-1,\beta-1)}\rangle_\rho}.$$

We can get the relation between $P_n^{(\alpha-1,\beta-1)}(x)$ and $P_{n+1}(x)$ by the Fourier expansion and the orthogonality with respect to $\rho\, dx$.

Lemma 3.3. *We have the following:*

$$P_{n+1}^{(\alpha-1,\beta-1)}(x) = P_{n+1}(x) + c_n P_n(x), \quad \textit{where } c_n = \frac{\langle P_{n+1}^{(\alpha-1,\beta-1)}, P_{n+1}^{(\alpha-1,\beta-1)}\rangle_\rho}{\langle P_n, P_n\rangle_{w_0}}.$$

Next, we quote the relation between $P_n(x)$ and $S_n(x)$ from [BdM].

Lemma 3.4 ([BdM])**.** *For $n \geq 1$, the following holds:*

$$\frac{P_{n+1}(x)}{n+1} + D_n \frac{P_n(x)}{n} = \frac{S_{n+1}(x)}{n+1} + d_n \frac{S_n(x)}{n}, \quad \textit{where } d_n = \frac{\langle P_n, P_n\rangle_{w_0}}{\langle S_n, S_n\rangle_S} D_n.$$

Finally in this section, we rewrite $\mathbf{F}S_n(x)$ in terms of $S_n(x)$.

Theorem 3.5. *The following holds:*

$$\mathbf{F}S_n(x) = S_{n+1}(x) + a_n S_n(x) + a_{n-1} S_{n-1}(x),$$

where

$$a_{n-1} = \frac{\langle S_n, S_n\rangle_S}{\langle S_{n-1}, S_{n-1}\rangle_S} \quad \textit{and} \quad a_n = \frac{n+1}{n} d_n - \frac{n+1}{n} D_n + b_n + c_n.$$

Proof. Consider the expansion

$$\mathbf{F}S_n(x) = S_{n+1}(x) + \sum_{i=0}^{n} a_i S_i(x).$$

We have

$$a_i = \frac{\langle \mathbf{F}S_n, S_i\rangle_S}{\langle S_i, S_i\rangle_S} = \frac{\langle S_n, \mathbf{F}S_i\rangle_S}{\langle S_i, S_i\rangle_S} = 0,$$

for $i = 0, \ldots, n-2$,

$$a_{n-1} = \frac{\langle S_n, \mathbf{F}S_{n-1}\rangle_S}{\langle S_{n-1}, S_{n-1}\rangle_S} = \frac{\langle S_n, S_n\rangle_S}{\langle S_{n-1}, S_{n-1}\rangle_S},$$

and, from Lemmas 3.2, 3.3 and 3.4,

$$\begin{aligned}
a_n &= \frac{\langle S_n, \mathbf{F}S_n\rangle_S}{\langle S_n, S_n\rangle_S} \\
&= \frac{\langle S_n, P_{n+1}^{(\alpha-1,\beta-1)} + b_n P_n^{(\alpha-1,\beta-1)}\rangle_S}{\langle S_n, S_n\rangle_S} \\
&= \frac{\langle S_n, P_{n+1}^{(\alpha-1,\beta-1)}\rangle_S}{\langle S_n, S_n\rangle_S} + b_n \frac{\langle S_n, P_n^{(\alpha-1,\beta-1)}\rangle_S}{\langle S_n, S_n\rangle_S} \\
&= \frac{\langle S_n, P_{n+1} + c_n P_n\rangle_S}{\langle S_n, S_n\rangle_S} + b_n = \frac{\langle S_n, P_{n+1}\rangle_S}{\langle S_n, S_n\rangle_S} + c_n \frac{\langle S_n, P_n\rangle_S}{\langle S_n, S_n\rangle_S} + b_n \\
&= \frac{\langle S_n, S_{n+1} + d_n \frac{n+1}{n} S_n - D_n \frac{n+1}{n} P_n\rangle_S}{\langle S_n, S_n\rangle_S} + b_n + c_n \\
&= \frac{n+1}{n} d_n - D_n \frac{n+1}{n} + b_n + c_n. \quad \square
\end{aligned}$$

4. Lemmas

We use this section to introduce some auxiliary results for the proofs of the theorems in section 2. We define the Nevai class $M(0,1)$ consisting of all measures ψ such that $\operatorname{supp}\psi = [-1,1]\backslash E$ with E a set which is at most denumerable and $E' \subset \{-1,1\}$.

Lemma 4.1 ([N]). *Let $\psi \in M(0,1)$, and let $\phi_n(x) = \eta_n x^n + \cdots$, $\eta > 0$, be the nth orthonormal polynomial with respect to ψ. Then*

$$\lim_{n\to\infty} \frac{\eta_{n-1}}{\eta_n} = \frac{1}{2},$$

$$\lim_{n\to\infty} \frac{\Phi_{n+1}(x)}{\Phi_n} = \phi(x) \quad \text{and} \quad \lim_{n\to\infty} \frac{\Phi_n'(x)}{n\Phi_n(x)} = \frac{1}{\sqrt{x^2-1}}$$

locally uniformly in $\mathbf{C}\backslash[-1,1]$, where $\Phi_n(x) = \phi_n(x)/\eta_n$.

We quote the following lemma for the relations between $P_n(x)$ and $P_n^{(\alpha-1,\beta-1)}(x)$.

Lemma 4.2 ([P]). *The following hold:*

$$(x-c)P_n(x) = P_{n+1}^{(\alpha-1,\beta-1)}(x) - \frac{P_{n+1}^{(\alpha-1,\beta-1)}(c)}{P_n^{(\alpha-1,\beta-1)}(c)} P_n^{(\alpha-1,\beta-1)}(x), \tag{4.1}$$

$$\lim_{n\to\infty} \frac{P_n(x)}{P_n^{(\alpha-1,\beta-1)}(x)} = \frac{\phi(x)-\phi(c)}{x-c} \tag{4.2}$$

and

$$\lim_{n\to\infty} \frac{Q_n(x)}{P_n(x)} = \frac{\phi(x)}{\sqrt{x^2-1}} \frac{x-c}{\phi(x)-\phi(c)} \tag{4.3}$$

locally uniformly in $\mathbf{C}\backslash[-1,1]$.

Lemma 4.3 ([P]). *Let $\psi \in M(0,1)$, and let $\Phi_n(x)$ be the nth monic orthogonal polynomial with respect to $d\psi$. For any polynomial sequence $\{u_n\}$, if $\lim_{n\to\infty} u_n(x)/\Phi_n(x)$ exists locally uniformly in $\mathbf{C}\backslash[-1,1]$, then*

$$\frac{u_n'(x)}{\Phi_n'(x)} - \frac{u_n(x)}{\Phi_n(x)} = \mathcal{O}\left(\frac{1}{n}\right), \quad \text{as } n \to \infty,$$

locally uniformly in $\mathbf{C}\backslash[-1,1]$.

For the relation between the Jacobi polynomials $p_n^{(\alpha-1,\beta-1)}(x)$ and $p_n(x)$, we have

Lemma 4.4 ([N]). *The following holds:*

$$\lim_{n\to\infty} \frac{\gamma_n^{(\alpha-1,\beta-1)}}{\kappa_n} = A,$$

where

$$A = \exp\left\{\frac{1}{2\pi}\int_{-1}^{1} \frac{\log(t-c)}{\sqrt{1-t^2}}\,dt\right\}.$$

Lemma 4.5. *For the leading coefficient of the Jacobi polynomial, we have* $\lim_{n\to\infty} \frac{\gamma_n^{(\alpha-1,\beta-1)}}{\gamma_n^{(\alpha,\beta)}} = \frac{1}{8}$.

Proof. It is easy to see the lemma holds from the following formula [Sz],

$$\gamma_n^{(\alpha,\beta)} = \frac{1}{2^n}\left(\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\beta+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\right)^{1/2}\binom{2n+\alpha+\beta}{n}. \quad \square$$

Lemma 4.6 ([P]). *The following is valid:*

$$d_n = \frac{D_n}{1+\frac{D_{n-1}^2 n^2\kappa_n^2}{(n-1)^2\kappa_{n-1}^2}+\frac{\lambda n^2\kappa_n^2}{[\gamma_{n-1}^{(\alpha,\beta)}]^2}-d_{n-1}\frac{D_{n-1}n^2\kappa_n^2}{(n-1)^2\kappa_{n-1}^2}}, \quad \textit{with } d_1 = \frac{\int d\mu}{\int d\mu+\int d\nu}D_1.$$

The last lemma in the section tells us $d_n \to 0$ as $n\to\infty$ and with rate $1/n^2$.

Lemma 4.7. *We have*

$$\lim_{n\to\infty} n^2 d_n = -\frac{4A^2}{\lambda\phi(c)}.$$

Proof. From the definition of $\langle S_n, S_n\rangle_S$, we have

$$\langle P_n, P_n\rangle_{w_0} \le \langle S_n, S_n\rangle_{w_0} \le \langle S_n, S_n\rangle_S.$$

So,

$$|d_n| = \left|\frac{\langle P_n, P_n\rangle_{w_0}}{\langle S_n, S_n\rangle_S}D_n\right| \le |D_n|.$$

Also, we know that (cf. [P])

$$\lim_{n\to\infty} D_n = -\frac{1}{4\phi(c)}. \tag{4.4}$$

Then d_n is bounded. Next, from Lemmas 4.1, 4.4 and 4.5,

$$\lim_{n\to\infty}\frac{\kappa_n}{\gamma_{n-1}^{(\alpha,\beta)}} = \lim_{n\to\infty}\frac{\kappa_n}{\gamma_n^{(\alpha-1,\beta-1)}}\frac{\gamma_n^{(\alpha-1,\beta-1)}}{\gamma_{n-1}^{(\alpha-1,\beta-1)}}\frac{\gamma_{n-1}^{(\alpha-1,\beta-1)}}{\gamma_{n-1}^{(\alpha,\beta)}} = \frac{1}{A}\cdot 2\cdot\frac{1}{8} = \frac{1}{4A}.$$

Thus, together with Lemmas 4.1 and 4.6 and (4.4),

$$\begin{aligned}\lim_{n\to\infty} n^2 d_n &= \lim_{n\to\infty}\frac{n^2 D_n}{1+\frac{D_{n-1}^2 n^2\kappa_n^2}{(n-1)^2\kappa_{n-1}^2}+\frac{\lambda n^2\kappa_n^2}{[\gamma_{n-1}^{(\alpha,\beta)}]^2}-d_{n-1}\frac{D_{n-1}n^2\kappa_n^2}{(n-1)^2\kappa_{n-1}^2}}\\ &= \lim_{n\to\infty}\frac{D_n}{\frac{\lambda\kappa_n^2}{[\gamma_{n-1}^{(\alpha,\beta)}]^2}} = -\frac{4A^2}{\lambda\phi(c)}. \quad \square\end{aligned}$$

5. Zero distribution

We investigate the zero distribution for $S_n(x)$ in this section.

Lemma 5.1 ([BSS]). *Let $\{w_n(x) = \prod_{j=1}^n (x-x_{n,j})\}_{n=0}^\infty$ be a sequence of monic polynomials of respective degree precisely n. If there are at most a finite number of $x_{n,i}$ outside of $[-1,1]$ for each n and*

$$\limsup_{n\to\infty}\|w_n(x)\|_{[-1,1]}^{1/n} \le 1/2,$$

then, in weak-* topology,

$$\lim_{n\to\infty}\mu_n = \frac{1}{\pi\sqrt{1-x^2}},$$

where μ_n is the discrete unit measure defined on the Borel sets having mass $1/n$ at each zero $x_{n,j}$.

The next lemma is the estimation for the norm of $P_n(x)$ in $[-1,1]$.

Lemma 5.2 ([Sz])**.** *There is a constant $C>0$, such that*

$$\|p_n^{(\alpha,\beta)}(x)\|_{[-1,1]} \le \begin{cases} Cn^{s+1/2}, & s=\max(\alpha,\beta)\ge -1/2,\\ C, & s=\max(\alpha,\beta)<-1/2, \end{cases} \qquad n\to\infty.$$

Lemma 5.3. *There is a constant $D>0$, such that*

$$\|p_n(x)\|_{[-1,1]} \le Dy(n),$$

where

$$y(n) = \begin{cases} (n+1)^{t+1/2}, & t=\max(\alpha-1,\beta-1)\ge -1/2,\\ 1, & t=\max(\alpha-1,\beta-1)<-1/2. \end{cases}$$

Proof. From (4.1), we have

$$\begin{aligned}\|p_n(x)\|_{[-1,1]} \le \frac{1}{|c|-1}\Bigg[&\left|\frac{\kappa_n}{\gamma_{n+1}^{(\alpha-1,\beta-1)}}\right| \|p_{n+1}^{(\alpha-1,\beta-1)}(x)\|_{[-1,1]}\\ &+\left|\frac{P_{n+1}^{(\alpha-1,\beta-1)}(c)}{P_n^{(\alpha-1,\beta-1)}(c)}\right| \frac{\kappa_n}{\gamma_n^{(\alpha-1,\beta-1)}}\|p_n^{(\alpha-1,\beta-1)}(x)\|_{[-1,1]}\Bigg].\end{aligned}$$

Together with Lemmas 4.1, 4.4 and 5.2, we show that the lemma holds. □

Next, we find the upper bound for the norm of $S_n(x)$ in $[-1,1]$.

Lemma 5.4. *Define $\hat{\kappa}_n = 1/\sqrt{\langle S_n, S_n\rangle_S}$ and the polynomial $g_n(x) = \hat{\kappa}_n S_n(x)$. Then there is a constant $E>0$, such that*

$$\|g_n(x)\|_{[-1,1]} \le Ey(n), \qquad n\to\infty.$$

Proof. From Lemma 3.4 and $\kappa_n = \hat{\kappa}_n\sqrt{D_n/d_n}$, we get

$$\begin{aligned} g_n(x) =&\sqrt{\frac{d_n}{D_n}}p_n(x) + D_{n-1}\frac{n}{n-1}\frac{\kappa_n}{\kappa_{n-1}}\sqrt{\frac{d_n}{D_n}}p_{n-1}(x)\\ &-\sqrt{d_{n-1}d_n}\frac{n}{n-1}\sqrt{\frac{D_{n-1}}{D_n}}\frac{\kappa_n}{\kappa_{n-1}}g_{n-1}(x). \end{aligned} \tag{5.1}$$

We assume that $\|g_{n-1}(x)\|_{[-1,1]} \le Ey(n-1)$. From Lemma 4.7, there is a constant $C_1>0$ such that $d_n \le \frac{C_1}{n^2}$. Notice that κ_n/κ_{n-1} and D_n are bounded. From (5.1) and Lemma 5.3, we obtain

$$\begin{aligned}&\|g_n(x)\|_{[-1,1]}\\ &\quad\le Ey(n)\Bigg\{\frac{D}{E}\sqrt{\frac{C_1}{|D_n|}}\frac{1}{n} + |D_{n-1}|\frac{\kappa_n}{\kappa_{n-1}}\frac{n}{n-1}\sqrt{\frac{C_1}{|D_n|}}\frac{D}{E}\frac{y(n-1)}{y(n)}\frac{1}{n}\\ &\qquad +C_1\frac{n}{n-1}\frac{\kappa_n}{\kappa_{n-1}}\sqrt{\frac{|D_{n-1}|}{|D_n|}}\frac{y(n-1)}{y(n)}\frac{1}{n-1}\Bigg\} \le Ey(n), \quad n\to\infty. \quad □\end{aligned}$$

We can prove the following zero distribution theorem now.

Theorem 5.5. *For the monic polynomials $S_n(x)$, we have, in weak-* topology,*

$$\lim_{n\to\infty} \nu_n = \frac{1}{\pi\sqrt{1-x^2}},$$

where ν_n is the discrete unit measure defined on the Borel sets having mass $1/n$ at each zero of $Q_n(x)$.

Proof. We know that at most one zero of $S_n(x)$ is outside $[-1,1]$ (cf. [BdM]). Thus, from Lemma 5.1, we only need to prove $\limsup_{n\to\infty} \|S_n(x)\|_{[-1,1]}^{1/n} \leq 1/2$. First, we know, from Lemma 4.1,

$$\lim_{n\to\infty} \kappa_n^{1/n} = 2. \tag{5.2}$$

Next, from Lemma 3.4, $S_n(x) = \frac{g_n(x)}{\hat{\kappa}_n} = \frac{g_n(x)}{\kappa_n}\sqrt{\frac{D_n}{d_n}}$. Thus, from (5.2) and Lemmas 5.4 and 4.7,

$$\limsup_{n\to\infty} \|S_n(x)\|_{[-1,1]}^{1/n} \leq \lim_{n\to\infty} \frac{1}{\kappa_n^{1/n}} \limsup_{n\to\infty} \|g_n(x)\|_{[-1,1]}^{1/n} \lim_{n\to\infty} \left(\sqrt{\frac{D_n}{d_n}}\right)^{1/n} = \frac{1}{2}. \quad \square$$

6. Proofs

We are now in a position to prove the theorems in Section 2.

Proof of Theorem 2.1. From Lemma 3.4, we have

$$\frac{\frac{P_{n+1}(x)}{n+1} + D_n \frac{P_n(x)}{n}}{\frac{P_{n+2}(x)}{n+2} + D_{n+1}\frac{P_{n+1}(x)}{n+1}} = \frac{t_{n+1}(x)[1+d_n t_n(x)]}{1+d_{n+1}t_{n+1}(x)},$$

where $t_n(x) = (n+1)S_n(x)/nS_{n+1}(x)$. Together with Lemma 4.1, we get

$$\lim_{n\to\infty} \frac{t_{n+1}(x)[1+d_n t_n(x)]}{1+d_{n+1}t_{n+1}(x)} = \frac{1}{\phi(x)}, \tag{6.1}$$

locally uniformly in $\mathbf{C}\backslash[-1,1]$. We claim that the sequence $\{t_n(x)\}$ is locally uniformly bounded in $\mathbf{C}\backslash[-1,1]$. Suppose there is $\Lambda \subset \mathbf{N}$, such that $t_{n+1}(x) \to \infty$ and $n \in \Lambda$, $x \in \mathbf{C}\backslash[-1,1]$. From (6.1) and Lemma 4.7, we know

$$\lim_{n\to\infty} (1 + d_n t_n(x)) = 0, \qquad n \in \Lambda. \tag{6.2}$$

Also from (6.1) and (6.2), we get

$$\lim_{n\to\infty} t_n(x)(1+d_{n-1}t_{n-1}(x)) = 0, \qquad n \in \Lambda. \tag{6.3}$$

We define $x_n \cong y_n$, if there are constants $C_2 > 0$ and $C_3 > 0$ such that $C_2 \leq \frac{x_n}{y_n} \leq C_3$. Notice that, from Lemmas 3.3 and 4.4,

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\kappa_n^2}{[\gamma_{n+1}^{(\alpha-1,\beta-1)}]^2} = \frac{1}{4A^2}. \tag{6.4}$$

Then, from Theorem 3.5,

$$\frac{\mathbf{F}S_n}{nS_n} \cong \frac{b_n}{n} + \frac{a_{n-1}(n-1)}{n^2} t_{n-1}(x). \tag{6.5}$$

Also,

$$\begin{aligned}
&a_{n-1}\frac{n-1}{n}t_{n-1}(x)+b_n\\
&\quad=\frac{\langle P_n,P_n\rangle_{w_0}D_nd_{n-1}}{D_{n-1}d_n\langle P_{n-1},P_{n-1}\rangle_{w_0}}\frac{n-1}{n}t_{n-1}+\frac{\langle P_n,P_n\rangle_{w_0}D_n}{d_n\langle P_n^{(\alpha-1,\beta-1)},P_n^{(\alpha-1,\beta-1)}\rangle_\rho}\\
&\quad=\frac{\langle P_n,P_n\rangle_{w_0}D_n}{d_n}\\
&\qquad\cdot\left[\frac{n-1}{n}t_{n-1}(x)d_{n-1}\frac{1}{D_{n-1}\langle P_{n-1},P_{n-1}\rangle_{w_0}}+\frac{1}{c_{n-1}\langle P_{n-1},P_{n-1}\rangle_{w_0}}\right]\\
&\quad=\frac{\langle P_n,P_n\rangle_{w_0}D_n}{\langle P_{n-1},P_{n-1}\rangle_{w_0}D_{n-1}d_n}\left[t_{n-1}(x)d_{n-1}+\frac{D_{n-1}}{c_{n-1}}-\frac{1}{n}t_{n-1}d_{n-1}\right]\\
&\quad=\frac{\langle P_n,P_n\rangle_{w_0}D_n}{\langle P_{n-1},P_{n-1}\rangle_{w_0}D_{n-1}}[1+t_{n-1}(x)d_{n-1}]\frac{1}{d_n}\\
&\qquad+\frac{\langle P_n,P_n\rangle_{w_0}D_n}{\langle P_{n-1},P_{n-1}\rangle_{w_0}D_{n-1}}\left[\frac{D_{n-1}}{c_{n-1}}-1\right]\frac{1}{d_n}\\
&\qquad-\frac{\langle P_n,P_n\rangle_{w_0}D_n}{\langle P_{n-1},P_{n-1}\rangle_{w_0}D_{n-1}}\frac{t_{n-1}(x)}{n}\frac{d_{n-1}}{d_n}.
\end{aligned}$$

Then, from (6.5),

$$\begin{aligned}
\frac{\mathbf{F}S_n}{nS_n}\cong{}&\frac{\langle P_n,P_n\rangle_{w_0}D_n}{\langle P_{n-1},P_{n-1}\rangle_{w_0}D_{n-1}}\left[\frac{(1+t_{n-1}(x)d_{n-1})t_n(x)}{t_n(x)d_n}\right]\frac{1}{n}\\
&+\frac{\langle P_n,P_n\rangle_{w_0}D_n}{\langle P_{n-1},P_{n-1}\rangle_{w_0}D_{n-1}}\left[\frac{D_{n-1}}{c_{n-1}}-1\right]\frac{1}{nd_n}\\
&-\frac{\langle P_n,P_n\rangle_{w_0}D_n}{\langle P_{n-1},P_{n-1}\rangle_{w_0}D_{n-1}}\frac{d_{n-1}t_{n-1}(x)}{n^2d_n}.
\end{aligned}\tag{6.6}$$

From (6.2), (6.3), (6.6), (6.4), Lemmas 4.1 and 4.7, we have

$$\begin{aligned}
\frac{\mathbf{F}S_n}{nS_n}&\cong\frac{1}{4}\left[\frac{D_{n-1}-c_{n-1}}{c_{n-1}}\right]\frac{1}{nd_n}-\frac{\lambda\phi(c)}{4A}\\
&\cong A^2[D_{n-1}-c_{n-1}]\frac{1}{nd_n}-\frac{\lambda\phi(c)}{4A},\qquad n\in\Lambda.
\end{aligned}\tag{6.7}$$

Then, from the definition of D_{n-1} and c_{n-1} in Lemma 3.3, we get, for $n\in\Lambda$,

$$\frac{\mathbf{F}S_n}{nS_n}\cong A^2\left[\frac{(n-1)P_{n-1}^{(\alpha,\beta)}(x)}{P'_{n-1}(x)}-\frac{n-1}{n}\frac{P'_n(x)}{P'_{n-1}(x)}-\frac{P_n^{(\alpha-1,\beta-1)}(x)}{P_{n-1}(x)}+\frac{P_n(x)}{P_{n-1}(x)}\right]\frac{1}{nd_n}.\tag{6.8}$$

It is well-known that (cf. [Sz]) $\frac{d}{dx}P_{n+1}^{(\alpha-1,\beta-1)}(x)=(n+1)P_n^{(\alpha,\beta)}(x)$. So, for $n\in\Lambda$,

$$\begin{aligned}
\frac{\mathbf{F}S_n}{nS_n}\cong A^2\Bigg[&n\left\{\frac{[P_n^{(\alpha-1,\beta-1)}(x)]'}{P'_{n-1}(x)}-\frac{P_n^{(\alpha-1,\beta-1)}(x)}{P_{n-1}(x)}\right\}\\
&-\frac{[P_n^{(\alpha-1,\beta-1)}(x)]'}{P'_{n-1}(x)}-n\left\{\frac{P'_n(x)}{P'_{n-1}(x)}-\frac{P_n(x)}{P_{n-1}(x)}\right\}+\frac{P'_n(x)}{P'_{n-1}(x)}\Bigg]\frac{1}{n^2d_n}.
\end{aligned}$$

This, together with Lemmas 4.1, 4.2, 4.3, 4.4, 4.7, yields

$$\frac{\mathbf{F}S_n(x)}{nS_n(x)} = \mathcal{O}(1), \qquad n \in \Lambda. \tag{6.9}$$

On the other hand, from Theorem 5.5, we have

$$\lim_{n\to\infty} \frac{S_n'(z)}{nS_n(z)} = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{d\nu_n(t)}{z-t} = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{z-t}\frac{dt}{\sqrt{1-t^2}}, \qquad z \in \mathbf{C}\backslash[-1,1]. \tag{6.10}$$

Then, we get

$$\lim_{n\to\infty} \frac{S_n(x)}{S_n'(x)} = 0. \tag{6.11}$$

From (6.10), we know that at most a finite number of zeros of S_n' lie outside of $[-1,1]$ and $\limsup_{n\to\infty} \|S_n'(x)\|_{[-1,1]}^{1/n} \le 1/2$, since $\limsup_{n\to\infty} \|S_n(x)\|_{[-1,1]}^{1/n} \le 1/2$. Thus from Lemma 5.1 again

$$\lim_{n\to\infty} \frac{S_n''(z)}{nS_n'(z)} = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{z-t}\frac{dt}{\sqrt{1-t^2}}, \qquad z \in \mathbf{C}\backslash[-1,1].$$

So,

$$\lim_{n\to\infty} \frac{S_n'(x)}{S_n''(x)} = 0. \tag{6.12}$$

From the definition of $\mathbf{F}S_n(x)$, (6.10) and (6.12), we have, for $z \in \mathbf{C}\backslash[-1,1]$,

$$\begin{aligned}&\lim_{n\to\infty} \frac{\mathbf{F}S_n(z)}{nS_n(z)} \\ &= \lim_{n\to\infty} \frac{zS_n(z)+\lambda\{z(\alpha+\beta)+c(\alpha+\beta)-\beta+\alpha\}S_n'(z)+\lambda\{z^2+2zc+c^2-1\}S_n''(z)}{nS_n(z)} = \infty.\end{aligned}$$

This contradicts (6.9). Thus the sequence $\{t_n(x)\}$ is locally uniformly bounded in $\mathbf{C}\backslash[-1,1]$. Pick any convergence sequence $t_{n+1}(x)$, $n \in \Lambda_1 \subset \mathbf{N}$, and assume that $\lim_{n\to\infty} t_{n+1}(x) = q(x)$, $n \in \Lambda_1$. Thus, notice that the sequence $\{t_n(x)\}$ is bounded; from (6.1) and Lemma 4.7, we have $q(x) = \frac{1}{\phi(x)}$. Since Λ_1 is arbitrary, we have $\lim_{n\to\infty} t_n(x) = \frac{1}{\phi(x)}$, locally uniform in $\mathbf{C}\backslash[-1,1]$. □

Proof of Theorem 2.2. Notice that

$$\begin{aligned} d_n b_n &= d_n \frac{\langle S_n, S_n\rangle_S}{\langle P_n^{(\alpha-1,\beta-1)}, P_n^{(\alpha-1,\beta-1)}\rangle_\rho} \\ &= \frac{\langle P_n, P_n\rangle_{w_0}}{\langle P_n^{(\alpha-1,\beta-1)}, P_n^{(\alpha-1,\beta-1)}\rangle_\rho} D_n = \left[\frac{\gamma_n^{(\alpha-1,\beta-1)}}{\kappa_n}\right]^2 D_n. \end{aligned} \tag{6.13}$$

Thus, from Lemmas 3.2, 4.1, 4.4, 4.7 and (4.4), we have

$$\lim_{n\to\infty} d_n \frac{\mathbf{F}S_n(x)}{P_{n+1}^{(\alpha-1,\beta-1)}(x)} = A^2\left[-\frac{1}{4\phi(c)}\right]\frac{1}{\phi(x)} = -\frac{A^2}{4\phi(c)\phi(x)}.$$

Together with Lemma 4.1 and (4.2), we get

$$\begin{aligned}\lim_{n\to\infty} d_n \frac{\mathbf{F}S_n(x)}{P_n(x)} &= \lim_{n\to\infty} d_n \frac{\mathbf{F}S_n(x)}{P_{n+1}^{(\alpha-1,\beta-1)}(x)} \frac{P_{n+1}^{(\alpha-1,\beta-1)}(x)}{P_n^{(\alpha-1,\beta-1)}(x)} \frac{P_n^{(\alpha-1,\beta-1)}(x)}{P_n(x)} \\ &= -\frac{A^2(x-c)}{4\phi(c)(\phi(x)-\phi(c))}.\end{aligned} \tag{6.14}$$

On the other hand,

$$\begin{aligned} d_n a_{n-1} &= d_n \frac{\langle S_n, S_n\rangle_S}{\langle S_{n-1}, S_{n-1}\rangle_S} \\ &= d_n \frac{\langle P_n, P_n\rangle_{w_0}}{\langle P_{n-1}, P_{n-1}\rangle_{w_0}} \frac{D_n}{D_{n-1}} \frac{d_{n-1}}{d_n} \\ &= d_n \frac{\kappa_{n-1}^2}{\kappa_n^2} \frac{D_n}{D_{n-1}} \frac{d_{n-1}}{d_n}.\end{aligned}$$

Thus, from Lemmas 4.1, 4.7 and (4.4), $\lim_{n\to\infty} d_n a_{n-1} = 0$. From Lemmas 3.5, 4.4, 4.7, (4.4), and (6.13), we have $\lim_{n\to\infty} d_n a_n = \lim_{n\to\infty} d_n b_n = -\frac{A^2}{4\phi(c)}$. So, from Theorems 2.1 and 3.5, $\lim_{n\to\infty} d_n \frac{\mathbf{F}S_n(x)}{S_n(x)} = \lim_{n\to\infty} d_n a_n = -\frac{A^2}{4\phi(c)}$. Together with (6.14),

$$\lim_{n\to\infty} \frac{S_n(x)}{P_n(x)} = \lim_{n\to\infty} \frac{S_n(x)}{d_n\mathbf{F}S_n(x)} \frac{d_n\mathbf{F}S_n(x)}{P_n(x)} = \frac{x-c}{\phi(x)-\phi(c)}. \quad \square$$

Proof of Theorem 2.3. From Lemma 4.4 and Theorem 2.2,

$$\lim_{n\to\infty} \frac{S_n(x)}{Q_n(x)} = \lim_{n\to\infty} \frac{S_n(x)}{P_n(x)} \frac{P_n(x)}{Q_n(x)} = \frac{\sqrt{x^2-1}}{\phi(x)}. \quad \square$$

Added in proof

After this paper was sent for publication a paper of A. Martinez-Finkelshtein, J. J. Moreno-Balcazar, T. E. Perez and M. A. Pinar was submitted for publication where they obtain the results independently. The reference is A. Martinez-Finkelshtein, J. J. Moreno-Balcazar, T. E. Perez and M. A. Pinar, *Asymptotics of Sobolev orthogonal polynomials for coherent pairs of measures.* Thanks to Professor M. A. Pinar for sending me the manuscript.

References

[BdM] K. J. Bruinsma, M. G. de Bruin and H. G. Meijer, *Zero of Sobolev orthogonal polynomials following from coherent pairs*, Report of FTMI 95-65, Delft University of Technology, 1995.

[BSS] H. P. Blatt, E. B. Saff and M. Simkani, *Jentzsch-Szegő type theorems for the zeros of best approximants*, J. London Math. Soc. (2) **38** (1988) 307–316. MR **90a:**30004

[C] E. A. Cohen, Jr., *Zero distribution and behavior of orthogonal polynomials in the Sobolev space* $W^{1,2}[-1,1]$, SIAM J. Math. Anal. **6** (1975), 105–116. MR **50:**13664

[IKNS] A. Iserles, P. E. Koch, P. Norsett and J. M. Sanz-Serna, *On polynomials with respect to certain Sobolev inner products*, J. Approx. Th. **65** (1991), 151–175. MR **92b:**42029

[M1] H. G. Meijer, *Coherent pairs and zeros of Sobolev-type orthogonal polynomials*, Indag. Math. N.S. **4** (1993), 163–176. MR **94i:**42030

[M2] H. G. Meijer, *Determination of all coherent pairs of functionals*, Report of FTMI 95-41, Delft University of Technology, 1995.

[N] P. Nevai, *Orthogonal Polynomials*, Memoirs Amer. Math. Soc. 213, Amer. Math. Soc., Providence, RI, 1979. MR **80k:**42025

[P] K. Pan, *On Sobolev orthogonal polynomials with coherent pairs: the Jacobi case*, J. Comp. and Appl. Math. **79** (1997), 249–262. CMP 97:13

[Sz] G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., **23**, Providence, RI (1975). MR **51:**8724

Department of Mathematics, Barry University, Miami Shores, Florida 33161
E-mail address: pan@euclid.barry.edu