

## SPECTRAL PROPERTIES OF THE OPERATOR OF RIESZ POTENTIAL TYPE

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. For the operator of the Riesz potential type we estimate the second term in the asymptotic behavior of the spectrum and find its “regularized” trace.

### 1. INTRODUCTION AND NOTATION

For some classes of integral operators there are methods of finding the first term in the asymptotic of the singular values or eigenvalues (see e.g. [2], [3]). However, finding higher terms is rather difficult and can be realized only in some situations, and depends essentially on the structure of the kernel of an operator. In this paper, we consider the operator of Riesz potential type  $A: L^2(-1, 1) \rightarrow L^2(-1, 1)$  defined by

$$Af(x) = \int_{-1}^1 |x - y|^{\alpha-1} f(y) dy, \quad 0 < \alpha < 1.$$

It occurs in the theory of Liouville fractional integrals (see [10]). It is well known (see e.g. [8]) that if  $0 < \alpha < 1$ , then operator  $A$  is positive and compact. Denote by  $\lambda_1(A)$ ,  $\lambda_2(A)$ ,  $\dots$  the eigenvalues of  $A$  arranged in decreasing order, according to their multiplicity.

Also it is known [2], [3], [6], [9] that

$$\lim_{n \rightarrow \infty} n^\alpha \lambda_n(A) = c(\alpha)$$

holds, where

$$c(\alpha) = 2\Gamma(\alpha) \cos \frac{\alpha\pi}{2} \left(\frac{2}{\pi}\right)^\alpha.$$

In what follows we denote by  $\int_a^b L(x, y) \cdot dy$  the integral operator acting on  $L^2(a, b)$ , whose kernel is  $L(x, y)$ .

For a compact operator  $T$  we denote by  $s_n(T)$  the  $n$ -th eigenvalue of the operator  $|T| = (T^*T)^{\frac{1}{2}}$ , i.e.  $s_n(T) = \lambda_n(|T|)$ .

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Received by the editors January 7, 1997.

1991 *Mathematics Subject Classification*. Primary 47B10.

*Key words and phrases*. Singular values, eigenvalues, regularized trace.

By  $c_p$  we denote the set of the compact operators  $T$  such that

$$|T|_p = \left( \sum_{k=1}^{\infty} s_k^p(T) \right)^{\frac{1}{p}} < \infty.$$

In particular  $c_1$  is the set of nuclear operators. From the asymptotic of the eigenvalues of  $A$  it follows that  $A$  is not nuclear.

## 2. RESULT

**Theorem.** a) For every  $0 < r < 1$  the following asymptotic formula holds:

$$\lambda_n(A) = \frac{c(\alpha)}{n^\alpha} \left( 1 + o\left(\frac{1}{n^r}\right) \right) \quad \left( c(\alpha) = 2\Gamma(\alpha) \cos \frac{\alpha\pi}{2} \left(\frac{2}{\pi}\right)^\alpha \right).$$

b) The series  $\sum_{n=1}^{\infty} (\lambda_n(A) - \frac{c(\alpha)}{n^\alpha})$  converges and

$$\sum_{n=1}^{\infty} \left( \lambda_n(A) - \frac{c(\alpha)}{n^\alpha} \right) = -4^\alpha \zeta(1 - \alpha)$$

holds, where  $\zeta$  is the Reimann zeta function.

*Proof of a).* For  $0 < \alpha < 1$  define the function

$$(1) \quad H(x, y) = \sum_{n=-\infty}^{\infty} (|x - y + 4n|^{\alpha-1} - |x + y + 4n + 2|^{\alpha-1})$$

and the operator  $B: L^2(-1, 1) \rightarrow L^2(-1, 1)$ ,

$$Bf(x) = \int_{-1}^1 H(x, y)f(y) dy.$$

Let  $\varphi_n(x) = \sin \frac{n\pi(1+x)}{2}$ ,  $n = 1, 2, 3, \dots$ , and let

$$K(\xi) = \int_{-\infty}^{\infty} e^{it\xi} |t|^{\alpha-1} dt \quad \left( = 2\Gamma(\alpha) \cos \frac{\alpha\pi}{2} |\xi|^{-\alpha} \right).$$

Observe that  $\{\varphi_n\}_{n=1}^{\infty}$  is an orthonormal basis of  $L^2(-1, 1)$ . By direct computation one finds that

$$\int_{-1}^1 H(x, y)\varphi_n(y) dy = K\left(\frac{n\pi}{2}\right)\varphi_n(x).$$

Therefore, the operator  $B$  is selfadjoint and positive and

$$\lambda_n(B) = \frac{c(\alpha)}{n^\alpha}.$$

From (1) we obtain

(2)

$$\begin{aligned} |x - y|^{\alpha-1} &= H(x, y) + |x + y + 2|^{\alpha-1} + |x + y - 2|^{\alpha-1} \\ &+ \left( -|x - y - 4|^{\alpha-1} - \sum_{n \neq 0, -1} (|x - y + 4n|^{\alpha-1} - |x + y + 4n + 2|^{\alpha-1}) \right). \end{aligned}$$

By the result of Laptev [7], we obtain

$$s_n \left( \int_{-1}^1 |x + y \pm 2|^{\alpha-1} \cdot dy \right) = O(e^{-c'_0 \sqrt{n}})$$

where  $c'_0 > 0$  is independent of  $n$ . On the other hand, by the Birman-Solomjak theorem [2], we have

$$s_n(W) = O(e^{-c''_0 n})$$

where  $W$  denotes the operator with the kernel

$$-|x - y - 4|^{\alpha-1} - \sum_{n \neq 0, -1} (|x - y + 4n|^{\alpha-1} - |x + y + 4n + 2|^{\alpha-1}),$$

and  $c''_0 > 0$  is independent of  $n$ .

Hence, by the properties of the singular values of the sum of two operators [5], from (2) it follows that

$$A = B + R$$

where

$$(3) \quad s_n(R) \leq c_1 e^{-c_0 \sqrt{n}}.$$

Fix  $r, 0 < r < 1$ . Choose  $\theta$  so that  $r < \theta < 1$  and let

$$k = k(n) = [n^\theta] - 1, \\ m = m(n) = [n^{1-\theta}].$$

Since every natural number  $n$  can be represented as  $n = (k+1)m + j, j = 0, 1, \dots, k$ , we obtain the estimate

$$0 \leq j = j(n) \leq n^\theta + n^{1-\theta}.$$

From the properties of the singular values of the sum two operators it follows that

$$s_{(k+1)m+j}(B + R) \leq s_{km+j}(B) + s_{m+1}(R),$$

i.e.

$$s_n(B + R) - \frac{c(\alpha)}{n^\alpha} \leq s_{km+j}(B) - \frac{c(\alpha)}{n^\alpha} + s_{m+1}(R),$$

i.e.

$$s_n(A) - \frac{c(\alpha)}{n^\alpha} \leq c(\alpha) \left( \frac{1}{(km + j)^\alpha} - \frac{1}{n^\alpha} \right) + s_{m+1}(R).$$

Hence

$$(4) \quad n^{\alpha+r} \left( s_n(A) - \frac{c(\alpha)}{n^\alpha} \right) \leq c(\alpha) n^{\alpha+r} \left( \frac{1}{(km + j)^\alpha} - \frac{1}{n^\alpha} \right) + n^{\alpha+r} s_{m+1}(R).$$

From the definition of the sequences  $k(n), m(n)$  and  $j(n)$  and from (3) it follows that

$$\overline{\lim}_{n \rightarrow \infty} n^{\alpha+r} \left( \frac{1}{(km + j)^\alpha} - \frac{1}{n^\alpha} \right) = 0$$

and

$$\overline{\lim}_{n \rightarrow \infty} n^{\alpha+r} s_{m+1}(R) = 0.$$

Hence, by (4)

$$\overline{\lim}_{n \rightarrow \infty} n^{\alpha+r} \left( s_n(A) - \frac{c(\alpha)}{n^\alpha} \right) \leq 0.$$

Similarly, starting from

$$s_{km+j}(B+R) \geq s_{(k+1)m+j}(B) - s_{m+1}(R),$$

we get

$$\underline{\lim}_{n \rightarrow \infty} n^{\alpha+r} \left( s_n(A) - \frac{c(\alpha)}{n^\alpha} \right) \geq 0;$$

thus  $\lim_{n \rightarrow \infty} n^{\alpha+r} (s_n(A) - \frac{c(\alpha)}{n^\alpha}) = 0$ . Since  $A > 0$  we have  $s_n(A) = \lambda_n(A)$  and therefore

$$\lambda_n(A) = \frac{c(\alpha)}{n^\alpha} \left( 1 + o\left(\frac{1}{n^r}\right) \right).$$

*Remark 1.* The above method does not make it possible to answer whether the following formula holds:

$$\lambda_n(A) = \frac{c(\alpha)}{n^\alpha} \left( 1 + O\left(\frac{1}{n}\right) \right).$$

*Proof of b).* From a) it follows that

$$\lambda_n(A) = \frac{c(\alpha)}{n^\alpha} + o\left(\frac{1}{n^{r+\alpha}}\right),$$

hence, for  $r > 1 - \alpha$ , the series  $\sum_{n=1}^{\infty} (\lambda_n(A) - \frac{c(\alpha)}{n^\alpha})$  converges. Moreover for all  $k = 1, 2, \dots$  the series

$$\sum_{n=1}^{\infty} \left( \lambda_n^k(A) - \left( \frac{c(\alpha)}{n^\alpha} \right)^k \right)$$

is convergent. For the proof of b) we need a lemma.

**Lemma.** *If  $C$  and  $D$  are positive, compact operators on some complex Hilbert space such that  $|C - D|^{\operatorname{Re} z} \in c_1$  ( $0 < \operatorname{Re} z < 1$ ), then  $C^z - D^z \in c_1$  and*

$$|C^z - D^z|_1 \leq \frac{|\sin \pi z|}{\sin(\pi \cdot \operatorname{Re} z)} \| |C - D|^{\operatorname{Re} z} \|_1.$$

*Proof.* If  $|C - D|^{\operatorname{Re} z} \in c_1$  for some  $z$ ,  $0 \leq \operatorname{Re} z < 1$ , then  $|C - D| \in c_1$  and hence  $C(I + tC)^{-1} - D(I + tD)^{-1} \in c_1$  for all  $t > 0$  ( $I$  is the identity operator). Since, for  $0 < \operatorname{Re} z < 1$ ,

$$C^z - D^z = \frac{\sin \pi z}{\pi} \int_0^\infty t^{-z} (C(I + tC)^{-1} - D(I + tD)^{-1}) dt$$

and since the function  $s \mapsto \frac{s}{1+ts}$  is operator monotone, according to a result of T. Ando [1], we have

$$\begin{aligned} |C^z - D^z|_1 &\leq \frac{|\sin \pi z|}{\pi} \int_0^\infty t^{-\operatorname{Re} z} |C(I+tC)^{-1} - D(I+tD)^{-1}|_1 dt \\ &\leq \frac{|\sin \pi z|}{\pi} \int_0^\infty t^{-\operatorname{Re} z} \|C - D\| (I+t|C - D|)^{-1}|_1 dt \\ &= \frac{|\sin \pi z|}{\pi} \sum_{k=1}^\infty s_k^{\operatorname{Re} z} (C - D) \cdot \int_0^\infty \frac{t^{-\operatorname{Re} z}}{1+t} dt \\ &= \frac{|\sin \pi z|}{\sin(\pi \operatorname{Re} z)} \|C - D\|^{\operatorname{Re} z}|_1. \end{aligned}$$

Applying the lemma to the operators  $C = A$ ,  $D = B$  we obtain from (3) that  $|A - B|^{\operatorname{Re} z} \in c_1$  for every  $z$ ,  $0 < \operatorname{Re} z < 1$ , and hence  $A^z - B^z \in c_1$ . If  $0 < \operatorname{Re} z < 2$ , then applying the lemma to  $C = A^2$ ,  $D = B^2$  we obtain  $(A^2)^{\frac{z}{2}} - (B^2)^{\frac{z}{2}} \in c_1$ , etc. Therefore  $A^z - B^z \in c_1$  for all  $z$  with  $\operatorname{Re} z > 0$ . In a similar way one shows that the function

$$z \mapsto \operatorname{tr}(A^z - B^z)$$

is analytic for  $\operatorname{Re} z > 0$ .

Consider the function

$$\Psi(z) = \sum_{n=1}^\infty \left( \lambda_n^z(A) - \left( \frac{c(\alpha)}{n^\alpha} \right)^z \right).$$

From the asymptotic of  $\lambda_n(A)$  it follows that  $\Psi(z)$  is analytic for  $\operatorname{Re} z > 0$ . If  $\operatorname{Re} z > \frac{1}{\alpha}$ , then both operators  $A^z$  and  $B^z$  are nuclear and hence

$$\operatorname{tr}(A^z - B^z) = \operatorname{tr} A^z - \operatorname{tr} B^z = \Psi(z).$$

Hence  $\Psi(z) = \operatorname{tr}(A^z - B^z)$  for  $\operatorname{Re} z > \frac{1}{\alpha}$ . It follows that  $\Psi(z) = \operatorname{tr}(A^z - B^z)$  for all  $z$ ,  $\operatorname{Re} z > 0$ , and hence

$$\sum_{n=1}^\infty \left( \lambda_n^k(A) - \left( \frac{c(\alpha)}{n^\alpha} \right)^k \right) = \operatorname{tr}(A^k - B^k), \quad k = 1, 2, \dots$$

It is especially simple to find the “first regularized” trace of  $A$ :

$$\sum_{n=1}^\infty \left( \lambda_n(A) - \frac{c(\alpha)}{n^\alpha} \right) = \operatorname{tr}(A - B) = \operatorname{tr} R.$$

The integral operator, on  $L^2(-1, 1)$ , with the continuous kernel

$$-|x - y - 4|^{\alpha-1} - \sum_{n \neq 0, -1} (|x - y + 4n|^{\alpha-1} - |x + y + 4n + 2|^{\alpha-1})$$

is nuclear and its trace is equal [5] to

$$\int_{-1}^1 \left[ -4^{\alpha-1} - \sum_{n \neq 0, -1} (|4n|^{\alpha-1} - |2x + 4n + 2|^{\alpha-1}) \right] dx.$$

The operators  $\int_{-1}^1 |x+y+2|^{\alpha-1} \cdot dy$  and  $\int_{-1}^1 |x+y-2|^{\alpha-1} \cdot dy$  are nuclear and, by [4], Theorem 3.1, it follows that

$$\begin{aligned} \operatorname{tr} \int_{-1}^1 |x+y+2|^{\alpha-1} \cdot dy &= \int_{-1}^1 |2x+2|^{\alpha-1} dx = \frac{4^\alpha}{2\alpha}, \\ \operatorname{tr} \int_{-1}^1 |x+y-2|^{\alpha-1} \cdot dy &= \int_{-1}^1 |2x-2|^{\alpha-1} dx = \frac{4^\alpha}{2\alpha}. \end{aligned}$$

Hence

$$\operatorname{tr} R = 2 \cdot \frac{1}{2\alpha} \cdot 4^\alpha + \int_{-1}^1 \left[ -4^{\alpha-1} - \sum_{n \neq 0, -1} (|4n|^{\alpha-1} - |2x+4n+2|^{\alpha-1}) \right] dx.$$

After simplification we get

$$\operatorname{tr} R = -4^\alpha \zeta(1-\alpha).$$

This completes the proof of the theorem.

*Remark 2.* A similar result can be obtained for the operator  $L: L^2(-1,1) \rightarrow L^2(-1,1)$  defined by

$$Lf(x) = -\frac{1}{\pi} \int_{-1}^1 \ln|x-y|f(y) dy.$$

It is known [8] that  $L > 0$  and  $\lambda_n(L) = \frac{2}{n\pi} (1 + O(\frac{\sqrt{\ln n}}{n^{\frac{1}{5}}}))$ . Starting from the function  $k(x) = \frac{1}{\pi} K_0(|x|)$  ( $K_0$  is the McDonald function), one can prove the formula

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \lambda_n(L) - \frac{1}{\sqrt{1 + (\frac{n\pi}{2})^2}} \right) \\ = -\frac{2}{\pi} (C + \ln 2) - \frac{2\sqrt{2}}{\pi} \int_0^{\infty} t^{-\frac{1}{2}} \left( 1 + \frac{t}{2} \right)^{-\frac{1}{2}} \frac{dt}{e^{4(1+t)} - 1} \end{aligned}$$

where  $C$  is the Euler constant.

*Remark 3.* By the same method one can find the “regularized” trace of the convolution operator with kernel of the form

$$k(x) = |x|^{\alpha-1} L\left(\frac{1}{|x|}\right) \quad (0 < \alpha < 1)$$

where  $L$  is a smooth function such that

$$x \mapsto \frac{xL'(x)}{L(x)}$$

decreases for  $x$  large enough and

$$\lim_{x \rightarrow \infty} \frac{xL'(x)}{L(x)} = 0.$$

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