

NORMS OF EMBEDDINGS OF LOGARITHMIC BESSEL POTENTIAL SPACES

DAVID E. EDMUNDS, PETR GURKA, AND BOHUMÍR OPIČ

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ABSTRACT. Let Ω be a subset of \mathbb{R}^n with finite volume, let $\nu > 0$ and let Φ be a Young function with $\Phi(t) = \exp(\exp t^\nu)$ for large t . We show that the norm on the Orlicz space $L_\Phi(\Omega)$ is equivalent to

$$\sup_{1 < q < \infty} (e + \log q)^{-1/\nu} \|f\|_{L^q(\Omega)}.$$

We also obtain estimates of the norms of the embeddings of certain logarithmic Bessel potential spaces in $L^q(\Omega)$ which are sharp in their dependences on q provided that q is large enough.

1. INTRODUCTION

In a series of recent papers ([EGO II], [EGO III], [EGO IV], [EOP]) an intensive study has been made of the logarithmic Bessel potential spaces

$$H^\sigma L_{p,q;\alpha,\beta}(\mathbb{R}^n) := \{u = g_\sigma * f; f \in L_{p,q;\alpha,\beta}(\mathbb{R}^n)\}.$$

Here $\sigma > 0$, $p \in (1, \infty)$, $q \in [1, \infty]$, and $\alpha, \beta \in \mathbb{R}$; g_σ is the usual Bessel kernel and $L_{p,q;\alpha,\beta}(\mathbb{R}^n)$ is the generalized Lorentz-Zygmund (GLZ) space of all functions f on \mathbb{R}^n such that

$$\|t^{1/p-1/q}(e + |\log t|)^\alpha \log^\beta(e + |\log t|)f^*(t)\|_{L^q(0,\infty)} < \infty,$$

f^* being the non-increasing rearrangement of f . (In Section 2 we shall see that GLZ spaces include many familiar objects including Lebesgue, Lorentz, Lorentz-Zygmund, and Zygmund spaces.) A good deal of this work was motivated by the need for embedding theorems more delicate than the classical ones and involving scales of spaces which could be more finely tuned than the Lebesgue scale. We refer to [ET] for applications of related ideas to the distribution of eigenvalues of degenerate elliptic operators.

One of the theorems obtained in [EGO II] (see also [EGO IV]) concerned the embedding of certain GLZ spaces in Orlicz spaces of double exponential type. To explain this, suppose that Ω is a subset of \mathbb{R}^n with finite volume, let $\nu > 0$

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and denote by $EE_\nu(\Omega)$ the Orlicz space $L_\Phi(\Omega)$ with Young function Φ given by $\Phi(t) = \exp(\exp t^\nu)$ for large t . It turns out that when $0 < \sigma < n$ and $\beta < 1/p'$, then $H^\sigma L_{n/\sigma,p;1/p',\beta}(\mathbb{R}^n)$ is continuously embedded in $EE_\nu(\Omega)$, where $1/\nu = 1/p' - \beta$. Another specialization of the general results of [EGO I, EGO II, EGO IV] gives analogous results concerning embeddings into Orlicz space of single exponential type, linking up with the celebrated embedding theorem of Trudinger [T] and Strichartz [S] and the more recent one due to Fusco, Lions and Sbordone [FLS].

The main results in the present paper are:

(a) We show that when $\nu > 0$ and $\Omega \subset \mathbb{R}^n$ has finite volume, then a norm on $EE_\nu(\Omega)$ equivalent to the usual Orlicz norm is

$$\sup_{1 < q < \infty} (e + \log q)^{-1/\nu} \|f\|_{L^q(\Omega)}.$$

This reduces the derivation of estimates in $EE_\nu(\Omega)$ to that of $L^q(\Omega)$ -estimates. The result should be compared with that in Orlicz spaces of single exponential type (see, for example, [Tr]), in which $(e + \log q)$ is replaced by q .

(b) We obtain estimates for the norms of the embeddings of

$$H^\sigma L_{n/\sigma,p;1/p',\beta}(\mathbb{R}^n) \quad \text{and} \quad H^\sigma L_{n/\sigma,p;\beta}(\mathbb{R}^n) \quad \text{in} \quad L^q(\Omega)$$

(when Ω has finite volume and $\beta < 1/p'$) which are sharp in their dependence on q for large values of q .

The analogues of (a) and (b) in the context of single exponential spaces have proved to be very useful in the estimation of the asymptotic behaviour of eigenvalues of degenerate elliptic operators by means of entropy number estimates (see [ET]). We anticipate that our results will lead to similar developments in which spaces of double exponential type play a prominent role.

2. NOTATION AND PRELIMINARIES

Let Ω be a measurable subset of \mathbb{R}^n (with respect to n -dimensional Lebesgue measure); by $|\Omega|_n$ we mean its n -volume; χ_Ω will represent the characteristic function of Ω .

For a measurable (real or complex) function f on Ω , its distribution function μ_f is given by

$$(2.1) \quad \mu_f(\lambda) = \mu_{f,\Omega}(\lambda) := |\{x \in \Omega; |f(x)| > \lambda\}|_n, \quad \lambda > 0,$$

and the non-increasing rearrangement f^* of f is defined by

$$(2.2) \quad f^*(t) = f_\Omega^*(t) := \inf\{\lambda > 0; \mu_{f,\Omega}(\lambda) \leq t\}, \quad t > 0.$$

If $p, q \in (0, \infty]$ and $\alpha, \beta \in \mathbb{R}$, the *generalized Lorentz-Zygmund space* $L_{p,q;\alpha,\beta}(\Omega)$ consists of all functions f on Ω such that

$$(2.3) \quad \begin{aligned} \|f\|_{p,q;\alpha,\beta} &= \|f\|_{p,q;\alpha,\beta;\Omega} \\ &= \|t^{1/p-1/q}(e + |\log t|)^\alpha \log^\beta(e + |\log t|) f_\Omega^*(t)\|_{q;(0,\infty)} < \infty, \end{aligned}$$

where $\|\cdot\|_{q;(0,\infty)}$ stands for the L^q -(quasi-) norm on $(0, \infty)$.

When $\alpha = \beta = 0$, the space $L_{p,q;\alpha,\beta}(\Omega)$ coincides with the classical Lorentz space $L^{p,q}(\Omega)$, which is just $L^p(\Omega)$ when $q = p$; $L_{p,q;\alpha}(\Omega) := L_{p,q;\alpha,0}(\Omega)$ is the Lorentz-Zygmund space $L^{p,q}(\log L)^\alpha(\Omega)$ introduced in [BR] and which, when $p = q$, is the Zygmund space $L^p(\log L)^\alpha(\Omega)$. If $p = q < \infty$ and $|\Omega|_n < \infty$, then $L_{p,q;\alpha,\beta}(\Omega)$ equals

$$L^p(\log L)^\alpha(\log \log L)^\beta(\Omega) = \left\{ f; \int_\Omega |f|^p \log^{\alpha p}(e + |f|) \cdot [\log \log(2e + |f|)]^{\beta p} < \infty \right\}.$$

These spaces were studied in [EGO II] and in [EOP], where the properties mentioned above can be found together with some other information.

The *Bessel kernel* g_σ ($\sigma > 0$) is defined to be that function on \mathbb{R}^n whose Fourier transform is

$$(2.4) \quad \hat{g}_\sigma(x) = (2\pi)^{-n/2}(1 + |x|^2)^{-\sigma/2},$$

where by the Fourier transform \hat{f} of a function f we mean

$$\hat{f}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot y} f(y) dy.$$

It is known that g_σ is a positive, integrable function which is analytic except at the origin (cf. [AS] or [Z]). We can now define the Bessel potential space with which we shall be concerned in this paper.

Let $\sigma > 0$, $p \in (1, \infty)$, $q \in [1, \infty]$, and $\alpha, \beta \in \mathbb{R}$. The *logarithmic Bessel potential space* $H^\sigma L_{p,q;\alpha,\beta}(\mathbb{R}^n)$ is defined by

$$(2.5) \quad H^\sigma L_{p,q;\alpha,\beta}(\mathbb{R}^n) := \{u = g_\sigma * f; f \in L_{p,q;\alpha,\beta}(\mathbb{R}^n)\},$$

and is equipped with the (quasi-) norm

$$(2.6) \quad \|u\|_{\sigma;p,q;\alpha,\beta} := \|f\|_{p,q;\alpha,\beta}.$$

Note that $H^\sigma L_{p,p;0,0}(\mathbb{R}^n)$ is simply the (fractional) Sobolev space of order σ .

When $k \in \mathbb{N}$, $p, q \in (1, \infty)$ and $\alpha, \beta \in \mathbb{R}$, then, by [EGO IV, Theorem 4.2], the space $H^k L_{p,q;\alpha,\beta}(\mathbb{R}^n)$ is equal to the *logarithmic Sobolev space*

$$(2.7) \quad W^k L_{p,q;\alpha,\beta}(\mathbb{R}^n) := \{u; D^\alpha u \in L_{p,q;\alpha,\beta}(\mathbb{R}^n) \text{ if } |\alpha| \leq k\},$$

equipped with the (quasi-) norm

$$(2.8) \quad \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p,q;\alpha,\beta},$$

and the corresponding (quasi-) norms are equivalent.

By a *Young function* Φ we shall mean a continuous, non-negative, strictly increasing, convex function on $[0, \infty)$ such that

$$(2.9) \quad \lim_{t \rightarrow 0^+} \frac{\Phi(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{\Phi(t)} = 0.$$

Given a Young function Φ and any measurable subset Ω of \mathbb{R}^n , $L_\Phi(\Omega)$ will denote the corresponding Orlicz space, equipped with the Luxemburg norm $\|\cdot\|_{\Phi;\Omega}$; for details of such spaces we refer to [A], [KJF], [BS].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $|\Omega|_n < \infty$ and let $\nu > 0$. Then:*

(i) *The space $L^\infty(\log L)^{-1/\nu}(\Omega) = L_{\infty,\infty;-1/\nu}(\Omega)$ coincides with the Orlicz space $L_\Phi(\Omega)$, where $\Phi(t) = \exp t^\nu$ for all $t \geq t_0$ with some $t_0 \in (0, \infty)$, and the corresponding (quasi-) norms are equivalent.*

(ii) *The space $L^\infty(\log \log L)^{-1/\nu}(\Omega) = L_{\infty,\infty;0,-1/\nu}(\Omega)$ coincides with the Orlicz space $L_\Phi(\Omega)$, where $\Phi(t) = \exp \exp t^\nu$ for all $t \geq t_0$ with some $t_0 \in (0, \infty)$, and the corresponding (quasi-) norms are equivalent.*

Proof. The proof of (i) can be found in [BR, Theorem D]; statement (ii) can be proved similarly (cf. [EGO I, Lemma 3.10]). The assertion about equivalent norms easily follows from [BS, Chapter 1, Thm. 1.8] and from the fact that all the spaces are (equivalent to) Banach function spaces. □

The lemma leads us to the following notation.

Notation 2.2. Let $\Omega \subset \mathbb{R}^n$, $|\Omega|_n < \infty$ and $\nu > 0$. Then we put

$$(2.10) \quad \begin{aligned} E_\nu(\Omega) &:= L^\infty(\log L)^{-1/\nu}(\Omega) = L_{\infty,\infty;-1/\nu}(\Omega), \\ \|\cdot\|_{E_\nu(\Omega)} &:= \|\cdot\|_{\infty,\infty;-1/\nu;\Omega}; \end{aligned}$$

$$(2.11) \quad \begin{aligned} EE_\nu(\Omega) &:= L^\infty(\log \log L)^{-1/\nu}(\Omega) = L_{\infty,\infty;0,-1/\nu}(\Omega), \\ \|\cdot\|_{EE_\nu(\Omega)} &:= \|\cdot\|_{\infty,\infty;0,-1/\nu;\Omega}. \end{aligned}$$

For non-negative expressions (i.e. functions or functionals) F_1, F_2 we use the symbols $F_1 \lesssim F_2$ and $F_1 \gtrsim F_2$, respectively, provided that $F_1 \leq CF_2$ and $CF_1 \geq F_2$, with some constant $C \in (0, \infty)$ independent of the variables in the expressions F_1, F_2 . If $F_1 \lesssim F_2$ and $F_1 \gtrsim F_2$, we write $F_1 \approx F_2$.

3. EQUIVALENT NORMS IN EXPONENTIAL SPACES

Theorem 3.1. *Let $\nu > 0$, and let $\Omega \subset \mathbb{R}^n$ be a domain with $|\Omega|_n < \infty$. Then*

$$(3.1) \quad \|f\|_{E_\nu(\Omega)} \approx \sup_{q \in (1, \infty)} q^{-1/\nu} \|f\|_{q;\Omega} \quad \text{for all } f \in E_\nu(\Omega)$$

and

$$(3.2) \quad \|f\|_{EE_\nu(\Omega)} \approx \sup_{q \in (1, \infty)} (e + \log q)^{-1/\nu} \|f\|_{q;\Omega} \quad \text{for all } f \in EE_\nu(\Omega).$$

Proof. We prove only (3.2); for the proof of (3.1) see e.g. [Tr]. The proof will be given in two steps.

STEP 1. We show that

$$(3.3) \quad \|f\|_{EE_\nu(\Omega)} \approx \sup_{\sigma \in (0, 1)} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}, \infty; \Omega}.$$

(Recall that $\|\cdot\|_{\frac{1}{\sigma}, \infty; \Omega}$ denotes the (quasi-) norm in the Lorentz (Marcinkiewicz) $L^{\frac{1}{\sigma}, \infty}(\Omega)$ -space, i.e. $\|f\|_{1/\sigma, \infty; \Omega} = \sup_{s \in (0, |\Omega|_n)} s^\sigma f_\Omega^*(s)$.)

First we show the relation “ \gtrsim ”. Suppose that $f \in EE_\nu(\Omega)$, i.e.

$$\sup_{s \in (0, |\Omega|_n)} \log^{-1/\nu}(e + |\log s|) f_\Omega^*(s) < \infty.$$

It is plain that, for $\sigma \in (0, 1)$,

$$(e - \log \sigma)^{-1/\nu} \approx \left(\frac{2}{\nu} - \log \sigma\right)^{-1/\nu},$$

and thus

$$(3.4) \quad \begin{aligned} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}, \infty; \Omega} &\approx \left(\frac{2}{\nu} - \log \sigma\right)^{-1/\nu} \sup_{s \in (0, |\Omega|_n)} s^\sigma f_\Omega^*(s) \\ &= \sup_{s \in (0, |\Omega|_n)} \left(\frac{2}{\nu} - \log \sigma\right)^{-1/\nu} s^\sigma f_\Omega^*(s). \end{aligned}$$

We prove the estimate

$$(3.5) \quad \left(\frac{2}{\nu} - \log \sigma\right)^{-1/\nu} s^\sigma \leq C \log^{-1/\nu}(e + |\log s|)$$

with a constant $C > 0$ independent of $\sigma \in (0, 1)$ and $s \in (0, |\Omega|_n)$.

If $|\Omega|_n > 1$, then, for any $s \in [1, |\Omega|_n)$ and $\sigma \in (0, 1)$,

$$(3.6) \quad \left(\frac{2}{\nu} - \log \sigma\right)^{-1/\nu} s^\sigma \leq \left(\frac{2}{\nu}\right)^{-1/\nu} |\Omega|_n \leq C \log^{-1/\nu}(e + |\log s|)$$

with $C > 0$ independent of σ and s . So we can assume that $|\Omega|_n \leq 1$.

Let $s \in (0, 1)$ be fixed. For $\tau \in (0, -\log s)$ define

$$F_s(\tau) = \left(\frac{2}{\nu} + \log(-\log s) - \log \tau\right)^{-1/\nu} e^{-\tau}.$$

Then

$$(3.7) \quad F_s(\tau) \leq C \log^{-1/\nu}(e - \log s)$$

with C independent of τ and s . Indeed, if $0 < \tau \leq \frac{1}{2}$, then (3.7) holds since $(-\log \tau) > 0$, $-\frac{1}{\nu} < 0$ and $e^{-\tau} < 1$. If $\frac{1}{2} < \tau < -\log s$, then the first derivative

$$\frac{dF_s}{d\tau}(\tau) = \frac{e^{-\tau}}{\tau} \left(\frac{2}{\nu} + \log(-\log s) - \log \tau\right)^{-\frac{1}{\nu}-1} \left[\frac{1}{\nu} - \tau \left(\frac{2}{\nu} + \log(-\log s) - \log \tau\right)\right]$$

is negative, since

$$\frac{1}{\nu} - \tau \left(\frac{2}{\nu} + \log(-\log s) - \log \tau\right) < \frac{1}{\nu} - \tau \cdot \frac{2}{\nu} < 0.$$

Hence, the function F_s is decreasing on the interval $[\frac{1}{2}, -\log s)$, which implies that for any $\tau \in (\frac{1}{2}, -\log s)$,

$$F_s(\tau) \leq F_s\left(\frac{1}{2}\right) = \left(\frac{2}{\nu} + \log(-\log s) + \log 2\right)^{-1/\nu} e^{-\frac{1}{2}} \leq C \log^{-1/\nu}(e - \log s)$$

with C independent of τ and s , and (3.7) is proved.

Now, putting $\tau = \sigma(-\log s)$ with $\sigma \in (0, 1)$, we obtain from (3.7) that

$$\begin{aligned} C \log^{-1/\nu}(e - \log s) &\geq F_s(\sigma(-\log s)) \\ &= \left(\frac{2}{\nu} + \log(-\log s) - \log(\sigma(-\log s))\right)^{-1/\nu} e^{\sigma \log s} = \left(\frac{2}{\nu} - \log \sigma\right)^{-1/\nu} s^\sigma. \end{aligned}$$

Together with (3.6), this verifies (3.5). The desired inequality follows from (3.4) and (3.5).

We next prove the relation “ \lesssim ” in (3.3). Suppose that

$$\sup_{\sigma \in (0,1)} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}, \infty; \Omega} =: A < \infty.$$

Fix $s \in (0, |\Omega|_n)$ and set $\sigma = 1/(e + |\log s|)$. Then $\sigma \in (0, 1)$, and we have, for any $s \in (0, |\Omega|_n)$,

$$\begin{aligned} A &\geq (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}, \infty; \Omega} \geq (e - \log \sigma)^{-1/\nu} s^\sigma f_\Omega^*(s) \\ &= (e + \log(e + |\log s|))^{-1/\nu} \exp\left(\frac{\log s}{e + |\log s|}\right) f_\Omega^*(s) \\ &\geq C \log^{-1/\nu}(e + |\log s|) f_\Omega^*(s) \end{aligned}$$

with a constant C independent of s . Passing to the supremum over all $s \in (0, |\Omega|_n)$, we obtain the desired estimate, and (3.3) is proved.

STEP 2. First we prove that

$$(3.8) \quad \sup_{\sigma \in (0,1)} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}, \infty; \Omega} \approx \sup_{\sigma \in (0,1)} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}; \Omega}$$

(recall that $\|\cdot\|_{p; \Omega}$ stands for the norm in the space $L^p(\Omega)$).

Assuming that

$$\sup_{s \in (0,1)} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}; \Omega} =: B < \infty$$

and using the well-known inequality

$$\|f\|_{\frac{1}{\sigma}, \infty; \Omega} \leq \|f\|_{\frac{1}{\sigma}; \Omega},$$

we immediately obtain

$$\sup_{\sigma \in (0,1)} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}, \infty; \Omega} \leq B.$$

We prove the converse inequality. If $0 < \sigma_1 < \sigma$, then

$$\begin{aligned} (3.9) \quad \|f\|_{\frac{1}{\sigma}; \Omega} &= \left(\int_0^{|\Omega|_n} (f_{\Omega}^*(t))^{\frac{1}{\sigma}} dt \right)^{\sigma} = \left(\int_0^{|\Omega|_n} t^{1-\frac{\sigma_1}{\sigma}} (t^{\sigma_1} f_{\Omega}^*(t))^{\frac{1}{\sigma}} \frac{dt}{t} \right)^{\sigma} \\ &\leq \left(1 - \frac{\sigma_1}{\sigma} \right)^{-\sigma} |\Omega|_n^{\sigma-\sigma_1} \|f\|_{\frac{1}{\sigma_1}, \infty; \Omega}^{\sigma}. \end{aligned}$$

The estimate

$$(e - \log \sigma)^{-1/\nu} \approx (e + \log 2 - \log \sigma)^{-1/\nu} = \left(e - \log \frac{\sigma}{2} \right)^{-1/\nu}, \quad \sigma \in (0, 1),$$

and (3.9) with $\sigma_1 = \sigma/2$ imply

$$(e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}; \Omega} \leq C (e - \log \sigma_1)^{-1/\nu} \|f\|_{\frac{1}{\sigma_1}, \infty; \Omega}, \quad \sigma \in (0, 1),$$

where C is independent of f . Consequently,

$$\begin{aligned} \sup_{\sigma \in (0,1)} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}; \Omega} &\lesssim \sup_{\sigma_1 \in (0, \frac{1}{2})} (e - \log \sigma_1)^{-1/\nu} \|f\|_{\frac{1}{\sigma_1}, \infty; \Omega} \\ &\lesssim \sup_{\sigma \in (0,1)} (e - \log \sigma)^{-1/\nu} \|f\|_{\frac{1}{\sigma}, \infty; \Omega}, \end{aligned}$$

and (3.8) is proved.

Now, (3.2) easily follows from (3.3) and (3.8) on putting $q = 1/\sigma$. □

Corollary 3.2. *Let $|\Omega|_n < \infty$ and $\nu > 0$.*

(i) *If $j_0 \in \mathbb{N}$ and $q_0 \geq 1$, then, for all $f \in E_{\nu}(\Omega)$,*

$$\|f\|_{E_{\nu}(\Omega)} \approx \sup_{j \in \mathbb{N}, j \geq j_0} j^{-1/\nu} \|f\|_{j; \Omega} \approx \sup_{q \geq q_0} q^{-1/\nu} \|f\|_{q; \Omega}.$$

(ii) *If $j_0 \in \mathbb{N}$, $j_0 \geq 2$ and $q_0 > 1$, then, for all $f \in EE_{\nu}(\Omega)$,*

$$(3.10) \quad \|f\|_{EE_{\nu}(\Omega)} \approx \sup_{j \in \mathbb{N}, j \geq j_0} (\log j)^{-1/\nu} \|f\|_{j; \Omega} \approx \sup_{q \geq q_0} (\log q)^{-1/\nu} \|f\|_{q; \Omega}.$$

Proof. We prove only (ii); the proof of (i) is analogous.

Let $q_0 > 1$ and $j_0 \geq 2$ be fixed. For $f \in EE_\nu(\Omega)$ we denote

$$\begin{aligned} S_1(f) &= \sup_{q \in (1, \infty)} (e + \log q)^{-1/\nu} \|f\|_{q; \Omega}, \\ S_2(f) &= \sup_{q \in (1, \infty)} (e + \log q)^{-1/\nu} |\Omega|_n^{-1/q} \|f\|_{q; \Omega}, \\ S_3(f) &= \sup_{q \geq q_0} (\log q)^{-1/\nu} \|f\|_{q; \Omega}, \\ \mathfrak{S}_1(f) &= \sup_{j \in \mathbb{N}, j \geq j_0} (\log j)^{-1/\nu} \|f\|_{j; \Omega}, \\ \mathfrak{S}_2(f) &= \sup_{j \in \mathbb{N}, j \geq j_0} (\log j)^{-1/\nu} |\Omega|_n^{-1/j} \|f\|_{j; \Omega}, \\ \mathfrak{S}_3(f) &= \sup_{j \in \mathbb{N}, j \geq [q_0] + 1} (\log j)^{-1/\nu} \|f\|_{j; \Omega}. \end{aligned}$$

If $q \in (1, \infty)$, we put $j = \max\{j_0, [q] + 1\}$ (where $[q]$ denotes the integer part of q) and choose $k \in \mathbb{N}$, $k \geq 2$, satisfying $2^k \geq j_0$. Then

$$j \leq j_0([q] + 1) \leq j_0(q + 1) \leq (4q)^k \leq (e^e q)^k$$

and hence

$$(3.11) \quad (\log j)^{-1/\nu} \geq k^{-1/\nu} (e + \log q)^{-1/\nu}.$$

Moreover, by the Hölder inequality, for any $f \in EE_\nu(\Omega)$,

$$|\Omega|_n^{-1/q} \|f\|_{q; \Omega} \leq |\Omega|_n^{-1/j} \|f\|_{j; \Omega},$$

which together with (3.11) implies that

$$(3.12) \quad S_2(f) \leq k^{1/\nu} \mathfrak{S}_2(f).$$

Further, it is obvious that, for any $f \in EE_\nu(\Omega)$,

$$S_1(f) \approx S_2(f), \quad \mathfrak{S}_1(f) \approx \mathfrak{S}_2(f),$$

and since $(e + \log j) \approx \log j$, $j \geq j_0$, we have

$$\mathfrak{S}_1(f) \lesssim S_1(f).$$

Together with (3.12) these estimates imply

$$(3.13) \quad \mathfrak{S}_1(f) \lesssim S_1(f) \approx S_2(f) \lesssim \mathfrak{S}_2(f) \approx \mathfrak{S}_1(f) \quad \text{for all } f \in EE_\nu(\Omega).$$

The first relation in (3.10) follows from Theorem 3.1 and (3.13). Moreover, by Theorem 3.1, $\mathfrak{S}_3(f) \lesssim \|f\|_{EE_\nu(\Omega)}$. Since $\mathfrak{S}_3(f) = \mathfrak{S}_1(f)$ if $j_0 := [q_0] + 1$, we have from the first relation in (3.10) that $\|f\|_{EE_\nu(\Omega)} \lesssim \mathfrak{S}_3(f)$, and the result follows. \square

4. NORMS OF EMBEDDINGS

Given two (quasi-) Banach spaces X and Y , we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding $\text{id} : X \rightarrow Y$ is continuous. The norm of the embedding is

$$\|\text{id}\| = \|\text{id}\|_{X \rightarrow Y} = \sup_{\|f\|_X \leq 1} \|f\|_Y.$$

In this section we shall consider embeddings of Bessel potential spaces (modelled on generalized Lorentz-Zygmund spaces) into L^q -spaces. Our aim will be to establish the dependence of norms of these embeddings on q .

Theorem 4.1. *Let $0 < \sigma < n$, $1 \leq p \leq \infty$ and $\beta < 1/p'$ (where $1/p' = 1 - 1/p$). Suppose that $\Omega \subset \mathbb{R}^n$ has non-empty interior and $|\Omega|_n < \infty$. Denote*

$$X_1 = H^\sigma L^{\frac{n}{\sigma}, p; \beta}(\mathbb{R}^n), \quad X_2 = H^\sigma L^{\frac{n}{\sigma}, p; \frac{1}{p'}, \beta}(\mathbb{R}^n).$$

Then

$$(4.1) \quad \|\text{id}\|_{X_1 \rightarrow L^q(\Omega)} \lesssim q^{1/p' - \beta}, \quad 1 < q < \infty,$$

$$(4.2) \quad \|\text{id}\|_{X_2 \rightarrow L^q(\Omega)} \lesssim (e + \log q)^{1/p' - \beta}, \quad 1 < q < \infty.$$

Moreover, there is $q_0 \in (1, \infty)$ such that

$$(4.1^*) \quad \|\text{id}\|_{X_1 \rightarrow L^q(\Omega)} \approx q^{1/p' - \beta} \quad \text{for all } q > q_0,$$

$$(4.2^*) \quad \|\text{id}\|_{X_2 \rightarrow L^q(\Omega)} \approx (\log q)^{1/p' - \beta} \quad \text{for all } q > q_0.$$

Proof. We prove only (4.2) and (4.2*); the proof of (4.1) and (4.1*) is similar and is left to the reader.

Let the assumptions of Theorem 4.1 be satisfied. Then, by [EGO II, Theorem 5.2],

$$H^\sigma L^{\frac{n}{\sigma}, p; \frac{1}{p'}, \beta}(\mathbb{R}^n) \hookrightarrow EE_\alpha(\Omega),$$

where $1/\alpha = 1/p' - \beta > 0$, i.e.,

$$(4.3) \quad \|f\|_{EE_\alpha(\Omega)} \leq C < \infty$$

for any $f \in X_2$ such that $\|f\|_{X_2} \leq 1$, with a constant C independent of f . By (4.3) and (3.2), for any $q \in (1, \infty)$,

$$(e + \log q)^{-1/\alpha} \|f\|_{q; \Omega} \leq C,$$

i.e.

$$\|f\|_{q; \Omega} \leq C(e + \log q)^{1/p' - \beta}.$$

Passing to the supremum over all $f \in X_2$ with $\|f\|_{X_2} \leq 1$, we get (4.2).

Without loss of generality we can assume that the ball $B_1 = \{x \in \mathbb{R}^n; |x| < 1\} \subset \Omega$. By [EGO III, Lemma 4.3 and Remark 4.5] there exist $r_1 \in (0, 1)$ and a function f_r , $r \in (0, r_1)$, satisfying

$$\text{supp } f_r \subset B_1,$$

$$(4.4) \quad |(g_\sigma * f_r)(x)| \geq C \log^{1/p' - \beta}(e - \log r) \quad \text{for all } |x| < r$$

(with $C \in (0, \infty)$ independent of r), and

$$(4.5) \quad \|f_r\|_{\frac{n}{\sigma}, p; \frac{1}{p'}, \beta} \leq 1.$$

Choose $q_0 \in (1, \infty)$ such that $\exp(e - q_0) \leq r_1$. If $q \in (q_0, \infty)$, set $r = \exp(e - q)$. Obviously, $r \in (0, r_1)$, and by (4.5) and (4.4) the function $F_q := g_\sigma * f_r$ satisfies,

$$F_q \in H^\sigma L^{\frac{n}{\sigma}, p; \frac{1}{p'}, \beta}(\mathbb{R}^n), \quad \|F_q\|_{\sigma; \frac{n}{\sigma}, p; \frac{1}{p'}, \beta} \leq 1,$$

$$\|F_q\|_{q; \Omega} \geq \left(\int_{\{|x| < r\}} |F_q(x)|^q dx \right)^{1/q} \gtrsim (\log q)^{1/p' - \beta},$$

and (4.2*) follows. □

Remark 4.1. If we take $\sigma = 1$, $p = n$ and $\beta = 0$, we obtain from Theorem 4.1 that for all large q ,

$$\|\text{id}\|_{W^{1,n}(\mathbb{R}^n) \rightarrow L^q(\Omega)} \approx q^{1/n'},$$

$$\|\text{id}\|_{W^1 L^n(\log L)^{1/n'}(\mathbb{R}^n) \rightarrow L^q(\Omega)} \approx (\log q)^{1/n'}.$$

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CENTRE FOR MATHEMATICAL ANALYSIS AND ITS APPLICATIONS, UNIVERSITY OF SUSSEX,
FALMER, BRIGHTON BN1 9QH, ENGLAND

E-mail address: d.e.edmunds@sussex.ac.uk

DEPARTMENT OF MATHEMATICS, CZECH UNIVERSITY OF AGRICULTURE, 165 21 PRAGUE 6,
CZECH REPUBLIC

E-mail address: gurka@tf.czu.cz

MATHEMATICAL INSTITUTE, ACADEMY OF SCIENCES OF THE CZECH REPUBLIC, ŽITNÁ 25, 115 67
PRAGUE 1, CZECH REPUBLIC

E-mail address: opic@math.cas.cz