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A SPLITTING THEOREM FOR EQUIFOCAL SUBMANIFOLDS IN SIMPLY CONNECTED COMPACT SYMMETRIC SPACES

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Abstract. A submanifold in a symmetric space is called equifocal if it has a globally flat abelian normal bundle and its focal data is invariant under normal parallel transportation. This is a generalization of the notion of isoparametric submanifolds in Euclidean spaces. To each equifocal submanifold, we can associate a Coxeter group, which is determined by the focal data at one point. In this paper we prove that an equifocal submanifold in a simply connected compact symmetric space is a non-trivial product of two such submanifolds if and only if its associated Coxeter group is decomposable. As a consequence, we get a similar splitting result for hyperpolar group actions on compact symmetric spaces. These results are an application of a splitting theorem for isoparametric submanifolds in Hilbert spaces by Heintze and Liu.

1. Introduction

Let $N = G/K$ be a Riemannian symmetric space, M an immersed submanifold of N, and $\nu(M)$ the normal bundle of M. The *end point map* or *normal exponential map* $\eta : \nu(M) \to N$ is the restriction of the exponential map exp on TN to $\nu(M)$. If $v \in \nu_x(M)$ is a singular point of η and the dimension of the kernel of $d_v\eta$ is k, then v is called a *multiplicity k focal normal* and $\exp(v)$ is called a *multiplicity k focal point* of M with respect to M in N. The normal bundle of M is called *abelian* if $\exp(\nu_x(M))$ is contained in some flat of N for each $x \in M$. It is called *globally flat* if the induced normal connection is flat and has trivial holonomy. Assume that M is connected and properly immersed in N. M is called *equifocal* if $\nu(M)$ is abelian and globally flat and the focal data is invariant under normal parallel transportation, i.e. if v is a parallel normal field on M such that $\eta_v(x_0)$ is a multiplicity k focal point of M with respect to x_0, then $\eta_v(x)$ is a multiplicity k focal point of M with respect to x for all $x \in M$. (This definition was first given in [TT].) If the ambient space N is $\mathbb{R}^n$, equifocal submanifolds are the same as *isoparametric* submanifolds, which have been studied since the 1930's. (See [PT] and [Te3] for further reference.)

Henceforth, we will assume that $N = G/K$ is a simply connected symmetric space of compact type. Suppose M is a codimension r equifocal submanifold of N. Then M is actually embedded in N, and for all $x \in M$, $T_x := \eta(\nu_x(M))$ is an r-dimensional flat torus in N. We can associate to M an affine Weyl group W, which acts isometrically on $\nu_x(M)$, $x \in M$. W can be obtained as a subgroup

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of $\mathrm{Isom}(\nu_x(M))$, generated by reflections, where the set of reflection hyperplanes is exactly the set of focal normals in $\nu_x(M)$. Furthermore, we have $M \cap T_x = \exp_x(W \cdot 0)$. (Cf. [TT], Theorem 1.8.)

It is a well-known fact (cf. [Bou]) that a Coxeter group (hence in particular an affine Weyl group) decomposes as a product of two or more Coxeter groups if and only if its Coxeter diagram is not connected. The property of a Coxeter group associated to a submanifold to be decomposable actually gives an information about the submanifold itself. Let us review this for the case of the ambient space being a vector space. If M is an isoparametric submanifold in a Hilbert space V, then we can associate to M a Coxeter group W in the same way as described above (cf. [Te1], [Te2]). W is a (finite) Weyl group if V is finite dimensional, and an affine Weyl group (i.e. infinite) if V is infinite dimensional and M not a product of a finite dimensional isoparametric submanifold with an affine subspace of V. M is called *decomposable* if there exist two proper closed affine subspaces V_1, V_2 of V and isoparametric submanifolds M_i in V_i, $i = 1, 2$, such that $V = V_1 \oplus V_2$ and $M = M_1 \times M_2$. We will call such a decomposition *trivial* if one of the components M_i has a trivial Coxeter group, i.e. if M_i is an affine subspace of V_i. It is known that M decomposes non-trivially if and only if its Coxeter group is decomposable. This has been proved by Terng [Te1] for the finite dimensional case, and by Heintze and Liu [HL] for the infinite dimensional case.

The purpose of this paper is to prove a similar result for equifocal submanifolds in symmetric spaces, as an application of the splitting theorem by Heintze and Liu. Let N be a symmetric space, M an equifocal submanifold in N. M is called *decomposable* if there exist symmetric spaces N_1, N_2 and equifocal submanifolds M_i in N_i, $i = 1, 2$, such that $N = N_1 \times N_2$ and $M = M_1 \times M_2$. Again, we will call such a decomposition *trivial* if one of the components M_i has a trivial Coxeter group. Note that if we assume N to be of compact type, the same holds for the N_i's, and so the only possibility for $M_i \subset N_i$ to have a trivial Coxeter group is that $M_i = N_i$.

Theorem 1. *An equifocal submanifold M in a simply connected symmetric space N of compact type decomposes non-trivially if and only if its associated affine Weyl group is decomposable. In particular, if N is irreducible, then M and its affine Weyl group are indecomposable.*

Due to this result, we can restrict any further investigation of equifocal submanifolds to the case of indecomposable ones. It is easy to see that if an equifocal submanifold decomposes non-trivially, then its affine Weyl group is decomposable. So we only have to prove the converse statement.

2. The parallel transport map

We will recall some facts about the method of studying submanifolds in symmetric spaces by lifting them to a Hilbert space, which was introduced in [TT]. Let G be a compact, semisimple Lie group, $\mathfrak{g}$ its Lie algebra, equipped with the inner product $\langle .,. \rangle$ given by the negative of the Killing form. Let $V := H^0([0,1];\mathfrak{g})$ denote the Hilbert space of L^2-integrable paths $u : [0,1] \to \mathfrak{g}$, where the L^2-inner product is defined by

$$\langle u, v \rangle_0 := \int_0^1 \langle u(t), v(t) \rangle dt.$$

We can view V as the space of connections of the trivial principal bundle $[0,1]\times G$ over $[0,1]$, by identifying the path $u(t)$ with the connection $u(t)dt$. For every $u\in V$, let $E_u(t)$ be the parallel translation corresponding to the connection defined by u, i.e. $E_u:[0,1]\to G$ is the unique solution to the initial value problem

$$\begin{cases} E^{-1}E' &= u,\\ E(0) &= e.\end{cases}$$

Define the *parallel transport map* $\Phi: V\to G$ by

$$\Phi(u):=E_u(1).$$

The Hilbert Lie group $H^1([0,1];G)$ of H^1-paths from $[0,1]$ to G acts on V isometrically via gauge transformations:

$$g*u:=gug^{-1}-g'g^{-1}.$$

For each subgroup H of $G\times G$, we denote by $P(G,H)$ the following subgroup of $H^1([0,1];G)$:

$$P(G,H):=\{g\in H^1([0,1];G)\,|\,(g(0),g(1))\in H\}.$$

Define

$$\Omega_e(G):=P(G,e\times e).$$

Then $\Phi: V\to G$ is a principal $\Omega_e(G)$-bundle via the gauge action ([TT], corollary 4.4).

Now let $\pi: G\to G/K$ be the natural Riemannian fibration of the symmetric space $N=G/K$, M a submanifold in N with globally flat abelian normal bundle, and M^* a connected component of $\pi^{-1}(M)$. Then M^* has a globally flat abelian normal bundle, and M is equifocal in N if and only if M^* is equifocal in G.

Let M^* be a submanifold in G with globally flat abelian normal bundle, and $\tilde{M}$ a connected component of $\Phi^{-1}(M^*)$. Then $\tilde{M}$ has a globally flat normal bundle, and M^* is equifocal in G if and only if $\tilde{M}$ is isoparametric in V.

These last two statements are part of theorems 1.9, 1.10 of [TT]. Note that if we assume G to be simply connected and K to be connected, then $\pi^{-1}(M)$ and $\Phi^{-1}(M^*)$ will be connected.

3. Proof of the theorem in the case $N=G$

As a first step, we will prove the splitting theorem for the case of an equifocal submanifold M in a simply connected compact semisimple Lie group G. Without loss of generality, assume $e\in M$. Let $\mathfrak{a}:=\nu_e(M)$, hence $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$. Let $\tilde{M}:=\Phi^{-1}(M)$. For each $x\in\mathfrak{g}$, we denote by $\hat{x}:[0,1]\to\mathfrak{g}$ the constant map of value x. Let $\hat{\mathfrak{a}}:=\{\hat{a}\,|\,a\in\mathfrak{a}\}$, hence $\hat{\mathfrak{a}}$ is the normal space of $\tilde{M}$ at $\hat{0}$. First we deal with the possibility of M having a trivial factor. Define

$$\begin{aligned} V' &:= \overline{\operatorname{span}\{\tilde{v}(u)\,|\,u\in\tilde{M},\ \text{and}\ \tilde{v}(u)\in\nu_u(\tilde{M})\}},\\ V_0 &:= (V')^{\perp}.\end{aligned}$$

Lemma 3.1. *If the dimension of V_0 is bigger than 0, then there exists a submanifold $\tilde{M}'\subset V'$ such that $\tilde{M}=\tilde{M}'\times V_0$ and $\tilde{M}'$ is an isoparametric submanifold of V'. In fact $\tilde{M}'=\tilde{M}\cap V'$.*

This is lemma 3.1 of [HL]. According to corollary 4.6 of [TT], we have

$$V' = \overline{\operatorname{span}\{gvx^{-1}g^{-1} \mid g \in P(G, e \times G),\ x \in M,\ g^{-1}(1) = x,\ v \in \nu_x(M)\}}.$$

The following lemma is easily checked.

Lemma 3.2. *Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, $A \subset \mathfrak{g}$ a subset. Let $\mathfrak{h}$ be the ideal of $\mathfrak{g}$ generated by A. Then $\mathfrak{h} = \operatorname{span}\{gag^{-1} \mid a \in A,\ g \in G\}$.*

Now define

$$\mathfrak{h} := \operatorname{span}\{gvx^{-1}g^{-1} \mid x \in M,\ v \in \nu_x(M),\ g \in G\}.$$

According to lemma 3.2, $\mathfrak{h}$ is the ideal of $\mathfrak{g}$ generated by all elements vx^{-1}, where $x \in M$, $v \in \nu_x(M)$.

Lemma 3.3. *We have $V' = H^0([0,1];\mathfrak{h})$.*

Proof. Clearly $V' \subseteq H^0([0,1];\mathfrak{h})$. Since $V = H^0([0,1];\mathfrak{h}) \oplus H^0([0,1];\mathfrak{h}^\perp)$, and also $V = V' \oplus V_0$, it is sufficient to show that $V_0 \subseteq H^0([0,1];\mathfrak{h}^\perp)$. Let $\alpha \in V_0$, i.e. $\alpha : [0,1] \to \mathfrak{g}$ such that for all $x \in M$, $v \in \nu_x(M)$, $\tilde{g} \in P(G, e \times G)$ with $\tilde{g}^{-1}(1) = x$,

$$\langle \alpha, \tilde{g}vx^{-1}\tilde{g}^{-1} \rangle_0 = \int_0^1 \langle \alpha(t), \tilde{g}(t)vx^{-1}\tilde{g}^{-1}(t) \rangle dt = 0.$$

Now assume that there is a $g \in G$, $x \in M$, $v \in \nu_x(M)$, and a $t \in [0,1]$ such that

$$\langle \alpha(t), gvx^{-1}g^{-1} \rangle \neq 0.$$

Choose a $\tilde{g} \in P(G, e \times G)$ with $\Phi(\tilde{g} * \hat{0}) = \tilde{g}^{-1}(1) = x$ and $\tilde{g}(\frac{1}{2}) = g$. Define

$$\varphi(t) := \langle \alpha(t), \tilde{g}(t)vx^{-1}\tilde{g}^{-1}(t) \rangle.$$

$\varphi : [0,1] \to \mathbb{R}$ is continuous, $\varphi(\frac{1}{2}) \neq 0$, and

$$\int_0^1 \varphi(t) dt = 0.$$

Hence there exists $\varepsilon \in (0, \frac{1}{2})$ such that

$$\int_{t_1}^{t_2} \varphi(t) dt \neq 0,$$

where $t_1 = \frac{1}{2} - \varepsilon$, $t_2 = \frac{1}{2} + \varepsilon$. Now define $\lambda : [0,1] \to [0,1]$ by

$$\lambda(t) := \begin{cases} 3t_2 t, & \text{for } 0 \leq t \leq \frac{1}{3}, \\ 2t_2 - t_1 - 3(t_2 - t_1)t, & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ t_1 - 2t_2 + 3t_2 t, & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases}$$

In other words, λ is running from 0 to t_2, back to t_1, and then on to 1. We have $\tilde{g} \circ \lambda \in P(G, e \times G)$ with $\Phi((\tilde{g} \circ \lambda) * \hat{0}) = \tilde{g}^{-1}(1) = x$, so

$$\int_0^1 \varphi(\lambda(t)) dt = 0,$$

by our assumption on α. On the other hand,

$$\int_0^1 \varphi(\lambda(t)) dt = \frac{1}{3t_2} \int_0^1 \varphi(t) dt + \frac{2t_2 - t_1}{3t_2(t_2 - t_1)} \int_{t_1}^{t_2} \varphi(t) dt \neq 0,$$

which is a contradiction. Hence for all $g \in G$, $t \in [0,1]$, $x \in M$, $v \in \nu_x(M)$, we have $\langle \alpha(t), gvx^{-1}g^{-1} \rangle = 0$, and so α is a path in $\mathfrak{h}^\perp$. □

Assume $V' \neq V$. Then

$$V_0 = H^0([0,1]; \mathfrak{k}),$$

where $\mathfrak{k} = \mathfrak{h}^\perp$, which is also an ideal in $\mathfrak{g}$. According to lemma 3.1, we have

$$\tilde{M} = \tilde{M}' \times V_0,$$

where $\tilde{M}' = \tilde{M} \cap V'$ is an isoparametric submanifold in V'. Let H, K be the connected Lie subgroups of G whose Lie algebras are $\mathfrak{h}$, $\mathfrak{k}$, respectively. Since G is simply connected, $G = H \times K$. Let $\Phi_1 := \Phi|V'$, $\Phi_2 := \Phi|V_0$. Then $V_0 = \Phi_2^{-1}(K)$, and $\tilde{M}' = \Phi_1^{-1}(M')$, where $M' \times e = M \cap (H \times e)$. M' is the image of the saturated submanifold $\tilde{M}'$ under the Riemannian submersion Φ_1, therefore it is a submanifold of H. M' inherits a globally flat abelian normal bundle from M, and so it is equifocal, since $\tilde{M}'$ is isoparametric. So we have $M = M' \times K$, i.e. M has a trivial factor. Hence we see that the following is true.

Lemma 3.4. *The following statements are equivalent:*

(i) *M contains a trivial factor,*
(ii) *there is an ideal $\mathfrak{k}$ in $\mathfrak{g}$ such that for all $x \in M$, $\mathfrak{k} \perp \nu_x(M)x^{-1}$ or, equivalently, $\mathfrak{k}x \subset T_xM$,*
(iii) *$\tilde{M}$ contains a Euclidean factor.*

Note that if M is homogeneous, then lemma 3.4 yields that M contains a trivial factor if and only if T_eM contains an ideal of $\mathfrak{g}$, so we can recognize the existence of a trivial factor by looking at the tangent space of M at a single point.

The following statement about Coxeter groups is easily checked (cf. [Bou], V.3.8).

Lemma 3.5. *Let P be a Euclidean space, $\{l_i \,|\, i \in I\}$ a set of affine hyperplanes, such that the reflections along the l_i's generate a Coxeter group $W \subset \mathrm{Isom}(P)$. For each $i \in I$ let n_i be a normal vector to l_i. Then W is decomposable if and only if there exist two proper linear subspaces P_1 and P_2 of P such that $P_1 \perp P_2$ and $P_1 \cup P_2$ contains all the n_i's.*

The affine Weyl group W associated to M is the same as the one associated to the isoparametric submanifold $\tilde{M}$ in V (cf. [TT], 1.8, 1.10, 6.17). Hence we can obtain W as follows. The normal bundle of $\tilde{M}$ is flat, hence the tangent bundle splits as

$$T\tilde{M} = \overline{\bigoplus \{E_i | i \in I\}}$$

into the direct sum of the simultaneous eigenspaces E_i of the shape operators $\tilde{A}$, where I is a countable index set. Let $\{n_i \,|\, i \in I\}$ be the corresponding *curvature normals* of $\tilde{M}$, i.e. the parallel normal vector fields on $\tilde{M}$ such that for all $u \in \tilde{M}$, $\tilde{v} \in \nu_u(\tilde{M})$,

$$\tilde{A}_{\tilde{v}}|E_i(u) = \langle \tilde{v}, n_i(u) \rangle \cdot \mathrm{id}_{E_i(u)}.$$

Let

$$l_i(u) := \{\tilde{v} \in \nu_u(\tilde{M}) | \langle \tilde{v}, n_i(u) \rangle = 1\}.$$

Then the union of all the affine hyperplanes $l_i(u)$ is exactly the set of focal normals in $\nu_u(\tilde{M})$, and W is isomorphic to the subgroup of $\mathrm{Isom}(\nu_u(\tilde{M}))$ generated by the reflections along the l_i's, for all $u \in \tilde{M}$.

Lemma 3.6. *Assume $V' = V$. Then $\tilde{M}$ is full, i.e. the curvature normals $n_i(u)$ span $\nu_u(\tilde{M})$ for all $u \in \tilde{M}$.*

Proof. Let $\tilde{P}$ be the subspace of $\nu_{\hat{0}}(\tilde{M})$ spanned by all $n_i(\hat{0})$. Let $P := \Phi_*(\tilde{P}) \subset \mathfrak{a}$, let $P^\perp$ be the orthogonal complement of P in $\mathfrak{a}$. If $z \in P^\perp$, there are no focal normals on the line $\mathbb{R}z$ in $\mathfrak{a} = \nu_e(M)$. Since G has only non-negative sectional curvature, this implies that the curvature operator R_z is zero, i.e. for all $v \in \mathfrak{g}$, $R_z(v) = -\frac{1}{4}\operatorname{ad}_z^2(v) = 0$. Now ad_z is skew-symmetric, hence it follows that $[z, v] = 0$ for all $v \in \mathfrak{g}$. So $P^\perp$ is an abelian ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $P^\perp$ must be zero, which means that $\tilde{M}$ is full. $\square$

Henceforth, we will assume that $V' = V$. This implies that $\tilde{M}$ is full, which is one of the assumptions in the proof of the splitting theorem for isoparametric submanifolds in Hilbert spaces in [HL], chap. 3.

If we now assume that W is decomposable, then there exist two proper linear subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathfrak{a}$, such that $\mathfrak{a}_1 \perp \mathfrak{a}_2$ and $\hat{\mathfrak{a}}_1 \cup \hat{\mathfrak{a}}_2$ contains all the curvature normals $n_i(\hat{0})$ of $\tilde{M}$ at $\hat{0}$.

Let P_i be the subbundle of $\nu(M)$ obtained by parallel transport of $\mathfrak{a}_i$ in the normal bundle. Let $\tilde{P}_i$ be the subbundle of $\nu(\tilde{M})$ obtained by parallel transport of $\hat{\mathfrak{a}}_i$ in the normal bundle, i.e. $\tilde{P}_i$ is the horizontal lift of P_i.

Lemma 3.7. *For all $g_1, g_2 \in G$, $x_1, x_2 \in M$, $v_i \in P_i(x_i)$, $i = 1, 2$, we have*

$$\langle g_1 v_1 x_1^{-1} g_1^{-1}, g_2 v_2 x_2^{-1} g_2^{-1} \rangle = 0.$$

Proof. From lemma 3.5 of [HL], we know that for all $u, w \in \tilde{M}$,

$$\tilde{P}_1(u) \perp \tilde{P}_2(w).$$

By corollary 4.6 of [TT] we know that for all $u \in \tilde{M}$, $i = 1, 2$,

$$\tilde{P}_i(u) = \{\tilde{g} v x^{-1} \tilde{g}^{-1} \,|\, \tilde{g} \in P(G, e \times G),\ \tilde{g} * \hat{0} = u,\ \Phi(u) = x,\ v \in P_i(x)\}.$$

So for all $\tilde{g}_i \in P(G, e \times G)$, $x_i = \Phi(\tilde{g}_i * \hat{0})$, $v_i \in P_i(x_i)$, $i = 1, 2$,

$$\langle \tilde{g}_1 v_1 x_1^{-1} \tilde{g}_1^{-1}, \tilde{g}_2 v_2 x_2^{-1} \tilde{g}_2^{-1} \rangle_0 = 0.$$

Now assume that there are $g_i \in G$, $x_i \in M$, $v_i \in P_i(x_i)$, $i = 1, 2$, such that

$$\langle g_1 v_1 x_1^{-1} g_1^{-1}, g_2 v_2 x_2^{-1} g_2^{-1} \rangle \neq 0.$$

Choose for $i = 1, 2$ a $\tilde{g}_i \in P(G, e \times G)$ with $\Phi(\tilde{g}_i * \hat{0}) = \tilde{g}_i^{-1}(1) = x_i$ and $\tilde{g}_i(\frac{1}{2}) = g_i$. Define

$$\varphi(t) := \langle \tilde{g}_1(t) v_1 x_1^{-1} \tilde{g}_1^{-1}(t), \tilde{g}_2(t) v_2 x_2^{-1} \tilde{g}_2^{-1}(t) \rangle.$$

Now we can apply exactly the same argument already used in lemma 3.3 to get a contradiction. $\square$

Now define for $i = 1, 2$,

$$\mathfrak{g}_i := \operatorname{span}\{g v x^{-1} g^{-1} \,|\, x \in M,\ v \in P_i(x),\ g \in G\}.$$

According to lemmas 3.2 and 3.7, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$, and $\mathfrak{g}_1 \perp \mathfrak{g}_2$. Since we assume that M has no trivial factors, lemma 3.4 implies that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let G_i, $i = 1, 2$, be the connected Lie subgroup of G whose Lie algebra is $\mathfrak{g}_i$. Since G is simply connected, $G = G_1 \times G_2$. This splitting of G then yields a splitting of M.

To see this, we have to take a closer look at the proof of the splitting theorem for isoparametric submanifolds in Hilbert spaces, as it is given in [HL], chap. 3. Define for $i = 1, 2$,

$$\begin{aligned} V_{\tilde{P}_i} &:= \overline{\operatorname{span}\{\tilde{v}(u) \mid u \in \tilde{M},\ \tilde{v}(u) \in \tilde{P}_i(u)\}} \\ &= \overline{\operatorname{span}\{gvx^{-1}g^{-1} \mid g \in P(G, e \times G),\ x = \Phi(g * \hat{0}) \in M,\ v \in P_i(x)\}}. \end{aligned}$$

Clearly $V_{\tilde{P}_i} \subseteq H^0([0,1]; \mathfrak{g}_i)$. But according to corollary 3.6 of [HL], we have $V = V_{\tilde{P}_1} \oplus V_{\tilde{P}_2}$, so $V_{\tilde{P}_i} = H^0([0,1]; \mathfrak{g}_i)$, $i = 1, 2$. Furthermore $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$, where $\tilde{M}_i = \tilde{M} \cap V_{\tilde{P}_i}$ is an isoparametric submanifold of $V_{\tilde{P}_i}$. If we define $\Phi_i := \Phi | V_{\tilde{P}_i}$ and $M_i := \Phi_i(\tilde{M}_i)$, we have $\tilde{M}_i = \Phi_i^{-1}(M_i)$ and $M = M_1 \times M_2$. M_i is the image of the saturated submanifold $\tilde{M}_i$ under the Riemannian submersion Φ_i, therefore it is a submanifold of G_i. M_i inherits a globally flat and abelian normal bundle from M, its lift $\tilde{M}_i$ is isoparametric in $V_{\tilde{P}_i}$, so it is equifocal in G_i, $i = 1, 2$.

4. Proof of the theorem in the case $N = G/K$

Let $N = G/K$ be a simply connected symmetric space of compact type, in particular G semisimple. We choose G to be simply connected, which implies that K is also simply connected. Let $\pi : G \to G/K$ be the natural projection, $M \subset N$ an equifocal submanifold, and $M^* := \pi^{-1}(M)$. Without loss of generality assume $eK \in M$, i.e. $e \in M^*$. M and M^* have the same affine Weyl group W associated to them (cf. [TT], 1.8, 1.9). We have seen that if W is decomposable, then M^* is decomposable. So we only have to check that such a decomposition of $M^* \subset G$ yields a decomposition of $M \subset G/K$.

Let $s : G \to G$ be the involution of G with K as the identity component of the fixed group of s. Let $\theta := d_e s : \mathfrak{g} \to \mathfrak{g}$ be the corresponding Cartan involution. Let $G = G_1 \times G_2$, $M_i^* \subset G_i$ such that $M^* = M_1^* \times M_2^*$. Let $\mathfrak{g}_i$ be the Lie algebra of G_i, i.e. $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Assume that $\theta(\mathfrak{g}_i) = \mathfrak{g}_i$ for $i = 1, 2$. Then we can define $\mathfrak{k}_i$ to be the 1-eigenspace of $\theta | \mathfrak{g}_i$, and K_i to be the connected Lie subgroup of G_i with Lie algebra $\mathfrak{k}_i$. K is simply connected, so $K = K_1 \times K_2$. If we define $N_i := G_i/K_i$, then N_i is a symmetric space, and $N = N_1 \times N_2$. Define $\pi_i := \pi | G_i$, $M_i := \pi_i(M_i^*)$. Then $M_i^* = \pi_i^{-1}(M_i)$, and M_i inherits a globally flat and abelian normal bundle from M, so M_i is equifocal in N_i, and $M = M_1 \times M_2$.

To conclude the proof of the theorem, we have to deal with the possibility of θ not respecting the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, i.e. $\theta(\mathfrak{g}_i) \neq \mathfrak{g}_i$ for $i = 1, 2$. The semisimple Lie algebra $\mathfrak{g}$ decomposes as a direct sum of its simple ideals

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r,$$

(cf. [Hel], II.6.3), and the ideals $\mathfrak{g}_1$, $\mathfrak{g}_2$ are each a direct sum of certain $\mathfrak{h}_\rho$, $1 \leq \rho \leq r$. On the other hand, the orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ decomposes as a direct sum of irreducible orthogonal symmetric Lie algebras $(\mathfrak{l}_\kappa, \theta_\kappa)$, $1 \leq \kappa \leq k$, where the $\mathfrak{l}_\kappa$'s are ideals in $\mathfrak{g}$ and $\theta_\kappa = \theta | \mathfrak{l}_\kappa$ (cf. [Hel], VIII.5.2). Each $\mathfrak{l}_\kappa$ is either a simple ideal or a sum of two simple ideals which are interchanged by θ_κ (cf. [Hel], VIII.5.3).

If we assume that $\theta(\mathfrak{g}_i) \neq \mathfrak{g}_i$ for $i = 1, 2$, then there must be a $\kappa \in \{1, \ldots, k\}$ such that $\mathfrak{l}_\kappa = \mathfrak{h}_\rho \oplus \mathfrak{h}_\sigma$ for some $1 \leq \rho, \sigma \leq r$, and $\theta_\kappa(\mathfrak{h}_\rho) = \mathfrak{h}_\sigma$, $\theta_\kappa(\mathfrak{h}_\sigma) = \mathfrak{h}_\rho$,

$\mathfrak{h}_\rho \subseteq \mathfrak{g}_1$, $\mathfrak{h}_\sigma \subseteq \mathfrak{g}_2$. Therefore $\mathfrak{k}$ contains all elements of the form $h + \theta_\kappa(h)$, where $h \in \mathfrak{h}_\rho$. Now on the one hand, $M^* = M_1^* \times M_2^*$, so $T_eM^* = T_eM_1^* \oplus T_eM_2^*$, where $T_eM_i^* \subset \mathfrak{g}_i$, $i = 1, 2$. On the other hand, $M^* = \pi^{-1}(M)$, so $\mathfrak{k} \subset T_eM^*$. Hence it follows that $\mathfrak{h}_\rho \subset T_eM_1^*$, $\mathfrak{h}_\sigma \subset T_eM_2^*$, i.e. $\mathfrak{l}_\kappa \subset T_eM^*$.

We will show that this implies the existence of a trivial factor of M (and M^*), which can be split off. According to lemma 3.4, we have to show that for all $y \in M^*$, the right translation $\mathfrak{l}_\kappa y$ is contained in T_yM^*. Let $y \in M^*$, consider the submanifolds $y^{-1}M^* \subset G$ and $y^{-1}M \subset N$. These are equifocal submanifolds, and $y^{-1}M^* = \pi^{-1}(y^{-1}M)$. The splitting of M^* is constructed by first splitting off trivial factors and then taking the ideals generated by all $P_i(x)x^{-1}$, $x \in M^*$. Say all trivial factors of M^* are in the G_2-part, then $\mathfrak{g}_1 = \operatorname{span}\{gvx^{-1}g^{-1} | x \in M^*,\ v \in P_1(x),\ g \in G\}$. Since for all $x \in M^*$, $\nu_x(M^*) = P_1(x) \oplus P_2(x)$, we have $\nu_{y^{-1}x}(y^{-1}M^*) = y^{-1}P_1(x) \oplus y^{-1}P_2(x)$. For $y^{-1}x \in y^{-1}M^*$, $y^{-1}v \in y^{-1}P_i(x)$, $g \in G$, we have $g(y^{-1}v)(y^{-1}x)^{-1}g^{-1} = (gy^{-1})vx^{-1}(gy^{-1})^{-1}$. Hence, if we apply the splitting procedure to $y^{-1}M^*$, we end up with the same $\mathfrak{g}_1, \mathfrak{g}_2$, and it follows for the same reason as above that $\mathfrak{l}_\kappa \subset T_e(y^{-1}M^*)$. Since $T_e(y^{-1}M^*) = y^{-1}T_yM^*$, it follows that $y\mathfrak{l}_\kappa \subset T_yM^*$. $\mathfrak{l}_\kappa$ is an ideal, so $y\mathfrak{l}_\kappa = \mathfrak{l}_\kappa y$. Now we can apply lemma 3.4 and conclude that M^* contains a trivial factor, namely the connected subgroup L_κ of G whose Lie algebra is $\mathfrak{l}_\kappa$. Furthermore, if we define $\mathfrak{k}_\kappa$ to be the 1-eigenspace of θ_κ, and K_κ to be the connected subgroup of G whose Lie algebra is $\mathfrak{k}_\kappa$, then $N_\kappa := L_\kappa/K_\kappa$ is an irreducible component of N, and M contains N_κ as a trivial factor. By splitting off all trivial factors obtained in that way, we eventually arrive at a situation where θ respects the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

5. Application to hyperpolar actions

Recall that an isometric action of a compact Lie group H on a Riemannian manifold N is called *hyperpolar* if there exists a closed flat connected submanifold Σ of M that meets all H-orbits and meets them orthogonally. Such a Σ is called a *section.* Define $N(\Sigma) := \{h \in H \,|\, h(\Sigma) \subset \Sigma\}$ and $Z(\Sigma) := \{h \in H \,|\, h(x) = x \text{ for all } x \in \Sigma\}$. Let $W(\Sigma) := N(\Sigma)/Z(\Sigma)$. $W(\Sigma)$ is called the *generalized Weyl group* associated to the H-action. Two isometric actions of groups H_i on Riemannian manifolds N_i for $i = 1, 2$ are called *ω-equivalent* if there exists an isometry $f : N_1 \to N_2$ such that $f(H_1 \cdot x) = H_2 \cdot f(x)$ for all $x \in N_1$. An isometric action of H on N is called *ω-decomposable* if there exist Riemannian H_i-manifolds N_i for $i = 1, 2$ such that $N = N_1 \times N_2$ and the action of H on N is ω-equivalent to the product action of $H_1 \times H_2$ on $N_1 \times N_2$. If one of the H_i-actions is transitive, then we say that this decomposition has a *trivial factor.* If G is a compact semisimple Lie group and H a closed subgroup of $G \times G$, H has a natural isometric action on G defined by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$ for $(h_1, h_2) \in H$, $g \in G$.

Assume that the action of H on G is hyperpolar. Let M be a principal H-orbit. Then M is equifocal ([TT], theorem 2.1). Let A be a torus section through $e \in G$ and let $\mathfrak{a}$ denote its Lie algebra. Let $\hat{W}$ be the affine Weyl group associated to M. We can view $\hat{W}$ as a subgroup of $\operatorname{Isom}(\mathfrak{a})$ generated by reflections. Let $\Lambda := \{\lambda \in \mathfrak{a} \,|\, \exp(\lambda) = e\}$ be the unit lattice of A. If we identify $\lambda \in \Lambda$ with the reflection along the hyperplane $\frac{1}{2}\lambda + \lambda^\perp$ in $\mathfrak{a}$, then Λ is a normal subgroup of $\hat{W}$, and $\hat{W}/\Lambda = W(A)$ (cf. [Te2], theorem 8.10, and [Te4], theorem 1.2).

Theorem 2. *Let G be a simply connected compact semisimple Lie group. Then the following statements are equivalent:*

(i) *the H-action on G is ω-decomposable into two non-trivial factors,*
(ii) *the isometric action of $W(A)$ on A is decomposable,*
(iii) $\hat{W}$ *is decomposable.*

Proof. The equivalence of (i) and (ii) is theorem B of [HL]. Furthermore it is clear that (i) implies (iii). If $\hat{W}$ is decomposable, then we know by the previous results that the $\hat{W}$-action on $\mathfrak{a}$ is a product action. The $\hat{W}$-action on $\mathfrak{a}$ induces the $W(A)$-action on A, so the latter one is also a product action. Hence (iii) implies (ii). □

More generally, let G/K be a compact symmetric space and H a closed subgroup of G that acts hyperpolarly on G/K. Then the subgroup $H \times K$ of $G \times G$ acts hyperpolarly on G (cf. [HPTT]), and the $H \times K$-orbits in G are the lifts of the H-orbits in G/K under $\pi : G \to G/K$. The principal H-orbits are equifocal in G/K (cf. [TT]). Let A be a torus section through $e \in G$ for the $H \times K$-action. Then $B := \pi(A)$ is a section through $eK \in G/K$ for the H-action. The projection $H \times K \to H$ induces isomorphisms $N(A) \to N(B)$ and $Z(A) \to Z(B)$, hence $W(A) = W(B)$, and for all $a \in A$ and $w \in W(A) = W(B)$, $\pi(w \cdot a) = w \cdot \pi(a)$. Let $\hat{W}$ be the affine Weyl group associated to a principal H-orbit, Λ the unit lattice of A, hence $W(B) = \hat{W}/\Lambda$.

Theorem 3. *Let G/K be a simply connected symmetric space of compact type. Then the following statements are equivalent:*

(i) *the H-action on G/K is ω-decomposable into two non-trivial factors,*
(ii) *the isometric action of $W(B)$ on B is decomposable,*
(iii) $\hat{W}$ *is decomposable.*

Proof. Clearly (i) implies (iii), and (iii) implies (ii). Let the action of $W(B)$ on B be decomposable. Then the action of $W(A)$ on A is also decomposable. By theorem 2 this implies that the $H \times K$-action on G is ω-decomposable. More precisely, let $G = G_1 \times G_2$ be the decomposition of G. In the proof of theorem 1 we have seen that we can choose this decomposition in a way that there are subgroups K_i of G_i, $i = 1, 2$, such that $K = K_1 \times K_2$, and $G/K = (G_1/K_1) \times (G_2/K_2)$. $G \times G$ decomposes as a product of $G_1 \times G_1$ and $G_2 \times G_2$. Let H_i be the projection of H to G_i, $i = 1, 2$. Then $H_i \times K_i$ is the projection of $H \times K$ to $G_i \times G_i$, and the $(H_1 \times K_1) \times (H_2 \times K_2)$-action on $G_1 \times G_2$ is ω-equivalent to the $H \times K$-action on G (cf. [HL], proof of theorem B). This implies that the $H_1 \times H_2$-action on $(G_1/K_1) \times (G_2/K_2)$ is ω-equivalent to the H-action on G/K. □

Note that a hyperpolar action can be ω-decomposable without being a product action. For example, let $G_0 := \mathbf{SO}(2n)$, $K_0 := \mathbf{U}(n)$. Then (G_0, K_0) is an irreducible symmetric pair of compact type (type D III in the list of [Hel], X.6). Let $G := G_0 \times G_0$, $K := K_0 \times K_0$. Then (G, K) is also a symmetric pair of compact type. Let $N := G/K$. K acts hyperpolarly on N by $k \cdot (gK) = kgK$, for $k \in K$, $g \in G$. Let $H := \mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$. H is a subgroup of K. More precisely,

$$H = \{(k_1, k_2) \in K_0 \times K_0 \mid \det(k_1)\det(k_2) = 1\}.$$

Let $k = (k_1, k_2) \in K$, let $\lambda := \det(k_1)\det(k_2)$. Then $h = (h_1, h_2) := (k_1, \lambda^{-\frac{1}{n}} k_2)$ is an element of H, and for all $g \in G$, we have $hgK = kgK$. Hence the actions of H and K have the same orbits, i.e. they are ω-equivalent. So the H-action is ω-decomposable, but it is not a product action.

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