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COMPLETELY CONTRACTIVE REPRESENTATIONS FOR SOME DOUBLY GENERATED ANTISYMMETRIC OPERATOR ALGEBRAS

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ABSTRACT. Contractive weak star continuous representations of the Fourier binest algebra $\mathcal{A}$ (of Katavolos and Power) are shown to be completely contractive. The proof depends on the approximation of $\mathcal{A}$ by semicrossed product algebras $A(\mathbb{D}) \times \mathbb{Z}_+$ and on the complete contractivity of contractive representations of such algebras. The latter result is obtained by two applications of the Sz.-Nagy–Foias lifting theorem. In the presence of an approximate identity of compact operators it is shown that an automorphism of a general weakly closed operator algebra is necessarily continuous for the weak star topology and leaves invariant the subalgebra of compact operators. This fact and the main result are used to show that isometric automorphisms of the Fourier binest algebra are unitarily implemented.

Ando's celebrated dilation theorem for pairs of commuting contractions implies that a contractive representation of the bidisc algebra $A(\mathbb{D}) \otimes A(\mathbb{D})$ is completely contractive. Here $A(\mathbb{D})$ is the usual algebra of functions which are analytic on the open unit disc and continuous on the closed disc. In this note we point out why some other doubly generated antisymmetric *noncommutative* operator algebras have this property, and we raise some general questions.

The Fourier binest algebra $\mathcal{A}$ was introduced recently in Katavolos and Power [5]. It may be defined directly as the algebra of operators A on $L^2(\mathbb{R})$ for which

$$AL^2(t,\infty) \subseteq L^2(t,\infty) \quad \text{and} \quad Ae^{itx}H^2(\mathbb{R}) \subseteq e^{itx}H^2(\mathbb{R})$$

for each t, $-\infty < t < \infty$. Here $H^2(\mathbb{R})$ is the Hardy space on the line corresponding to analytic functions in the upper half plane and $L^2(t,\infty)$ is the subspace of functions vanishing on $(-\infty, t]$. Clearly $\mathcal{A}$ is the intersection of the two nest algebras corresponding to the Volterra nest $\mathcal{N}_v$, consisting of the subspaces $L^2(t,\infty)$, and to the analytic nest $\mathcal{N}_a$ comprising the subspaces $e^{itx}H^2(\mathbb{R})$. Amongst the interesting properties of $\mathcal{A}$ is the fact that it is the closure in the weak operator topology of the Hilbert-Schmidt pseudo-differential operators $Op(a)$ of the form

$$(Op(a)f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(x,y) e^{-ixy} \tilde{f}(y)\, dy,$$

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where $a(x, y)$ is bianalytic in the sense that $a(x, y)$ belongs to the Hilbert space $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$. Here $\tilde{f} = F^* f$, where F is the usual Fourier transform unitary on $L^2(\mathbb{R})$.

We recall some terminology. A contractive representation $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$, with $\mathcal{H}$ a separable Hilbert space, is completely contractive if the induced representations $M_n(\mathcal{B}) \to M_n(\mathcal{L}(\mathcal{H}))$ of the spatially normed algebras $M_n(\mathcal{B})$ are contractive for each n. Complete contractivity is often demonstrated (as here) by the construction of a dilation π for ρ, this being a star representation $\pi : C^*(\mathcal{B}) \to \mathcal{L}(\mathcal{K})$, where $\mathcal{K} \supseteq \mathcal{H}$ and $\rho(B) = P_{\mathcal{H}} \pi(B)|_{\mathcal{H}}$ for all B in $\mathcal{B}$.

Theorem 1. *Let ρ be a σ-weakly continuous contractive representation of $\mathcal{A}$ on $\mathcal{H}$. Then ρ is completely contractive. Furthermore, there is a σ-weakly continuous representation $\pi : \mathcal{L}(L^2(\mathbb{R})) \to \mathcal{L}(\mathcal{K})$, with $\mathcal{K} \supseteq \mathcal{H}$, such that $\rho(A) = P_{\mathcal{H}} \pi(A)|_{\mathcal{H}}$ for all A in $\mathcal{A}$.*

As in the case of nest algebras [9], as well as tensor products of nest algebras [8], the proof of the theorem is based upon approximation of $\mathcal{A}$ by simpler algebras. However $\mathcal{A}$ contains no finite-rank operators and, furthermore, is antisymmetric in the sense that $\mathcal{A} \cap \mathcal{A}^* = \mathbb{C}I$. Consequently the semidiscreteness methods of [1], [8] and [9] are not available. However, we can approximate $\mathcal{A}$ by doubly generated subalgebras $\mathcal{A}_n$ whose generators u_n, v_n satisfy the commutation relations

$$u_n v_n = e^{in^{-2}} v_n u_n.$$

The complete contractivity of contractive representations of the algebras $\mathcal{A}_n$ will be obtained by two applications of the Sz.-Nagy–Foias lifting theorem, an equivalent form of Ando's theorem, together with some elementary intertwining constructions. The result appears to be one of a range of covariant versions of Ando's theorem which, to the author's knowledge, have not yet been investigated. In this direction we also give the following more general result for semicrossed products of the disc algebra, another family of doubly generated antisymmetric noncommutative operator algebras.

Theorem 2. *Let $A(\mathbb{D}) \times_\alpha \mathbb{Z}_+$ be the semicrossed product algebra associated with a biholomorphic automorphism α of $\mathbb{D}$. Then every contractive Hilbert space representation of $A(\mathbb{D}) \times_\alpha \mathbb{Z}_+$ is completely contractive.*

It is well-known that contractive representations of the semicrossed product algebras $C(X) \times_\alpha \mathbb{Z}_+$ are completely contractive. See, for example, McAsey and Muhly [7]. However, the analytic semicrossed products of Theorem 2, which were considered recently in [4] in a different context, have a bianalytic character (they are generated by two isometries) and this requires the subtler techniques of commutant lifting, and additional arguments. There are commutant lifting theorems associated with the algebras $C(X) \times_\alpha \mathbb{Z}_+$ (see [6]) which are similar in spirit to Theorem 2 and its proof, but this context is also different.

Proof of Theorem 1. First we show that a pair of contractions satisfying the rotation commutation relations can be lifted to a joint unitary power dilation. This is achieved with four successive dilations, starting with two applications of the Sz.-Nagy–Foias lifting theorem.

Let λ be a unimodular complex number and let U, V be contraction operators on the Hilbert space $\mathcal{H}$ satisfying the commutation relations $UV = \lambda VU$. Let V_1

be an isometric operator on $\mathcal{K} \supseteq \mathcal{H}$ which is a (power) dilation of V, so that λV_1 is an isometric dilation of λV. By the intertwining form of the Sz.-Nagy–Foias lifting theorem there is a contractive (power) dilation U_1 of U on the space $\mathcal{K}$ with $U_1 V_1 = (\lambda V_1) U_1$. The isometric dilation of V_1^* is unitary, and so we may repeat the dilation process above for the equation $U_1^*(\lambda V_1)^* = V_1^* U_1^*$ to obtain for the unitary dilation V_2 of V_1, acting on $\mathcal{G} \supseteq \mathcal{K}$ and a contractive power dilation U_2 of U_1 with $U_2 V_2 = \lambda V_2 U_2$. At this point we could observe that V_2 and U_2 provide a contractive representation of $C(X) \times_{\overline{\lambda}} \mathbb{Z}_+$ and appeal to [7]. However, the argument may be completed in the following elementary fashion.

Consider now the standard isometric dilation U_3 of U_2 on the Hilbert space $\tilde{\mathcal{G}} = \mathcal{G} \oplus \mathcal{G} \oplus \dots$ given by $U_3(g_1, g_2, \dots) = (U_2 g_1, D_2 g_1, g_2, \dots)$ where $D_2 = (I - U_2^* U_2)^{\frac{1}{2}}$. Since V_2 is a unitary operator, $V_2^* U_2 V_2 = \lambda U_2$. Let V_3 be the block diagonal operator on $\tilde{\mathcal{G}}$ given by

$$V_3(g_1, g_2, g_3, \dots) = (V_2 g_1, \bar{\lambda} V_2 q_2, \bar{\lambda}^2 V_2 g_3, \dots).$$

Note that $V_2^* D_2 V_2 = D_2$, from which it follows that $V_3^* U_3 V_3 = \lambda U_3$. Repeat the same trick, with the isometric (and hence unitary) dilation of U_3^*, for the equation $U_3^* V_3^* = \lambda V_3^* U_3^*$ to obtain, finally, unitary dilations U_4, V_4 of U, V acting on a Hilbert space $\mathcal{R} \supseteq \mathcal{H}$ satisfying the relation $U_4 V_4 = \lambda V_4 U_4$.

In fact the dilation operators satisfy the stronger property of being a joint power dilation in the sense that, for $n, m \geq 0$,

$$U^n V^m = P_{\mathcal{H}} U_4^n V_4^m | \mathcal{H}$$

where $P_{\mathcal{H}}$ is the canonical projection onto $\mathcal{H}$. Indeed, this is known to be a feature of the construction of the Sz.-Nagy–Foias lifting theorem, and can be also seen to hold for the constructions of U_3, V_3 and U_4, V_4.

We have now obtained the following special case of Theorem 2. The last part follows from standard arguments with Arveson's dilation theorem for completely contractive maps.

Theorem 3. *Let $\lambda \in \mathbb{C}$ be unimodular and let A_λ be the non-self-adjoint closed subalgebra of the rotation C^*-algebra $C(\partial\mathbb{D}) \times_\lambda \mathbb{Z}$ generated by the canonical C^*-algebra generators. If $\rho : A_\lambda \to \mathcal{L}(\mathcal{H})$ is a contractive representation, then ρ is completely contractive and there exists a *-dilation $\pi : C(\partial\mathbb{D}) \times_\lambda \mathbb{Z} \to \mathcal{L}(\mathcal{R})$ such that $\rho(a) = P_{\mathcal{H}} \pi(a) | \mathcal{H}$ for all a in A_λ.*

Continuing with the proof of Theorem 1, let $\mathcal{A}_n$ be the subalgebra of the Fourier binest algebra generated by the multiplication operator M_{ϕ_n}, where $\phi_n(x) = e^{ix/n}$, and the translation operator $D_{1/n}$ with $(D_{1/n} f)(x) = f(x - n^{-1})$. Then $M_{\phi_n} D_{1/n} = \lambda_n D_{1/n} M_{\varphi_n}$, where $\lambda_n = e^{in^{-2}}$, and so $\mathcal{A}_n$ is completely isometrically isomorphic to the rotation algebra A_{λ_n}.

In [5] it was shown that the Hilbert–Schmidt operators of the Fourier binest algebra were precisely those of the form $Op(a)$ where $a \in H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$. From this it follows that $\mathcal{A}$ is the weak star closed operator algebra generated by the operators M_α, D_β for $\alpha \geq 0, \beta \geq 0$. To see this observe first that

$$\begin{aligned} w^*\text{-}span\{M_\alpha D_\beta : \alpha, \beta \geq 0\} &\supseteq w^*\text{-}span\{M_h D_k : h, k \in H^\infty \cap H^2\} \\ &= w^*\text{-}span\{Op(a) : a(x,y) = h(x)k(y), h, k \in H^2\} \\ &= w^*\text{-}span\{\mathcal{A} \cap \mathcal{C}_2\}. \end{aligned}$$

Furthermore, $\mathcal{A} \cap \mathcal{C}_2$ contains the weak star topology approximate identity (X_n), where X_n is the Hilbert Schmidt operator

$$X_n = M_{\frac{ni}{x+ni}} D_{\frac{ni}{y+ni}},$$

and so any operator Z in $\mathcal{A}$ is the weak star limit of the sequence ZX_n.

It now follows that $\mathcal{A}$ is the weak star closed union of the subalgebras $\mathcal{A}_1, \mathcal{A}_2, \dots$. The representation ρ is completely contractive on this union. Thus, by weak star continuity and weak star density, ρ is completely contractive on $\mathcal{A}$. This latter verification is elementary.

Proof of Theorem 2. Let z be the generator of $A(\mathbb{D})$ and $u \in A(\mathbb{D}) \times_\alpha \mathbb{Z}_+$ the canonical unitary, so that $zu = u\alpha(z)$, and let $\rho : A(\mathbb{D}) \times_\alpha \mathbb{Z} \to \mathcal{L}(\mathcal{H}_\rho)$ be a contractive representation with $T = \rho(z), U = \rho(u)$. Let T_1 be the unitary dilation of the contraction T on $\mathcal{H} \supseteq \mathcal{H}_\rho$, so that $\alpha(T_1)$ is the unitary dilation of $\alpha(T)$. By the Sz.-Nagy–Foias lifting theorem (applied twice) there is a dilation U_1 of U on $\mathcal{H}$ such that $T_1 U_1 = U_1 \alpha(T_1)$ and the associated representation $\sigma : A(\mathbb{D}) \times_\alpha \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ is a dilation of the representation ρ. Since T_1 is unitary, the representation σ extends to a representation $\tilde{\sigma} : C(\partial\mathbb{D}) \times_\alpha \mathbb{Z}_+ \to \mathcal{L}(\mathcal{H})$ with $\tilde{\sigma}(u)$ a contraction. By [7] $\tilde{\sigma}$ is completely contractive, and so ρ is completely contractive.

Remarks. Note that since H^∞ is singly generated as a weakly closed operator algebra, it follows from the density assertions above that the Fourier binest algebra is doubly generated. Indeed, it is the weakly closed operator algebra generated by the pair $\{M_f, D_g\}$, where f and g are generators of H^∞.

By the main result of Paulsen and Power in [8], a contractive weak star continuous representation of the tensor product of two commuting nest algebras is completely contractive. Such a tensor product can be viewed as the intersection of two nest algebras, and so, in view of Theorem 1 above, it is natural to ask the following general question. What hypotheses ensure that the intersection of two nest algebras has the property that (weak star) contractive representations are completely contractive? Certainly some assumptions are needed, as the following finite-dimensional example shows.

Let $\mathcal{A} \subseteq M_6$ be the algebra of matrices of the form

$$A = \begin{bmatrix} D_1 & B \\ 0 & D_2 \end{bmatrix},$$

where D_1, D_2 are diagonal matrices in $M_3(\mathbb{C})$ and $B \in M_3(\mathbb{C})$. It is straightforward to express $\mathcal{A}$ as the intersection of two nest algebras. Nevertheless, it is shown in Davidson, Paulsen and Power [1] that not all contractive representations are completely contractive.

Automorphisms of the Fourier binest algebra

In [5] we obtained a detailed characterisation of the unitary automorphisms of the Fourier binest algebra $\mathcal{A}$. (In fact this group is a Lie group generated by shift automorphisms, Fourier shift automorphisms and dilation automorphisms.) We now point out how Theorem 1 above can be used in showing that the unitary automorphisms are precisely the isometric automorphisms of $\mathcal{A}$. By Theorem 4 below, which may be of independent interest, it is enough to show that the weak star continuous isometric automorphisms are unitarily implemented.

By Theorem 1 an isometric weak star continuous automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ is completely contractive and so induces a completely positive bijection $\tilde{\alpha} : \mathcal{A} + \mathcal{A}^* \to \mathcal{A} + \mathcal{A}^*$. By the universal properties of C^*-envelopes this bijection has a unique extension to a C^*-algebra automorphism of the C^*-envelope. However, the C^*-envelope is a quotient of $C^*(\mathcal{A})$ (see [3]), and $C^*(\mathcal{A})$ is equal to the algebra $\mathcal{L}(L^2(\mathbb{R}))$, and so it follows that the C^*-envelope must be $\mathcal{L}(L^2(\mathbb{R}))$ itself. Thus the extension and the original automorphism are unitarily implemented.

Theorem 4. *Let $\mathcal{A}$ be a weakly closed operator algebra containing a bounded sequence of compact operators which converges to the identity in the strong operator topology. If α is a continuous automorphism of $\mathcal{A}$, then α maps compact operators to compact operators and is continuous in the weak star topology.*

Proof. Let $\{K_n\}$ be the given sequence of compact operators and let K be an arbitrary compact operator in $\mathcal{A}$. The set $K(\mathrm{ball}(\mathcal{A}))K$ is precompact, since K is compact, and so, by the continuity of α and the open mapping theorem, the set $\alpha(K)(\mathrm{ball}(\alpha(\mathcal{A})))\alpha(K)$ is also precompact. Thus the sequence $\alpha(K)(I - K_n)\alpha(K)$ has a norm-convergent subsequence. The limit is necessarily zero, since $I - K_n$ tends to zero weakly, and so $\alpha(K)^2$ is a compact operator.

Now, if T is a compact operator in $\mathcal{A}$, then $TK_n^2 - T \to 0$ in norm, and so $\alpha(T)\alpha(K_n)^2 - \alpha(T) \to 0$ in norm. Thus $\alpha(T)$ is compact. Since α preserves the compact subalgebra, we can now argue as in [2]. Suppose that $\{T_\gamma\}$ is a bounded net in $\mathcal{A}$ which converges to T in the weak star topology. Then $K_nT_\gamma K_n \to K_nTK_n$ in norm for each n. Thus $\alpha(K_n)\alpha(T_\gamma)\alpha(K_n) \to \alpha(K_n)\alpha(T)\alpha(K_n)$ in norm for each n. Therefore $T_\gamma \to T$ in the weak operator topology, and hence weak star, by the boundedness of $\{T_\gamma\}$. Thus α is weak star continuous on the unit ball of $\mathcal{A}$ and hence, by the Krein-Smulyan theorem, is weak star continuous.

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