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LOGARITHMIC SOBOLEV INEQUALITIES AND THE GROWTH OF L^p NORMS

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ABSTRACT. We show that many of the recent results on exponential integrability of Lip 1 functions, when a logarithmic Sobolev inequality holds, follow from more fundamental estimates of the growth of L^p norms under the same hypotheses.

There have been a number of recent papers [1], [2], [3], [4] showing that a probability measure m on a Riemannian manifold M satisfying a logarithmic Sobolev inequality

$$(1)\quad \rho\int_M |\nabla f|^2 dm \geq \int_M |f|^2 \ln |f|^2 dm - \int_M |f|^2 dm \ln \int_M |f|^2 dm - \tau \int_M |f|^2 dm$$

has rather strong decay properties near infinity. These results take the form of integrability for exponentials of the powers of Lip 1 functions. Typically one has

$$(2)\qquad \int_M e^{\lambda f^2} dm < \infty$$

for λ sufficiently small, provided f is Lip 1 on M. Weaker conditions on f also yield weaker conclusions than (2). Some methods yield precise bounds for the left side of (2) and some techniques only seem to work under the stronger condition that the parameter τ ($\tau \geq 0$, the so called defect) in (1) is zero; in any case these techniques give considerably weaker results when $\tau > 0$.

We intend to show that the exponential integrability holds in the defective case to the same extent as in the nondefective, despite the fact that the defective inequality does not in general give a mass gap or a Poincaré inequality. To achieve our goal, we use an approach quite different from the ones typically used by others, who generally establish a result such as (2) by proving a differential equality for one or the other of the Laplace transforms $E(e^{\lambda f})$ or $E(e^{\lambda f^2})$. Rather, we use the defective log Sobolev inequality to relate $E(f^{p+2})$ inductively to $E(f^p)$, which in turn yields bounds on L^{2n} norms and good estimates of $E(e^{\lambda f^2})$. Our method works as well in the case $\tau > 0$ as the case $\tau = 0$, though it does not give as sharp inequalities in the case $\tau = 0$ as those obtained from previous techniques in [1]–[4].

It is well known how to approximate a globally Lip 1 function on a Riemannian manifold by smooth functions without increasing the Lip 1 norm. For simplicity of statement and without loss of generality, our theorems will be stated for a smooth function of compact support. Even more specifically, if μ is Lip 1, so also is $|\mu|$, with

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the same Lip 1 norm, and so we will suppose in all our arguments and statements that μ is a smooth, nonnegative, compactly supported Lip 1 function of Lip 1 norm at most one; that is, $|\nabla u(x)| \leq 1$ for all $x \in M$. The results carry over readily to general Lip 1 functions. Generalizations of the present technique to more abstract settings such as those in [1]–[4] are also easily made.

Our main theorem is the following. For μ as just above, define

$$K(\lambda) = \int_M \mu^\lambda dm \tag{3}$$

Theorem 1. *There is an absolute constant d such that*

$$K(2n) \leq \frac{e^{-\tau} n^n \rho^n}{e^n} \Big(1 + \frac{c}{\sqrt{\pi n}}\Big)^n,$$

where

$$c = \frac{2e^{\tau+1} K(2)}{\rho} + d.$$

This is approximately comparable to estimates obtained from [1], [2], [4]. The proof of this theorem will be given below. Our inequality immediately yields the integrability of $e^{\alpha\mu^2}$ for $\alpha < \frac{1}{\rho}$ and can obviously be used to bound $\int_M e^{\alpha\mu^2} dm$. Indeed, using the well known facts that

$$n! \geq \frac{1}{\sqrt{2\pi}} \; \frac{(n+1)^{n+1/2}}{e^{n+1}},$$

that $(1+A)^{1/A} \leq e$, $(1-A)^{1/A} \leq \frac{1}{e}$, and that

$$\int_M e^{\alpha\mu^2} dm = \sum_{n=2}^{\infty} \alpha^n \frac{K(2n)}{n!},$$

we have:

Corollary 2. *$e^{\alpha\mu^2}$ is integrable for $\alpha < \frac{1}{\rho}$. Moreover,*

$$\int_M e^{\alpha\mu^2} dm \leq 1 + \sqrt{2\pi} e^{-\tau} \; \sum_{n=1}^{\infty} \; \frac{\sqrt{n+1}}{n} \; e^{c\sqrt{n/\pi}} (\alpha\rho)^n,$$

where c is as in Theorem 1.

There is a somewhat simpler route for establishing only the integrability of $e^{\alpha\mu^2}$ without an effective bound — we present this at the end of the paper. For now, we begin the proof proper of Theorem 1. We return then to our defective log–Sobolev inequality; replace f by $\mu^{\lambda/2}$, $\lambda \geq 2$, with μ a smooth nonnegative compactly supported function of Lip 1 norm at most one. We obtain

$$\begin{aligned}\frac{\rho\lambda^2}{4} K(\lambda - 2) &\geq \lambda \int \mu^\lambda \ln \mu dm - K(\lambda) \ln K(\lambda) - \tau K(\lambda) \\ &= \lambda K'(\lambda) - K(\lambda) \ln K(\lambda) - \tau K(\lambda)\end{aligned}$$

It is well known that $\ln K(\lambda)$ is convex. Applying then the monotonicity of its derivative, or simply using the convexity of $x \ln x$, one readily obtains:

$$K'(\lambda) \equiv \int \mu^\lambda \ln \mu dm \geq K(\lambda) \frac{\ln K(\lambda) - \ln K(\lambda - 2)}{2}.$$

Using this in the previous inequality gives:

Lemma 3. $\frac{\rho\lambda^2}{2}K(\lambda-2) \geq (\lambda-2)K(\lambda)\ln K(\lambda) - \lambda K(\lambda)\ln K(\lambda-2) - 2\tau K(\lambda).$

Next define

$$L(\lambda) = e^{\tau}K(\lambda).$$

[If we replace $K(\lambda)$ by $e^{-\tau}L(\lambda)$, we can "eliminate" the defect.] ($L(\lambda)$ is still log–convex.) Then

$$\frac{\rho\lambda^2}{2}L(\lambda-2) \geq (\lambda-2)L(\lambda)\ln L(\lambda) - \lambda L(\lambda)\ln L(\lambda-2),$$

and holding $\lambda > 2$, a little more manipulation yields

$$\rho\ \frac{\lambda^2}{2(\lambda-2)}\ L(\lambda-2)^{2/\lambda-2} \geq \frac{L(\lambda)}{L(\lambda-2)^{\lambda/\lambda-2}}\ \ln\ \frac{L(\lambda)}{L(\lambda-2)^{\lambda/\lambda-2}}. \tag{4}$$

The function $x\ln x$ is monotone increasing for $x \geq 1$, mapping into $[0,\infty)$. Let F be the inverse function. Thus from (4) we readily obtain (paying some attention to the graph of $x\ln x$ for $0 < x \leq 1$):

Lemma 4. $L(\lambda) \leq L(\lambda-2)^{\lambda/\lambda-2}F\Big(\rho\ \frac{\lambda^2}{2(\lambda-2)}\ L(\lambda-2)^{-2/\lambda-2}\Big).$

It is readily proved by elementary calculus that $x^aF(Ax^{1-a})$ is increasing (if $a, A > 0$). Thus, if $M(\lambda) \geq L(\lambda)$ for $0 < \lambda \leq 2$, the dynamical system recursively determining $M(\lambda)$ in stretches of length 2 from the preceding stretch of length 2 by the formula

$$M(\lambda) = M(\lambda-2)^{\lambda/\lambda-2}F\Big(\rho\ \frac{\lambda^2}{2}\ M(\lambda-2)^{-2/\lambda-2}\Big)$$

will give an $M(\lambda)$ dominating $L(\lambda)$ for all $\lambda > 0$. We have not explored this avenue, and pursue it no further. Instead, we will linearize the convex function $x\ln x$ about the fixed point of the map $x \to x\ln x$, and obtain

$$x\ln x \geq 2x - e.$$

If we use this inequality in the left–hand side of (4), we get

Lemma 5. $L(\lambda) \leq \frac{e}{2}L(\lambda-2)^{\lambda/\lambda-2} + \rho\ \frac{\lambda^2}{4(\lambda-2)}\ L(\lambda-2).$

The inequality, holding for $\lambda > 2$, obviously holds for $\lambda = 2$ as well. Next replace $L(\lambda)$ by $U(\lambda)\lambda^{\lambda/2}e^{A\lambda}$, where $A = \frac{1}{2}\ln\rho - \frac{1}{2}\ln 2 - \frac{1}{2}$ to get

$$U(\lambda) \leq \frac{e}{2}\Big(1-\frac{2}{\lambda}\Big)^{\lambda/2}U(\lambda-2)^{\lambda/\lambda-2} - \frac{e}{2}\Big(\frac{\lambda}{\lambda-2}\Big)^2\Big(1-\frac{2}{\lambda}\Big)^{\lambda/2}U(\lambda-2).$$

Using the obvious fact that $(1-x)^{1/x}$ increases to $1/e$ as x decreases to zero, the last gives

$$U(\lambda) \leq \frac{1}{2}U(\lambda-2)^{\lambda/\lambda-2} + \frac{1}{2}\Big(\frac{\lambda}{\lambda-2}\Big)^2U(\lambda-2).$$

(We could have done slightly better here, since in fact $(1-x)^{1/x-1/2}$ also increases to $1/e$.) Next, setting $a_n = U(2n)$, we have:

Lemma 6. $a_n \leq \frac{1}{2}\, a_{n-1}^{n/n-1} + \frac{1}{2}\Big(\frac{n}{n-1}\Big)^2 a_{n-1}$.

Now we will manipulate the last inequality to get suitable estimates of the growth of a_n. First, setting $b_n = a_n^{1/n}$, we have

$$\frac{a_n}{a_{n-1}} \leq \frac{1}{2}\Big(\frac{n}{n-1}\Big)^2 + \frac{1}{2}b_{n-1},$$

whence

$$b_n^n = a_n \leq \Big[\frac{1}{2}\Big(\frac{n}{n-1}\Big)^2 + \frac{1}{2}(b_{n-1})\Big] \times \cdots \times \Big[\frac{1}{2}\Big(\frac{2}{1}\Big)^2 + \frac{1}{2}b_1\Big] a_1,$$

and then, using the inequality of the arithmetic and geometric mean,

$$b_n \leq \frac{\frac{1}{2}\sum_1^{n-1} b_v + \frac{1}{2}\sum_1^{n-1}\Big(\frac{v+1}{v}\Big)^2 + b_1}{n}.$$

If the last inequality is replaced by an equality, and we start at the same or a larger value of b_1, the new sequence dominates the original. Thus, for the purpose of estimating from above, we may suppose that

$$nb_n = \frac{1}{2}\sum_{v=1}^{n-1}\Big[b_v + \Big(\frac{v+1}{v}\Big)^2\Big] + b_1,$$

whence

$$nb_n = \Big(n - \frac{1}{2}\Big)b_{n-1} + \frac{1}{2}\Big(\frac{n}{n-1}\Big)^2, \quad (n > 1).$$

Putting $b_n = \dfrac{(2n)!}{2^{2n} n!^2} c_n =: \mu_n c_n$, we obtain

$$c_n = c_{n-1} + \frac{1}{2}\Big(\frac{n}{n-1}\Big)^2 \frac{1}{n\mu_n}$$

or

$$c_n = c_1 + \frac{1}{2}\sum_{v=1}^{n-1}\Big(\frac{v+1}{v}\Big)^2 \frac{1}{(v+1)\mu_{v+1}},$$

where

$$c_1 = \frac{2e^{\tau+1}K(2)}{\rho}.$$

By careful tracking of the error terms in Stirling's inequality for the gamma function, it is possible to get good asymptotic upper bounds for c_n. We prefer only rough asymptotics, using

$$n! = \sqrt{2\pi}n^{n+1/2}e^{-n}\Big(1 + O\Big(\frac{1}{n}\Big)\Big),$$

whence

$$\mu_n = \frac{(2n)!}{2^{2n}n!^2} = \frac{1}{\sqrt{\pi n}}\Big(1 + O\Big(\frac{1}{n}\Big)\Big),$$

yielding

$$c_n \leq c_1 + \sqrt{\pi n} + d, \ \ d \text{ an absolute constant},$$

and also

$$b_n \leq 1 + \frac{c}{\sqrt{\pi n}}, \quad c = c_1 + d',\ d' \text{ also an absolute constant.}$$

Since $K(2n) \leq \dfrac{e^{-\tau} n^n \rho^n}{e^n} b_n^n$, we finally obtain our originally announced Theorem 1.

As we noted earlier, there is a straightforward way to get integrability of $e^{\alpha\mu^2}$, indeed an estimate of the values of the integral $\int e^{\alpha\mu^2} dm$, in terms of earlier values, given a defective log–Sobolev inequality, which uses a simple extension of familiar arguments. Namely, put $f = e^{\frac{\lambda}{2}\mu^2}$ in the defective log–Sobolev inequality and set $P(\lambda) = \int e^{\lambda\mu^2} dm$. Since

$$\int |\nabla f|^2 \leq \lambda^2 \int u^2 e^{\lambda\mu^2} dm = \lambda^2 P'(\lambda) \quad (\text{because } |\nabla\mu|_\infty \leq 1),$$

and

$$\int |f|^2 \ln |f|^2 = \lambda P'(\lambda),$$

one obtains the inequality

$$(\lambda - \rho\lambda^2) P'(\lambda) \leq P(\lambda) \ln P(\lambda) + \tau P(\lambda).$$

Since $P(\lambda) \geq 1$, we may write

$$\frac{P'(\lambda)}{P(\lambda) \ln P(\lambda) + \tau P(\lambda)} \leq \frac{1}{\lambda - \rho\lambda^2} = \frac{1}{\lambda} + \frac{\rho}{1 - \rho\lambda}.$$

Integrating both sides from ε to λ ($\varepsilon < \lambda < \frac{1}{\rho}$) gives

$$\ln(\tau + \ln P(\lambda)) - \ln(\tau + \ln P(\varepsilon)) \leq \ln \frac{\lambda}{\varepsilon} - \ln \frac{1 - \rho\lambda}{1 - \rho\varepsilon}$$

or

$$\frac{\tau + \ln P(\lambda)}{\tau + \ln P(\varepsilon)} \leq \frac{\lambda}{1 - \rho\lambda} \cdot \frac{1 - \rho\varepsilon}{\varepsilon}.$$

The last inequality may be rewritten as

$$\ln P(\lambda) \leq \frac{\lambda - \varepsilon}{\varepsilon(1 - \rho\lambda)}\ \tau + \frac{\lambda}{1 - \rho\lambda}\ \frac{1 - \rho\varepsilon}{\varepsilon} \ln P(\varepsilon).$$

Ledoux has shown [4] that $P(\varepsilon)$ exists for sufficiently small ε if there is a defective log–Sobolev inequality. Hence we have another demonstration of the desired integrability of $e^{\alpha\mu^2}$.

It is interesting to note that this inequality implies $(1 - \rho\lambda) \ln P(\lambda)$ stays bounded as $\lambda \to \frac{1}{\rho}$.

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