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ON JB^*-TRIPLES WHICH ARE M-IDEALS IN THEIR BIDUALS

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ABSTRACT. The object of this paper is to investigate JB^*-triples which are M-ideals in their biduals.

M-ideals in their biduals is a subject of current interest (see [9], [10], [11], [12], [14]), c_0 and the compact operators on a Hilbert space being representative examples.

Let us recall that a Banach space X is an M-ideal in its bidual (in short, M-ideal) if

$$\|\varphi\| = \|\pi(\varphi)\| + \|\varphi - \pi(\varphi)\|, \text{ for all } \varphi \in X^{***},$$

where π is the canonical projection of X.

M-ideals can also be characterized by intersection properties of balls. For further information we refer to [2]. In particular, Å. Lima proved (see e.g. [12, p. 43] that a Banach space X is M-ideal in its bidual if, and only if, X has the 2-ball property in its bidual.

On the other hand, it is known (see [14, Theorem 2.6]) that Banach spaces which are M-ideals in their biduals are Asplund spaces (see [5, Chapter VII, Section 5] for a definition).

A JB^*-triple is a complex Banach space J together with a continuous triple product $\{.,.,.\} : J \times J \times J \longrightarrow J$, which satisfies:

1. $\{x, y, z\}$ is bilinear and symmetric in x and z and conjugate-linear in y.
2. The Jordan identity
$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$
3. For each $x \in J$, the operator $x\Delta x : J \longrightarrow J$ defined by $x\Delta x(y) = \{x, x, y\}$ is a hermitian operator with non-negative spectrum.
4. $\|x\Delta x\| = \|x\|^2$, for each $x \in J$.

As an example, any C^*-algebra with the triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ is a JB^*-triple.

We recall that the bidual J^{**} of a JB^*-triple J is in a natural way a JB^*-triple containing J as a subtriple (see [6]).

An (algebraic) ideal in a JB^*-triple J is a complex subspace F satisfying $\{x, y, F\} \subseteq F$ and $\{x, F, y\} \subseteq F$, for all $x, y \in J$. Observe that it is enough to take $x = y$ in this definition.

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For the statement of our main result, let us fix some notation.

Given a closed subspace X of a Banach space Y, and $y \in Y$, we write $P_X(y)$ for the set of best approximants of y in X:

$$P_X(y) = \{x \in X : \|x - y\| = \|y + X\|\}.$$

Also for $x \in X$ and $\epsilon > 0$, $B_X^\circ(x, \epsilon)$ will mean the open ball in X with center at x and radius ϵ.

In what follows, we identify, if there is no ambiguity, a Banach space Z with $j_Z(Z)$ and Z° with $j_Z(Z)^\circ$, where j_Z denotes the natural injection of Z into Z^{**}.

We say that the natural projection $Z^{***} \longrightarrow Z^*$ is of *best approximation* if, for every $\varphi \in Z^{***}$, we have that $\pi(\varphi) \in P_{Z^*}(\varphi)$.

Now, we state and prove our main result.

Theorem 1. *Let J be a JB^*-triple. The following assertions are equivalent:*

1) *J is an (algebraic) ideal in J^{**}.*
2) *J is an M-ideal in J^{**}.*
3) *There exists $t > 1$ such that*

$$B_J^\circ(0, t\|F + J\|) \subseteq P_J(F) - P_J(F)$$

for all $F \in J^{**}$.

4) *There exists $\epsilon > 0$ such that*

$$\|w\| + \epsilon\|f\| \leq \|w + f\|$$

for all $w \in J^\circ$ and $f \in J^*$.

5) *J is an Asplund space and the natural projection $J^{***} \longrightarrow J^*$ is of best approximation.*

The following results are crucial for the proof of Theorem 1.

Lemma 2 ([3, Lemma 1])**.** *Let X be a closed subspace of a Banach space Y and $t \geq 0$. If*

$$B_X^\circ(0, t\|y + X\|) \subseteq P_X(y) - P_X(y), \forall y \in Y,$$

then

$$\|w\| \leq \|h\| + (1 - t)\|h + X^\circ\|, \forall h \in Y^*, w \in P_{X^\circ}(h).$$

Lemma 3 (see [4, Proposition 2.5 and Theorem 4.2])**.** *Let X be a Banach space and $\epsilon > 0$ such that*

$$\|w\| + \epsilon\|f\| \leq \|w + f\|, \forall w \in X^\circ, f \in X^*.$$

Then, the following statements are true:

1. *For all $\varphi \in X^{***}$, $P_{X^*}(\varphi) = \{\pi(\varphi)\}$.*
2. *X is an Asplund space.*
3. *If Z is a Banach space for which the natural projection π_Z is of best approximation and I is any isometric linear mapping from X^{**} onto Z^{**}, then I is the bitranspose of an isometric linear mapping from X onto Z.*

Proof of Lemma 3. 1. Let $\varphi \in X^{***}$ and $x^* \in X^*$ with $\pi(\varphi) \neq x^*$. Then

$$\|\varphi - x^*\| \geq \|(\varphi - x^*) - \pi(\varphi - x^*)\| + \varepsilon\|\pi(\varphi - x^*)\| > \|\varphi - \pi(\varphi)\|.$$

Therefore,

$$P_{X^*}(\varphi) = \{\pi(\varphi)\}, \ \forall \varphi \in X^{***}.$$

2. If Z is a separable subspace of X, again Z satisfies (cf. [12, p.111])

$$\|\chi\| \geq \|\chi - \pi_Z(\chi)\| + \varepsilon\|\pi_Z(\chi)\|, \ \forall \chi \in Z^{***},$$

where π_Z is the canonical projection of Z.

On the other hand, it is clear that, for every $\chi \in Z^{***}$,

$$\|\chi - 2\pi_Z(\chi)\| \leq \|\chi - \pi_Z(\chi)\| + \varepsilon\|\pi_Z(\chi)\| + (1-\varepsilon)\|\pi_Z(\chi)\| \leq (2-\varepsilon)\|\chi\|,$$

and so, by [8, Proposition 2.8], Z^* is separable, that is, X is an Asplund space.

3. Let $I : X^{**} \to Z^{**}$ be an isometric isomorphism. Since X and Z contain no copy of l_1 (see [8, Proposition 2.6]), by [8, Lemma 5.6] and [7, Corollary 5.5], I is w^*-w^*-continuous. In particular, $I^*(Z^*) = X^*$. It is clear that

$$\|\chi + Z^*\| = \|I^*(\chi) + X^*\|,$$

for all $\chi \in Z^{***}$ (of course, $\pi_Z(\chi) \in P_{Z^*}(\chi)$), and so,

$$I^*\pi_Z = \pi I^*.$$

Hence,

$$I^*(Z^\circ) = X^\circ.$$

Therefore, by the Hahn-Banach theorem, $I(X) = Z$. Now, we can define $H : X \to Z$ by

$$H(x) = j_Z^{-1} I j_X(x), \ \forall x \in X.$$

The operator H is continuous and H^{**} coincides with I on X. Since both operators are w^*-w^*-continuous, $H^{**} = I$. □

Proof of Theorem 1. The equivalence 1) ⇔ 2) follows from the well-known fact that the closed ideals of a JB^*-triple J are precisely the M-ideals of J (see [1, Theorem 3.2]). The implications 2) ⇒ 3) ⇒ 4) ⇒ 5) have sense for any Banach space J and actually are true in this more general context. Indeed, 2) ⇒ 3) (with $t = 2$) have been proved in [14, Theorem 1.2]. Next, we show that 3) ⇒ 4). Assume that π is the natural projection of J. Note that

$$\|\varphi - (\varphi - \pi(\varphi))\| = \|\pi(\varphi)\| = \|\pi(\varphi - w)\| \leq \|\varphi - w\|, \forall \varphi \in J^{***}, w \in J^\circ$$

so,

$$\varphi - \pi(\varphi) \in P_{J^\circ}(\varphi) \text{ and } \|\pi(\varphi)\| = \|\varphi + J^\circ\|.$$

(In particular, if $w \in J^\circ$, then $w \in P_{J^\circ}(f + w), \forall f \in J^*$.) Therefore, by Lemma 2, taking $\epsilon = t - 1$, $X = J$ and $Y = J^{**}$, the implication 3) ⇒ 4) follows.

The implication 4) ⇒ 5) follows from the assertions 1 and 2 of Lemma 3.

Finally, assume that condition 5) holds for J. Since J^* is the predual of the JBW^*-triple J^{**} and J^* has the RNP, it follows from [13, Theorem 11] that J^* is isometric to the dual space of a JB^*-triple X which is an ideal (so an M-ideal) in its bidual. Now applying to X the proved implication 2) ⇒ 4) (with $Z = J$) and the assertion 3 of Lemma 3, X is isometric to J, so J is an M-ideal in J^{**}, and so 2) holds. □

Remark 4. 1. Theorem 1 is true if we change the JB^*-triple J by a noncommutative JB^*-algebra A and 1) by "A is a two-sided ideal in its bidual".

Let us recall that an n. c. (not necessarily commutative) JB^*-algebra is a complex Banach space A which is also a complex n. c. Jordan algebra (i.e. $(ab)a =$

$a(ba)$ and $(a^2b)a = a^2(ba), \forall a, b \in A)$ equipped with a conjugate linear algebra involution * such that

$$\|ab\| \leq \|a\|\|b\| \text{ and } \|a(a*a) + (a*a)a - a*a^2\| = \|a\|^3, \forall a, b \in A.$$

If A is an n. c. JB^*-algebra, then A can be regarded as a JB^*-triple under suitable triple product and the given norm. Since, for n. c. JB^*-algebras, M-ideals and closed two-sided ideals are the same [15, Theorem 4] and the bidual of any JB^*-algebra is a JB^*-algebra containing the given one as a subalgebra [15, Theorem 1.7], the above announced result follows directly from our theorem.

2. Assertions 2), 3) are not equivalent in an arbitrary Banach space, in fact we have the following

Proposition 5. *For every $r \in [0, 2[$, there is a Banach space X failing to be M-ideal and satisfying*

$$B_X^\circ(0, r\|x^{**} + X\|) \subseteq P_X(x^{**}) - P_X(x^{**}), \forall x^{**} \in X^{**}.$$

Proof. In the first place, we recall (see [14, Theorem 1.2 and Proposition 1.5]) that a Banach space X is an M-ideal if, and only if,

$$B_X^\circ(0, 2\|x^{**} + X\|) \subseteq P_X(x^{**}) - P_X(x^{**}), \forall x^{**} \in X^{**}, \tag{1}$$

in particular (cf. [17, Theorem 4] and [16, Proposition 3]), X is a proximinal subspace in X^{**}, and, for every $x^{**} \in X^{**}$ and $x \in X$,

$$\|x^{**} - x\| = \|x^{**} + X\| + d(x, P_X(x^{**})) \tag{2}$$

holds.

Let us consider l_∞ with the usual norm $\|\cdot\|$ and denote, for every $F \in l_\infty$,

$$| F | = \|F + c_0\| \text{ and } P(F) = P_{c_0}(F).$$

Let $0 \leq t \leq 1$ and consider in $c_0 \times c_0$ the following norm:

$$\|(x, y)\|_t = \max\{\|x\|, \|y\| + t\|x\|, (1+t)\|y\|\}, \forall x, y \in c_0.$$

We will denote $X_t = (c_0 \times c_0, \|\cdot\|_t)$.

It is easy to show the following assertions:

1. $l_\infty \times l_\infty$ with the norm

$$\|(F, G)\|_t = \max\{\|F\|, \|G\| + t\|F\|, (1+t)\|G\|\}, \forall F, G \in l_\infty$$

is the bidual of X_t.

2. For every $(F, G) \in X_t^{**}$, we have that

$$\|(F, G) + X_t\|_t = \max\{| F |, | G | + t | F |, (1+t) | G |\}.$$

3. $P(F) \times P(G) \subseteq P_t(F, G), \forall F, G \in l_\infty$, where $P_t(F, G) = P_{X_t}(F, G)$.

We will need the following technical lemmas.

Lemma 6. *X_t is a proximinal subspace in its bidual and satisfies the following property:*

$$B_{X_t}^\circ(0, 2(1-t)\|(F, G) + X_t\|_t) \subseteq P_t(F, G) - P_t(F, G), \forall F, G \in l_\infty.$$

Proof. The case $t = 0$ is trivial. Let $t > 0$ and $x, y \in c_0$ satisfy

$$\|(x, y)\|_t < 2(1-t)\|(F, G) + X_t\|_t.$$

It is clear that one of the following assertions holds:

1. $| F | \leq | G |$.
2. $| G | + t | F | \geq | F | > | G |$.
3. $| F | > | G | + t | F |$.

Case 1. ($\|(F, G) + X_t\|_t = (1 + t) | G |$).

By assumption on (x, y),

$$\|y\| < 2 | G |$$

(and so, by (1), there are $y_1, y_2 \in P(G)$ with $y = y_1 - y_2$), and

$$\|x\| < 2(1 - t^2) | G | .$$

If $(1-t^2) | G | < | F |$, then, by (1), there are $x_1, x_2 \in P(F)$ such that $x = x_1 - x_2$, and so, $(x_i, y_i) \in P(F) \times P(G) \subseteq P_t(F, G)$, for $i = 1, 2$.

In the other case, if $\alpha = \frac{|F|}{(1-t^2)|G|}$, then, by (1), there are $z_1, z_2 \in P(F)$ such that $\alpha x = z_1 - z_2$.

Let $x_1 = z_1 + \frac{1-\alpha}{2} x$ and $x_2 = z_2 - \frac{1-\alpha}{2} x$. It is clear that $x = x_1 - x_2$ and

$$\|F - x_i\| \leq \|F - z_i\| + \frac{1 - \alpha}{2} \|x\|$$

$$\leq | F | + (1 - \alpha)(1 - t^2) | G | = (1 - t^2) | G |, \text{ for } i = 1, 2$$

and so,

$$\|(F, G) - (x_i, y_i)\|_t \leq (1 + t) | G |, \text{ for } i = 1, 2;$$

that is, in any case, $(x_i, y_i) \in P_t(F, G)$, for $i = 1, 2$, as required.

Case 2. ($\|(F, G) + X_t\|_t = | G | + t | F |$).

By assumption,

$$(1 + t)\|y\| < 2(1 - t)(| G | + t | F |) \leq 2 | G |,$$

and

$$\|x\| < 2(1 - t)(| G | + t | F |) \leq 2 | F | .$$

Therefore, by (1), $(x, y) \in P(F) \times P(G) - P(F) \times P(G) \subseteq P_t(F, G) - P_t(F, G)$, as required.

Case 3. ($\|(F, G) + X_t\|_t = | F |$).

Again, by assumption on (x, y),

$$\|x\| < 2 | F |$$

(and so, by (1), there are $x_1, x_2 \in P(F)$ such that $x = x_1 - x_2$), and

$$\|y\| < 2(1 - t) | F | .$$

If $\alpha = \frac{|G|}{(1-t)|F|}$, then, by (1), there are $z_1, z_2 \in P(G)$ such that $\alpha y = z_1 - z_2$.

Let $y_1 = z_1 + \frac{1-\alpha}{2} y$ and $y_2 = z_2 - \frac{1-\alpha}{2} y$. It is clear that $y = y_1 - y_2$ and

$$\|(F, G) - (x_i, y_i)\|_t \leq | F |, \text{ for } i = 1, 2;$$

that is, $(x_i, y_i) \in P_t(F, G)$, for $i = 1, 2$, as required. □

Lemma 7. *X_t is an M-ideal if, and only if, $t = 0$.*

Proof. If $t = 0$, then X_0 is an M-ideal.

Suppose that $t > 0$. Let $(F, G) \in X_t^{**}$ such that

$$0 \in P(F) \setminus P(G) \text{ and } \mid G \mid +t \mid F \mid \geq \mid F \mid > \|G\|.$$

Then,

$$\|(F, G) + X_t\|_t = \mid G \mid +t \mid F \mid \text{ and } \|(F, G)\|_t = \|G\| + t\|F\|.$$

In this case, it is easy to show that

$$P_t(F, G) = P(F) \times P(G),$$

and

$$d_t(0, P_t(F, G)) \geq (1 + t)d(0, P(G)).$$

Therefore, by (2), we have that

$$\|(F, G) + X_t\|_t + d_t(0, P_t(F, G))$$

$$\geq \mid G \mid +t \mid F \mid +(1 + t)d(0, P(G))$$

$$= \|G\| + t \mid F \mid +td(0, P(G)) > \|(F, G)\|_t.$$

In particular, again by (2), X_t is not an M-ideal. □

Now, to conclude the proof of Proposition 5, it is enough to take $r = 2(1 - t)$, with $0 < t \leq 1$. □

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