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ON THE MERGELYAN APPROXIMATION PROPERTY ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

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ABSTRACT. Let Ω be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. We prove the Mergelyan approximation property in various topologies on Ω when the estimates for $\overline{\partial}$-equation are known in the corresponding topologies.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $H(\Omega)$ denote the functions holomorphic on Ω. We say Ω has the Mergelyan property if every $f \in H(\Omega) \cap C(\overline{\Omega})$ can be approximated uniformly on $\overline{\Omega}$ by functions in $H(\overline{\Omega})$, where $H(\overline{\Omega})$ denotes the functions holomorphic in a neighborhood of $\overline{\Omega}$.

For $n > 1$, Henkin [9], Kerzman [10] and Lieb [11] proved the Mergelyan property on strongly pseudoconvex domains in $\mathbb{C}^n$. The key technical tools needed to prove the Mergelyan property were:

(1) existence of a Stein neighborhood basis for $\overline{\Omega}$, and
(2) Lipschitz estimates and their stability for $\overline{\partial}$ on this Stein neighborhood basis.

Notice that the author has constructed a Stein neighborhood basis of $\overline{\Omega}$ when Ω is a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$ [3].

Let $\Lambda^\alpha(\Omega)$ denote the usual Lipschitz space of order $\alpha \geq 0$ with norm $\| \ \|_{\Lambda^\alpha(\Omega)}$ and let $L^p_k(\Omega)$ denote the space of functions on Ω that are in $L^p(\Omega)$ along with all their derivatives up to order k, with norm denoted by $\| \ \|_{L^p_k(\Omega)}$. Lipschitz estimates for $\overline{\partial}$ on pseudoconvex domains in $\mathbb{C}^n$ are known for some special kinds of domains; that is, smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$ [2], [7], and smoothly bounded pseudoconvex domain Ω of finite type in $\mathbb{C}^n$ such that the Levi-form of $b\Omega$ is diagonizable [8], etc. However, it is difficult and sometimes tedious to prove the stability of the estimates for $\overline{\partial}$ even though the estimates are known [6], [12].

In this paper, we will present a new method to prove the Mergelyan property in various topologies. That is, if $\overline{\Omega}$ has a Stein neighborhood basis with the estimates for $\overline{\partial}$ in $\Lambda^\alpha(\Omega)$, $\alpha \geq 0$, or in $L^p_k(\Omega)$, $1 < p < \infty$, $k \geq 0$, we will prove the Mergelyan

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property in the corresponding topologies. Here we are not assuming the stability of the estimates for $\overline{\partial}$ in the corresponding spaces. Nevertheless, we will use the known result of the stability of estimates for $\overline{\partial}$ in $L^2_k(\Omega)$ spaces [4].

We will state and prove our results only on smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$. The same (or similar) results hold for the domains of finite type in $\mathbb{C}^n$ with estimates for $\overline{\partial}$ in the corresponding spaces.

Theorem 1. *Let Ω be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$. Assume $f \in H(\Omega) \cap L^p_k(\Omega)$, where $1 < p < \infty$ and k is a non-negative integer. Then there is a sequence $\{g_n\} \subset H(\overline{\Omega})$ such that*

$$\lim_{n\to\infty} \|g_n - f\|_{L^p_k(\Omega)} = 0.$$

Let $\mathbb{N}$ denote the set of natural numbers.

Theorem 2. *Let Ω be as in Theorem 1 and assume $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$. Then for each $\alpha' < \alpha$ ($\alpha' = \alpha$ if $\alpha \in \{0\} \cup \mathbb{N}$) arbitrarily given, there is a sequence $\{g_n\} \subset H(\overline{\Omega})$ such that*

$$\lim_{n\to\infty} \|g_n - f\|_{\Lambda^{\alpha'}(\Omega)} = 0.$$

Remark 3. In [5], the author proved that every $f \in H(\Omega) \cap L^2_k(\Omega)$ can be approximated by functions in $H(\overline{\Omega})$ in the $L^2_k(\Omega)$ topologies, $k \geq 0$, when Ω is a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. When $n = 2$, Cho and others [6] also proved the Mergelyan property in Lipschitz spaces $\Lambda^\alpha(\Omega)$, $0 \leq \alpha < 1/m$, for $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$. Here m is the type of $b\Omega$. Both of these results depend on the solvability and stability of the estimates for $\overline{\partial}$ in the corresponding topologies [4], [6].

The key ingredients needed to prove Theorem 1 and Theorem 2 are the $L^p_k(\Omega)$ and Lipschitz estimates for $\overline{\partial}$ on $\Omega \Subset \mathbb{C}^2$ [2], [7], [8], and the smooth bumping theorem for pseudoconvex domains of finite type in $\mathbb{C}^n$ [2], [3], [4].

2. Approximation by smooth functions

In this section, we prove that any holomorphic function in $L^p_k(\Omega)$ or $\Lambda^\alpha(\Omega)$ can be approximated by smooth functions on $\overline{\Omega}$ in appropriate topologies.

Let U_j, $j = 0, 1, \ldots, N$, be a finite collection of open sets with the following properties:

(a) $\overline{\Omega} \subset \bigcup_{j=0}^N U_j$.
(b) $U_0 \Subset \Omega$.
(c) On each U_j, $j = 1, 2, \ldots, N$, there are holomorphic coordinates z_1^j, z_2^j with the property that $\partial r/\partial x_2^j > 0$, where $z_2^j = x_2^j + iy_2^j$.

Let ζ_j, $j = 0, 1, \ldots, N$, be a partition of unity subordinate to the covering $\{U_j\}$. For all sufficiently small $\delta > 0$, and for a given function $f \in H(\Omega)$, let f_δ be given by

$$f_\delta(z) = \zeta_0(z) f(z) + \sum_{j=1}^N \zeta_j(z) f(z_1^j, z_2^j - \delta). \tag{1}$$

Observe that $f_\delta \in C^\infty(\overline{\Omega})$.

Proposition 4. *Suppose that* $f \in H(\Omega) \cap L^p_k(\Omega)$, $1 < p < \infty$, $k \geq 0$. *Then* $\|f_\delta - f\|_{L^p_k(\Omega)} \longrightarrow 0$ *and* $\|\overline{\partial} f_\delta\|_{L^p_k(\Omega)} \longrightarrow 0$ *as* $\delta \longrightarrow 0$.

Proof. Note that $D^\alpha(f - f_\delta) \in L^p(\Omega)$, $|\alpha| \leq k$. So $\|f_\delta - f\|_{L^p_k(\Omega)}$ converges to zero as δ does, by the Lebesgue dominated convergence theorem. Also note that

$$D^\alpha(\overline{\partial} f_\delta) = D^\alpha(\overline{\partial}(f_\delta - f)) = D^\alpha(\sum_{j=1}^N (\overline{\partial}\zeta_j)(f(z_1^j, z_2^j - \delta) - f(z_1^j, z_2^j))).$$

So $\|\overline{\partial} f_\delta\|_{L^p_k(\Omega)} \longrightarrow 0$ as $\delta \longrightarrow 0$ by the same reasoning. □

The functions $f_\delta \in C^\infty(\overline{\Omega})$ defined in (1) also approximate f in the $\Lambda^\alpha(\Omega)$-topology:

Proposition 5. *Suppose that* $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$. *Then for each* $\alpha' < \alpha$ ($\alpha' = \alpha$ *if* $\alpha \in \{0\} \cup \mathbb{N}$), $\|f_\delta - f\|_{\Lambda^{\alpha'}(\Omega)}$ *and* $\|\overline{\partial} f_\delta\|_{\Lambda^{\alpha'}(\Omega)}$ *converge to zero as* δ *does.*

Proof. It is clear that the conclution holds if $\alpha \in \{0\} \cup \mathbb{N}$. Without loss of generality, we may assume that $0 < \alpha' < \alpha < 1$. If x, y satisfy $\delta \leq |x - y|$, we use the Lipschitz continuity condition on $(f_\delta - f)(x)$ and $(f_\delta - f)(y)$, and if $\delta > |x - y|$, we use the same condition on $f(x) - f(y)$ and $f_\delta(x) - f_\delta(y)$. In both cases, we will get $\|f_\delta - f\|_{\Lambda^\alpha(\Omega)} \lesssim \delta^{\alpha - \alpha'}$. Since $\alpha' < \alpha$, this proves that $\|f_\delta - f\|_{\Lambda^\alpha(\Omega)}$ converges to zero as δ does. Similarly, we can prove that $\|\overline{\partial} f_\delta\|_{\Lambda^\alpha(\Omega)}$ converges to zero as δ does. □

Remark 6. If $\alpha \notin \mathbb{N} \cup \{0\}$, the functions f_δ defined in (1) does not converge to f in the $\Lambda^\alpha(\Omega)$ norm in general.

Definition 7. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain with C^∞ defining function r. By a smooth bumping family of Ω we mean a family of smoothly bounded pseudoconvex domains $\{\Omega_\tau\}_{0 \leq \tau \leq 1}$ satisfying the following properties:

(1) $\Omega_0 = \Omega$,
(2) $\Omega_{\tau_1} \Subset \Omega_{\tau_2}$ if $\tau_1 < \tau_2$,
(3) $\{b\Omega_\tau\}_{0 \leq \tau \leq 1}$ is a C^∞ family of real hypersurfaces in $\mathbb{C}^n$,
(4) the boundary defining functions r_τ of Ω_τ vary smoothly with respect to τ, and $r_\tau \to r$ as $t \to 0$ in the C^∞ topology.

In [3], the author constructed a smooth bumping family of Ω if $b\Omega$ is of finite type.

For the final remark of this section, we state the following theorem which gives the stability of L^2-estimates for $\overline{\partial}$ on Ω [4].

Theorem 8. *Let* $\{\Omega_\tau\}_{0 \leq \tau \leq 1}$ *be a smooth bumping family of pseudoconvex domains in* $\mathbb{C}^n$. *Then for each* m *there exists a constant* C_m, *independent of* τ, *such that*

$$\|f^\tau\|_m \leq C_m \|\Box_\tau f^\tau\|_m,$$

for all $f^\tau \in Dom(\Box_\tau) \cap C^\infty(\overline{\Omega}_\tau)$ *with* $f^\tau \perp H^{0,1}(\overline{\Omega}_\tau)$. *Here* $\Box_\tau$ *denotes the complex Laplacian on* Ω_τ.

3. Approximation by holomorphic functions

Let Ω be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$, and let N be the Neumann operator associated with the $\overline{\partial}$-Neumann problem. Then the main results in [2], [7] show that

$$\overline{\partial}^* N : L^p_k(\Omega) \longrightarrow L^p_k(\Omega), \quad 1 < p < \infty, \quad k \geq 0, \tag{2}$$

and

$$\overline{\partial}^* N : \Lambda^\alpha(\Omega) \longrightarrow \Lambda^\alpha(\Omega), \quad \alpha \geq 0, \tag{3}$$

are bounded operators on the corresponding spaces.

Now let us prove Theorem 1 and Theorem 2. Here we will only present a proof of Theorem 2. The proof of Theorem 1 follows the same (or similar) lines. Let us state Theorem 2 again:

Theorem 2. *Let Ω be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$. Assume $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$. Then for each $\alpha' < \alpha$ ($\alpha' = \alpha$ if $\alpha \in \{0\} \cup \mathbb{N}$) arbitrarily given, there is a sequence $\{g_n\} \subset H(\overline{\Omega})$ such that*

$$\lim_{n\to\infty} \|g_n - f\|_{\Lambda^{\alpha'}(\Omega)} = 0.$$

Proof. Let f_δ be defined by (1). By Proposition 5, $f_\delta \in C^\infty(\overline{\Omega})$ converge to f in the $\Lambda^{\alpha'}(\Omega)$ topology as δ goes to zero. For each fixed $\delta > 0$, let us solve $\overline{\partial} u_\delta = \overline{\partial} f_\delta$ on Ω. Then by Catlin's global regularity theorem for the $\overline{\partial}$-equation on pseudoconvex domains of finite type in $\mathbb{C}^n$ [1], it follows that $u_\delta \in C^\infty(\overline{\Omega})$. Also, by (3), u_δ satisfies

$$\|u_\delta\|_{\Lambda^{\alpha'}(\Omega)} \leq C_\alpha \|\overline{\partial} f_\delta\|_{\Lambda^{\alpha'}(\Omega)},$$

where C_α does not depend on δ. Since $\|\overline{\partial} f_\delta\|_{\Lambda^{\alpha'}(\Omega)}$ converges to zero as δ does, it follows that $\|u_\delta\|_{\Lambda^{\alpha'}(\Omega)} \to 0$ as $\delta \to 0$. Set

$$h_\delta = f_\delta - u_\delta.$$

Then $h_\delta \in H(\Omega) \cap C^\infty(\overline{\Omega})$ and

$$\lim_{\delta\to 0} \|h_\delta - f\|_{\Lambda^{\alpha'}(\Omega)} = 0. \tag{4}$$

Let $\{\Omega_\tau\}$ be a smooth pseudoconvex bumping family of Ω. We extend h_δ to Ω_τ and set it equal to h^τ_δ on Ω_τ. Notice that $\overline{\partial} h^\tau_\delta \equiv 0$ on Ω, and $\overline{\partial} h^\tau_\delta \in C^\infty(\overline{\Omega})$. Hence $\overline{\partial} h^\tau_\delta$ vanishes to order infinity on $b\Omega$ as $\tau \to 0$. Set $k = [\alpha] + 4$, where $[\alpha]$ denotes the greatest integer less than or equal to α. Let us solve the following $\overline{\partial}$-equation in $L^2_k(\Omega)$ spaces (with weighted estimates of $\overline{\partial}$):

$$\overline{\partial} p^\tau_\delta = \overline{\partial} h^\tau_\delta \quad \text{on} \quad \Omega_\tau.$$

From Theorem 8 (stability of $L^2_k(\Omega_\tau)$-estimates of $\overline{\partial}$-equation), it follows that

$$\|p^\tau_\delta\|_{L^2_k(\Omega_\tau)} \leq C_\alpha \|\overline{\partial} h^\tau_\delta\|_{L^2_k(\Omega_\tau)}, \tag{5}$$

where C_α does not depend on τ. Set

$$g^\tau_\delta = p^\tau_\delta - h^\tau_\delta \in H(\Omega_\tau).$$

Then (4), (5) and the Sobolev embedding theorem imply that

$$\|g^\tau_\delta - f\|_{\Lambda^{\alpha'}(\Omega)} \longrightarrow 0 \quad \text{as} \quad \delta, \tau \to 0.$$

This proves Theorem 2. □

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