PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 8, August 1998, Pages 2481–2491
S 0002-9939(98)04451-7

AN INDEX THEORY FOR $\mathbb{Z}$-ACTIONS

IN-SOOK KIM

(Communicated by Jozef Dodziuk)

ABSTRACT. This paper concerns an index theory for $\mathbb{Z}$-actions induced by a homeomorphism of a compact space. We give a definition of a genus for uniform spaces and prove that the genus for compact spaces is an index. To this end we show a $\mathbb{Z}$-version of the Borsuk-Ulam theorem and the existence of a continuous equivariant extension for these $\mathbb{Z}$-actions.

1. Introduction

An index theory for free $\mathbb{Z}_2$-actions is due to M.A. Krasnosel'skij and has many applications (cf. [12], [13]). The concept of genus for compact Lie groups G was introduced by E.R. Fadell [8]. In the case of $G = S^1$, the genus for complex Banach spaces is equivalent to the S^1-index.

The aim of this paper is to extend index theory for compact Lie groups to $\mathbb{Z}$-actions. The additional condition of Lyapunov stability enables us to give an index theory for $\mathbb{Z}$-actions induced by a homeomorphism of a compact space, i.e. almost periodic actions, where the compactness of the Lie group is not required.

In a dynamical system there can coexist a variety of orbit types. Considering almost periodic orbits of the same type in Section 3, we define a genus for uniform spaces, which is analogous to the geometrical S^1-index for Hilbert spaces of V. Benci [2]. A $\mathbb{Z}$-version of the Borsuk-Ulam theorem implies the dimension property of the genus. We show the existence theorem about continuous equivariant extensions for $\mathbb{Z}$-actions, as the basis of further results in Section 4. The almost periodicity allows us to do that. Moreover, we observe the relation between rotation numbers by a continuous equivariant map. Therefore, we can prove in Section 5 that the genus for compact spaces is indeed an index introduced in Section 2.

Since the size of sets having desired properties is measured by an index, index theory plays an important role in differential equations and variational analysis (cf. [1], [3], [9], [10], [15]). This kind of application is one of the main reasons to study the index theory for $\mathbb{Z}$-actions.

2. Preliminaries

We consider a discrete dynamical system

$$\pi : \mathbb{Z} \times X \to X, \qquad (n, x) \mapsto f^n(x)$$

Received by the editors January 22, 1997.
1991 *Mathematics Subject Classification.* Primary 58G10, 58F27, 34D20, 58E40; Secondary 34C35.
Key words and phrases. Index, genus, almost periodic, Lyapunov stable, group actions.

induced by a homeomorphism $f : X \to X$ of a compact space X. This $\mathbb{Z}$-space will be denoted by (X, f). Throughout this paper, a compact space will be assumed to be a Hausdorff space.

Let (X, f) and (Y, g) be $\mathbb{Z}$-spaces. A map $\Phi : X \to Y$ is said to be *equivariant*, denoted by $\Phi : (X, f) \to (Y, g)$, if $\Phi \circ f = g \circ \Phi$. $\mathbb{Z}$-spaces (X, f) and (Y, g) are said to be *isomorphic*, denoted by $(X, f) \simeq (Y, g)$, if there is a homeomorphism $\Phi : (X, f) \to (Y, g)$.

A $\mathbb{Z}$-index is defined in the following setting. For a $\mathbb{Z}$-space (X, f), we denote

$$\mathcal{P}(X, f) := \{A \subset X \,:\, A \text{ is } f\text{-invariant, i.e. } f(A) = A\ \},$$

$$\overline{\mathcal{P}}(X, f) := \{A \in \mathcal{P}(X, f) \,:\, A \text{ is closed in } X\},$$

$$\overset{\circ}{\mathcal{P}}(X, f) := \{A \in \mathcal{P}(X, f) \,:\, A \text{ is open in } X\}.$$

Definition 1. A $\mathbb{Z}$-*index* for f is a map

$$i : \mathcal{P}(X, f) \to \mathbb{N}_0 \cup \{\infty\}$$

which has the following properties

(a) $i(A) = 0$ if and only if $A = \emptyset$.
(b) If $A, B \in \mathcal{P}(X, f)$ and if $\Phi : (A, f) \to (B, f)$ is a continuous map, then $i(A) \leq i(B)$.
(c) If $A \in \overline{\mathcal{P}}(X, f)$, then there exists a $U \in \overset{\circ}{\mathcal{P}}(X, f)$ such that $A \subset U$ and $i(A) = i(U)$. (Continuity)
(d) If $A, B \in \overline{\mathcal{P}}(X, f)$, then $i(A \cup B) \leq i(A) + i(B)$. (Subadditivity)

We introduce the concepts of almost periodicity and Lyapunov stability on uniform spaces $(X, \mathcal{U})$, where $\mathcal{U}$ is a uniformity on X.

Definition 2. A map $F : \mathbb{Z} \to X$ is said to be *almost periodic* if for every $M \in \mathcal{U}$,

$$D(M) := \ \{\, p \in \mathbb{Z} \,:\ (\,F(n), F(n + p)) \in M \quad \text{for all} \ \ n \in \mathbb{Z}\,\}$$

is relatively dense in $\mathbb{Z}$, i.e. there exists a finite set K in $\mathbb{Z}$ such that $D(M) + K = \mathbb{Z}$.

Let $f : X \to X$ be a bijection, and let A be a subset of X such that $f(A) = A$. The set A is said to be *almost periodic* with respect to f if for every $M \in \mathcal{U}$,

$$\{\, p \in \mathbb{Z} \,:\ (x, f^p(x)) \in M \quad \text{for all} \ \ x \in A\,\}$$

is relatively dense in $\mathbb{Z}$.

In this case, it is obvious that for each $x \in A$, the map $\mathbb{Z} \to A$, $n \mapsto f^n(x)$, is almost periodic. The set $O(x) := \{f^n(x) : n \in \mathbb{Z}\}$ will be called *an almost periodic orbit* of x.

Definition 3. Let $f : X \to X$ be a bijection, and let $A \subset X$ be a set such that $f(A) = A$. The set A is said to be *Lyapunov stable* with respect to f if for every $M \in \mathcal{U}$, there exists an $N \in \mathcal{U}$ such that for all $x, y \in A$ and for all $n \in \mathbb{Z}$

$$(x, y) \in N \ \text{ implies } \ \big(f^n(x), f^n(y)\big) \in M.$$

For a homeomorphism $f : X \to X$ on a compact space X, Lyapunov stability is equivalent to almost periodicity (cf. [18]). In this case, $\mathbb{Z}$-actions on (X, f) will be called *almost periodic actions*.

For example, taking $f : S^{2k-1} \to S^{2k-1}$, $f(x) := e^{2\pi i\alpha}x$ $(\alpha \in \mathbb{R}\setminus\mathbb{Q})$, we will denote the $\mathbb{Z}$-spaces (S^{2k-1}, f) often by $(S^{2k-1}, e^{2\pi i\alpha})$, in order to emphasize the almost periodicity of these canonical actions.

To show the subadditivity for an index, we need the following concept.

Let $\Delta^{n-1} := \{(t_1, \cdots, t_n) \in [0,1]^n : \sum_{i=1}^n t_i = 1\}$ be the standard $(n-1)$-simplex. For any topological spaces $Y_1, \cdots, Y_n$ we define the join

$$Y_1 * \cdots * Y_n := \Big(\Delta^{n-1} \times \prod_{i=1}^n Y_i\Big)/\!\sim$$

with the following equivalence relation:

$$(t_1, \cdots, t_n, y_1, \cdots, y_n) \sim (s_1, \cdots, s_n, y_1', \cdots, y_n')$$
$$\text{if } t_i = s_i \text{ and } (y_i = y_i' \text{ when } t_i \neq 0) \text{ for } i = 1, \cdots, n.$$

In the case of $Y_1, \cdots, Y_n$ compact, the quotient topology on $Y_1 * \cdots * Y_n$ coincides with the weakest topology for which the map

$$q : Y_1 * \cdots * Y_n \to \Delta^{n-1}, \quad \big[(t_1, \cdots, t_n, y_1, \cdots, y_n)\big] \mapsto (t_1, \cdots, t_n)$$

and partial functions

$$p_j : Y_1 * \cdots * Y_n \rightarrowtail Y_j, \quad \big[(t_1, \cdots, t_n, y_1, \cdots, y_n)\big] \mapsto y_j$$

are continuous.

Given $\mathbb{Z}$-spaces $Y_1, \ldots, Y_n$, the $\mathbb{Z}$-action on $\prod_{i=1}^n Y_i$ carries over in a canonical wise to $Y_1 * \cdots * Y_n$. For further information about joins we refer to R. Brown [5].

3. Main result

In this section we give a definition of a genus for uniform spaces and a main result that the genus for compact spaces is an index. Then we show a $\mathbb{Z}$-version of the Borsuk-Ulam theorem for the dimension and answer the question of when a restricted condition of orbits for this definition is satisfied.

Theorem 4. *Let X be a Hausdorff uniform space and let $f : X \to X$ be a continuous bijection. Let*

$$\begin{aligned}\Sigma(X, f) \ := \ & \{A \subset X : A \ f\text{-invariant, Lyapunov stable w.r.t. } f \text{ and } \big(\overline{O(x)}, f\big) \\ & \simeq (S^1, e^{2\pi i\alpha}) \text{ for all } x \in X \text{ and for some } \alpha \in [0,1] \cap (\mathbb{R}\setminus\mathbb{Q})\}.\end{aligned}$$

We define the genus $g : \Sigma(X, f) \to \mathbb{N}_0 \cup \{\infty\}$ by

$$\begin{aligned}g(A, f) \ := \ \min\{k \in \mathbb{N} \ & : \text{there exist an } m \in \mathbb{N} \text{ and a continuous map} \\ & \varphi : (A, f) \longrightarrow (S^{2k-1}, e^{2\pi i m\alpha})\}.\end{aligned}$$

for $A \in \Sigma(X, f) \setminus \{\emptyset\}$, $g(A, f) := \infty$ if such a map φ does not exist, and $g(\emptyset, f) := 0$.

If X is a compact space, then g is an index in the sense of Definition 1.

Theorem 14 below proves this main result, for which we make some preparations in the next section.

The following new result, which is a $\mathbb{Z}$-version of the Borsuk-Ulam theorem, tells us that $g\,(S^{2k-1}, f) = k$, where $f : S^{2k-1} \to S^{2k-1}, f(x) := e^{2\pi i\alpha}x$ for $x \in S^{2k-1}$ ($\alpha \in [0,1] \cap (\mathbb{R}\setminus\mathbb{Q})$). See [4] for the Borsuk-Ulam theorem.

Theorem 5. *Let $\alpha \in \mathbb{R}\setminus\mathbb{Z}$. For every $k, l \in \mathbb{N}$ with $k > l$ there is no continuous map $h : (S^{2k-1}, e^{2\pi i\alpha}) \longrightarrow (S^{2l-1}, e^{2\pi i\alpha})$.*

Proof. For $\alpha \in \mathbb{Q}\backslash\mathbb{Z}$ the theorem is well-known, e.g. [7]. Now let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Suppose that there is a continuous map $h : (S^{2k-1}, e^{2\pi i\alpha}) \longrightarrow (S^{2l-1}, e^{2\pi i\alpha})$ for $k, l \in \mathbb{N}$. Since $\overline{\{e^{2\pi in\alpha} : n \in \mathbb{Z}\}} = S^1$, there exists a sequence (n_j) of integers such that $\lim\limits_{j\to\infty} e^{2\pi in_j\alpha} = -1$. As h is continuous and equivariant, we have

$$h(-x) \ = \ h(\lim_{j\to\infty} e^{2\pi in_j\alpha}x) \ = \ \lim_{j\to\infty} e^{2\pi in_j\alpha}h(x) \ = \ -h(x)$$

for every $x \in S^{2k-1}$. The Borsuk-Ulam theorem implies that $k \leq l$. $\square$

We now state when almost periodic orbits have the same type. For the theory of a homeomorphism of S^1 into itself, due to H. Poincaré, we use the methods by topological dynamics and then consider a classification of almost periodic orbits in the number theoretic sense.

Definition 6. The set of n real numbers $\alpha_1, \cdots, \alpha_n$ is said to be *rationally dependent* if the relation $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$ holds for some rational numbers $c_1, \cdots, c_n$, not all zero.

Proposition 7. *Let g be a homeomorphism of S^1 onto itself with dense orbit. Then there exists an $\alpha \in [0,1] \cap (\mathbb{R}\backslash\mathbb{Q})$ such that $(S^1, g) \simeq (S^1, e^{2\pi i\alpha})$.*

The proof of this proposition can be found in N.G. Markley [14], J. de Vries [18].

Theorem 8. *Let $f : X \to X$ be a continuous bijection on a Hausdorff uniform space $(X,\mathcal{U})$. Let $A \subset X$ be connected, f-invariant and Lyapunov stable w.r.t. f such that $\overline{O(x)}$ is homeomorphic to S^1 for all $x \in A$. Then the following statements hold:*

(a) *For every $x \in A$, there is an $\alpha_x \in [0,1] \cap (\mathbb{R}\backslash\mathbb{Q})$ such that $(\overline{O(x)}, f) \simeq (S^1, e^{2\pi i\alpha_x})$.*
(b) *For $x, y \in A$, the set $\{1, \alpha_x, \alpha_y\}$ is rationally dependent.*
(c) *We can choose all $\alpha_x \in [0, \frac{1}{2}]$. If, for all pairs $x, y \in A$, every continuous map $p : \big(\overline{O(x)}, f\big) \to \big(\overline{O(y)}, f\big)$ is not a multiple covering map, then the α_x, $x \in A$, are equal.*

Proof. I. (a) For $x \in A$, let $\Phi_x : \overline{O(x)} \to S^1$ be a homeomorphism, and define a map $g_x : S^1 \to S^1$ by $g_x(y) := \Phi_x \circ f \circ \Phi_x^{-1}(y)$ for $y \in S^1$.

Since all assumptions of Proposition 7 are satisfied, there are a homeomorphism $\Psi_x : S^1 \to S^1$ and an $\alpha_x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$ such that $\Psi_x \circ g_x = e^{2\pi i\alpha_x} \circ \Psi_x$. Hence we have

$$(\Psi_x \circ \Phi_x) \circ f|_{\overline{O(x)}} \ = \ e^{2\pi i\alpha_x} \circ (\Psi_x \circ \Phi_x). \tag{1}$$

This proves statement (a).

We first provide two claims for the proof of statements (b) and (c).

II. For $M, N \subset X \times X$ and $a \in X$, we denote $M^{-1} := \{(x,y) \in X \times X : (y,x) \in M\}$, $M \circ M := \{(x,y) \in X \times X : (x,z) \in M$ and $(z,y) \in M$ for some $z \in X\}$ and $N(a) := \{x \in X : (a,x) \in N\}$.

Claim 1. Let $a \in A$. Then there exists an $N \in \mathcal{U}$ such that for every $u \in N(a) \cap A$, the set $\{1, \alpha_a, \alpha_u\}$ is rationally dependent.

Proof. Let $a \in A$. Let $M, W \in \mathcal{U}$, $M = M^{-1}$ and $M \circ M \circ M \circ M \subset W$ be such that $(a, w) \notin W$ for some $w \in O(a)$. By the Lyapunov-stability of A w.r.t. f, there is an $N \in \mathcal{U}, N \subset M$, such that for all $x \in A$ and for all $n \in \mathbb{Z}$

$$(a, x) \in N \quad \text{implies} \quad \big(f^n(a), f^n(x)\big) \in M. \tag{2}$$

Let $u \in N(a) \cap A$. Assume that $\{1, \alpha_a, \alpha_u\}$ is rationally independent. Then, for all $(z_1, z_2) \in (S^1)^2$, the set $\{(e^{2\pi i n\alpha_a} z_1, e^{2\pi i n\alpha_u} z_2) : n \in \mathbb{Z}\}$ is dense in $(S^1)^2$, since $\{(e^{2\pi i n\alpha_a}, e^{2\pi i n\alpha_u}) : n \in \mathbb{Z}\}$ is dense in $(S^1)^2$ (cf. [11], [18]). Hence, it follows from (1) that there exists an $n_0 \in \mathbb{Z}$ such that $\big(w, f^{n_0}(a)\big) \in M$ and $\big(u, f^{n_0}(u)\big) \in M$. Since $(a, u) \in N$, we have $(a, w) \in N \circ M \circ M^{-1} \circ M^{-1} \subset W$, which leads to a contradiction. This proves Claim 1.

Claim 2. Let $a \in A$. Then there exists an $N \in \mathcal{U}$ such that for every $u \in N(a) \cap A$ there are integers q, r with $\alpha_a + q\alpha_u + r = 0$.

Proof. Let $a \in A$. Let $M, W \in \mathcal{U}$, $M = M^{-1}$ and $M \circ M \circ M \subset W$ be such that $(a, y) \notin W$ for all $y \in R$, where

$$R := (\Psi_a \circ \Phi_a)^{-1}\Big(\big\{e^{2\pi i s}(\Psi_a \circ \Phi_a)(a) : \tfrac{1}{4} \leq s \leq \tfrac{3}{4}\big\}\Big).$$

Then there exists an $N \in \mathcal{U}, N \subset M$, such that the condition (2) is satisfied. Let $u \in N(a) \cap A$. By Claim 1, there are relatively prime numbers $p, q \in \mathbb{Z}$ and an $r \in \mathbb{Q}$ such that $p\alpha_a + q\alpha_u + r = 0$.

(i) To show that $r \in \mathbb{Z}$, assume that $r = \frac{m}{n} \in \mathbb{Q} \setminus \mathbb{Z}$ for relatively prime numbers $m \in \mathbb{Z}, n \in \mathbb{N}$. Then there is an $s \in \mathbb{N}$ such that

$$\operatorname{dist}(\tfrac{rs}{p}, \mathbb{Z}) \ \geq\ \tfrac{1}{3},$$

where we denote $\operatorname{dist}(a, \mathbb{Z}) := \inf\{|a - n| : n \in \mathbb{Z}\}$ for $a \in \mathbb{R}$.

Since α_u is irrational, there exists a sequence (v_k) in $s + np\mathbb{Z}$ such that

$$\operatorname{dist}(v_k \tfrac{\alpha_u}{p}, \mathbb{Z}) \to 0.$$

In particular, we can choose a $k_0 \in \mathbb{N}$ such that $\big(u, f^{v_{k_0}}(u)\big) \in M$ and $f^{v_{k_0}}(a) \in R$. This is a contradiction to the choice of N, M and W.

(ii) To show that $|p| = 1$, assume that $|p| \geq 2$ and $r \in \mathbb{Z}$. Since p, q are relatively prime, there exist $\beta_1, \beta_2 \in \mathbb{Z}$ such that $r = p\beta_1 + q\beta_2$; hence we have $p(\alpha_a + \beta_1) + q(\alpha_u + \beta_2) = 0$. Similarly, as in the proof of (i), we can choose an $s \in \mathbb{N}$ such that

$$\operatorname{dist}(\tfrac{qs}{p}, \mathbb{Z}) \ \geq\ \tfrac{1}{3}$$

and a sequence (v_k) in $\mathbb{Z}$ so that

$$\operatorname{dist}\big(v_k(\alpha_u + \beta_2), s + p\mathbb{Z}\big) \to 0,$$

showing $\big(u, f^{v_{k_0}}(u)\big) \in M$ and $f^{v_{k_0}}(a) \in R$ for some $k_0 \in \mathbb{N}$. We again have a contradiction. Consequently, Claim 2 is proved.

III. (b) For arbitrary $a \in A$, let

$$B := \{x \in A : \{1, \alpha_a, \alpha_x\} \text{ is rationally dependent}\}.$$

Then B is nonempty. We shall show that B is open. Let $b \in B$. By Claim 1, there exists an $N \in \mathcal{U}$ such that for every $u \in N(b) \cap A =: U$, the set $\{1, \alpha_b, \alpha_u\}$ is rationally dependent. As $\{1, \alpha_a, \alpha_b\}$ is rationally dependent, we obtain that for every $u \in U$, $\{1, \alpha_a, \alpha_u\}$ is rationally dependent; hence we have $U \subset B$. A similar argument establishes the closedness of B. Since A is connected, we conclude that $B = A$; thus statement (b) follows.

(c) Now let $H : A \to \mathbb{R}, \quad x \mapsto \beta_x \ , \ \beta_x := \begin{cases} \alpha_x & \text{for } 0 \le \alpha_x \le \frac{1}{2}, \\ 1-\alpha_x & \text{for } \frac{1}{2} \le \alpha_x \le 1. \end{cases}$

Then we have $(\overline{O(x)}, f) \simeq (S^1, e^{2\pi i \beta_x})$. Let $a \in A$. By Claim 2, there exists an $N \in \mathcal{U}$ such that for every $u \in N(a) \cap A$, there are $q, r \in \mathbb{Z}$ with $\beta_a + q\beta_u + r = 0$. By hypothesis, we have $q = \pm 1$. It follows that $\beta_a = \beta_u$ for all $u \in N(a) \cap A$. Since A is connected, we conclude that $\beta_a = \beta_x$ for all $x \in A$. This completes the proof of statement (c). □

4. Continuous equivariant extensions

The main tool for showing the continuity and the subadditivity for an index is the theorem about existence of a continuous equivariant extension. The existence theorem for compact Lie groups G has been proved by using Bochner integrals; see the Tietze-Gleason Theorem [16]. In general this does not hold for $\mathbb{Z}$-actions, but the almost periodicity shows that.

In the following we always assume that $f : X \to X$ is a continuous bijection on a compact space X, $X \in \Sigma(X, f)$ and $\alpha \in [0, 1] \cap (\mathbb{R} \backslash \mathbb{Q})$ is given by $\Sigma(X, f)$, and $k, m \in \mathbb{N}$.

Theorem 9. *For every $A \in \overline{\mathcal{P}}(X, f)$ and for every continuous map $\varphi : (A, f) \to (\mathbb{C}^k, e^{2\pi i m\alpha})$, φ has a continuous extension*

$$\Phi : (X, f) \longrightarrow (\mathbb{C}^k, e^{2\pi i m\alpha}).$$

Proof. Without loss of generality, we may suppose that $m = 1$.

Let $A \in \overline{\mathcal{P}}(X, f)$, and let $\varphi : (A, f) \to (\mathbb{C}^k, e^{2\pi i\alpha})$ be a continuous map. Since A is a closed subset of normal space X, there is, by the extension theorem of Tietze-Urysohn, a continuous map $\hat{\varphi} : X \to \mathbb{C}^k$ such that $\hat{\varphi}|_A = \varphi$. Let us define $\Phi : X \to \mathbb{C}^k$ by

$$\Phi(x) := \lim_{k\to\infty} \frac{1}{2k+1} \sum_{|n| \le k} e^{-2\pi i n\alpha}\, \hat{\varphi}(f^n(x)) \qquad \text{for each } x \in X.$$

Then Φ is well-defined because $F : \mathbb{Z} \to \mathbb{C}^k$, $n \mapsto e^{-2\pi i n\alpha}\hat{\varphi}(f^n(x))$ is almost periodic implying existence of the mean value of F, namely Φ (cf. C. Corduneanu [6]).

It is easy to verify that Φ is an equivariant map and $\Phi|_A = \varphi$.

To show that Φ is continuous, let $\epsilon > 0$ and $V_\epsilon := \{ (y, y') \in \mathbb{C}^k \times \mathbb{C}^k : \| y - y' \| < \epsilon \}$. As $\hat{\varphi}$ is uniformly continuous, there exists an $M \in \mathcal{U}$ such that for all $z, z' \in X$

$$(z, z') \in M \ \text{ implies } \ (\, \hat{\varphi}(z), \ \hat{\varphi}(z')\,) \in V_\epsilon.$$

By the Lyapunov-stability of X w.r.t. f, there is an $N \in \mathcal{U}$ such that for all $x, y \in X$ and for all $n \in \mathbb{Z}$

$$(x, y) \in N \ \text{ implies } \ (\, f^n(x), \ f^n(y)\,) \in M;$$

hence we have

$$\Big(\hat{\varphi}\big(f^n(x)\big), \ \hat{\varphi}\big(f^n(y)\big)\Big) \in V_\epsilon.$$

Consequently, we obtain that for all $x, y \in X$ with $(x, y) \in N$

$$\begin{aligned} \|\Phi(x)-\Phi(y)\| &= \Big\| \lim_{k\to\infty} \frac{1}{2k+1} \sum_{|n|\le k} e^{-2\pi i n\alpha}\Big(\hat{\varphi}\big(f^n(x)\big)-\hat{\varphi}\big(f^n(y)\big)\Big)\Big\| \\ &\le \limsup_{k\to\infty} \frac{1}{2k+1} \sum_{|n|\le k} \Big\| e^{-2\pi i n\alpha}\Big(\hat{\varphi}\big(f^n(x)\big)-\hat{\varphi}\big(f^n(y)\big)\Big)\Big\| \\ &\le \lim_{k\to\infty} \frac{1}{2k+1}\cdot(2k+1)\epsilon = \epsilon, \end{aligned}$$

i.e. $(\,\Phi(x), \Phi(y)\,) \in V_\epsilon$. This completes the proof. $\square$

For the proof of the following theorem we use the fact that S^{2k-1} is a neighborhood retract of $\mathbb{C}^k$. See [16] for compact Lie groups.

Theorem 10. *For every $A \in \overline{\mathcal{P}}(X, f)$ and for every continuous map $\varphi : (A, f) \to (\,S^{2k-1}, e^{2\pi i m\alpha})$, there are a neighborhood $U \in \overset{\circ}{\mathcal{P}}(X, f)$ of A and a continuous extension*

$$\Phi:\ (\,U,\ f\,)\ \longrightarrow\ (\,S^{2k-1}, e^{2\pi i m\alpha}\,)$$

of φ.

Proof. Let $A \in \overline{\mathcal{P}}(X, f)$ and let $\varphi : (A, f) \to (S^{2k-1}, e^{2\pi i m\alpha})$ be a continuous map. Since S^{2k-1} is a invariant neighborhood retract of $\mathbb{C}^k$, there are an open invariant neighborhood $V \subset \mathbb{C}^k$ of S^{2k-1} and a continuous map $r : (\,V, e^{2\pi i m\alpha}) \to (\,S^{2k-1}, e^{2\pi i m\alpha})$ such that $r|_{S^{2k-1}} = id_{S^{2k-1}}$.

With the inclusion $i : S^{2k-1} \hookrightarrow \mathbb{C}^k$, the map $i\circ\varphi :\ (\,A, f) \to (\,\mathbb{C}^k, e^{2\pi i m\alpha})$ is continuous. By Theorem 9, there exists a continuous map $\Phi : (\,X, f\,) \to (\,\mathbb{C}^k, e^{2\pi i m\alpha})$ such that $\Phi|_A = i \circ \varphi$.

Now set $U := \Phi^{-1}(V)$. Then U is invariant, open in X, and $A \subset U$. Put $\Phi' := r \circ \Phi|_U : U \to S^{2k-1}$. Hence, Φ' is also equivariant and continuous, and $\Phi'|_A = \varphi$. Thus the proof is complete. $\square$

Finally, we obtain the following result by using the concept of join and its topological properties presented in Section 2.

Theorem 11. *For $j = 1, 2$, let $A_j \in \overline{\mathcal{P}}(X, f)$ and let*

$$\varphi_j : (A_j, f) \to (S^{2k_j-1}, e^{2\pi i m\alpha})$$

be continuous. Then there exists a continuous map

$$\Phi:\ (\,A_1 \cup A_2,\ f\,)\ \longrightarrow\ (\,S^{2k_1-1} * S^{2k_2-1},\ e^{2\pi i m\alpha}\,),$$

where $S^{2k_1-1} * S^{2k_2-1}$ has the initial topology with respect to $q : S^{2k_1-1} * S^{2k_2-1} \to \triangle^1$ and partial functions $p_j : S^{2k_1-1} * S^{2k_2-1} \rightarrowtail S^{2k_j-1}\,(j = 1, 2)$. Furthermore, there is a continuous map

$$\Psi : (\,A_1 \cup A_2,\ f)\ \longrightarrow\ (\,S^{2(k_1+k_2)-1},\ e^{2\pi i m\alpha}\,).$$

Proof. It suffices to prove the result for the case $m = 1$. For $j = 1, 2$, by Theorem 10, there are a neighborhood $U_j \in \overset{\circ}{\mathcal{P}}(X, f)$ of A_j and a continuous extension $\Phi_j : (\,U_j,\ f\,) \to (\,S^{2k_j-1}, e^{2\pi i\alpha}\,)$ of φ_j.

Since X is normal, there exists a continuous function $\gamma_j' : X \to [\,0, 1]$ such that

$$\gamma_j'(x)\ =\ \begin{cases} 1 & \text{for } x \in A_j, \\ 0 & \text{for } x \in X\backslash U_j. \end{cases}$$

Let

$$\gamma_j : X \to \mathbb{R},\ x \mapsto \lim_{k\to\infty} \frac{1}{2k+1} \sum_{|n|\le k} \gamma_j'(f^n(x)).$$

Then γ_j is well-defined, continuous and invariant, i.e. $\gamma_j \circ f = \gamma_j$ (see proof of Theorem 9). It follows that $\gamma_j|_{A_j} = 1,\ \gamma_j|_{X\setminus U_j} = 0,\ \gamma_j(X) \subset [0,1]$.

Let $\tilde\gamma :\ A_1 \cup A_2 \to \triangle^1 = \{\, (t_1,t_2) \in [0,1]^2 :\ t_1 + t_2 = 1\,\}$,

$$\tilde\gamma(x) := (\tilde\gamma_1(x),\, \tilde\gamma_2(x)), \quad \tilde\gamma_j(x) := \frac{\gamma_j(x)}{\gamma_1(x)+\gamma_2(x)} \quad \text{for } j = 1,2\,.$$

Then $\tilde\gamma$ is continuous, invariant, and $\tilde\gamma_j|_{(A_1\cup A_2)\setminus U_j} = 0$.

Define $\Phi : A_1 \cup A_2 \to S^{2k_1-1} * S^{2k_2-1}$ by

$$\Phi(x) := [\,(\tilde\gamma_1(x), \tilde\gamma_2(x), \tilde\Phi_1(x), \tilde\Phi_2(x))\,],$$

where $\tilde\Phi_j : X \to S^{2k_j-1}$ is an arbitrary extension of Φ_j. Then Φ is well-defined, continuous and equivariant, since $\tilde\Phi_j|_{U_j} = \Phi_j$ is equivariant and since $\tilde\gamma_j$ invariant and $\tilde\gamma_j|_{(A_1\cup A_2)\setminus U_j} = 0$. Hence the first part is proved.

Note that the function

$$h :\ S^{2k_1-1} * S^{2k_2-1} \to\ S^{2(k_1+k_2)-1}, \quad [(t,s,x,y)] \mapsto\ (\, x\sin(t\pi/2),\, y\sin(s\pi/2)\,)$$

is a homeomorphism. Let $\Psi := h\circ\Phi :\ A_1\cup A_2 \to S^{2(k_1+k_2)-1}$. Then Ψ is continuous and equivariant, since h is equivariant with respect to the canonical almost periodic actions. This completes the proof. □

5. Proof of main result

Our goal is to prove that the genus g above is an index in the sense of Definition 1. For this we need the following two lemmas and apply the results obtained in Section 4.

Lemma 12. *Let $\alpha_A,\ \alpha_B \in [0,1]$ be the irrational numbers related to $A, B \in \Sigma(X,f)$, respectively. If there exists a continuous map $h : (A,f) \to (B,f)$, then $\{1, \alpha_A,\, \alpha_B\}$ is rationally dependent, and there are also $q_A, q_B \in \mathbb{Z}\setminus\{0\}$ such that $q_A\alpha_A - q_B\alpha_B \in \mathbb{Z}$.*

Proof. Assume that $\{1, \alpha_A,\, \alpha_B\}$ is rationally independent. Then we have $c_A\alpha_A + c_B\alpha_B \in \mathbb{R}\setminus\mathbb{Q}$ for all $c_A, c_B \in \mathbb{Z}\setminus\{0\}$. Hence there is a sequence (n_k) in $\mathbb{Z}$ such that

$$\lim_{k\to\infty} e^{2\pi i n_k\alpha_A} \ =\ e^{2\pi i\alpha_A} \quad \text{and} \quad \lim_{k\to\infty} e^{2\pi i n_k c_B\alpha_B} \ =\ e^{2\pi i(-c_A\alpha_A)}.$$

As h is continuous and equivariant and $e^{2\pi i(-c_A\alpha_A)} \ne e^{2\pi i c_B\alpha_B}$, we have

$$h(f^{c_B}(x)) \ =\ h(\lim_{k\to\infty} f^{n_k c_B}(x)) \ =\ \lim_{k\to\infty} f^{n_k c_B}(h(x)) \ \ne\ f^{c_B}(h(x)).$$

for $x \in A$. This contradicts the hypothesis of equivariance of h, and the proof is complete. □

Lemma 13. *Let $A \in \Sigma(X,f)$, and let $\varphi : (A,f) \to (\,S^{2k-1},\, e^{2\pi i\alpha}\,)$ be a continuous map. Then, for each $m \in \mathbb{N}$, there is a continuous map*

$$\psi : (A,f)\ \to\ (\,S^{2k-1},\, e^{2\pi i m\alpha}\,)\,.$$

This lemma follows immediately from elementary observations.

Now we can prove that the genus g has the following properties, whence g for compact spaces is an index in the sense of Definition 1.

Theorem 14. *Let $f : X \to X$ be a continuous bijection on a Hausdorff uniform space X. Then the following statements hold:*

(a) *If $A, B \in \Sigma(X, f)$ and there exists a continuous map $h : (A, f) \to (B, f)$, then $g\,(A, f) \leq g\,(B, f)$.*

(b) *If $B \in \Sigma(X, f)$ and $A \subset B$, then $g\,(A, f) \leq g\,(B, f)$.*

In addition let X be a compact space and $X \in \Sigma(X, f)$. Then the following statements hold:

(c) *If $A_1, A_2 \in \overline{\mathcal{P}}(X, f)$, then $g\,(A_1 \cup A_2, f) \;\leq\; g\,(A_1, f) \;+\; g\,(A_2, f)$.*

(d) *If $A \in \overline{\mathcal{P}}(X, f)$, then there is a neighborhood $U \in \overset{\circ}{\mathcal{P}}(X, f)$ of A such that $g\,(A, f) \;=\; g\,(U, f)$.*

Proof. (a) Let α_A, $\alpha_B \in [0, 1]$ be the irrational numbers related to $A, B \in \Sigma(X, f)$, such that $q_A\alpha_A = q_B\,\alpha_B + \ell$ for some $q_A, q_B \in \mathbb{N}$, $\ell \in \mathbb{Z}$ (This is possible, because an α_i, $i \in \{A, B\}$, can be replaced by $1 - \alpha_i$ if q_A und q_B have not the same sign; cf. Lemma 12). If $g\,(B, f) = \infty$, the statement (a) holds. Let $g\,(B, f) = k < \infty$. By definition of g, there is an $m \in \mathbb{N}$ and a continuous map $\varphi : (B, f) \to (\,S^{2k-1},\, e^{2\pi i m\alpha_B}\,)$. Let $\psi := \varphi \circ h :\;\; A \to S^{2k-1}$. Then ψ is equivariant and continuous. By Lemma 13, there is a continuous map

$$\Psi :\; (A, f) \;\to\; (\,S^{2k-1},\, e^{2\pi i m q_B\alpha_B}\,).$$

Since $q_A\alpha_A = q_B\alpha_B + \ell$, it follows that $g\,(A, f) \;\leq\; k \;=\; g\,(B, f)\,$.

(b) follows from (a), with the inclusion $i : A \hookrightarrow B$.

In the following suppose that $X \in \Sigma(X, f)$ is a compact space and $\alpha \in [0, 1] \cap (\mathbb{R}\backslash\mathbb{Q})$ is given by $\Sigma(X, f)$.

(c) By hypothesis, we have $A_1, A_2 \in \Sigma(X, f)$. If $g\,(A_1, f) = \infty$ or $g\,(A_2, f) = \infty$, then the conclusion is clear by (b). Let $g\,(A_1, f) = k_1$ and $g\,(A_2, f) = k_2$. Then there are $m_1, m_2 \in \mathbb{N}$ and continuous maps

$$\varphi_1 :\; (A_1, f) \;\to\; (\,S^{2k_1-1},\, e^{2\pi i m_1\alpha}\,),$$
$$\varphi_2 :\; (A_2, f) \;\to\; (\,S^{2k_2-1},\, e^{2\pi i m_2\alpha}\,)\,.$$

By Lemma 13, there are continuous maps

$$\psi_1 :\; (A_1, f) \;\to\; (\,S^{2k_1-1},\, e^{2\pi i m_2 m_1\alpha}\,),$$
$$\psi_2 :\; (A_2, f) \;\to\; (\,S^{2k_2-1},\, e^{2\pi i m_1 m_2\alpha}\,)\,.$$

By Theorem 11, there exists a continuous map

$$\Psi :\; (A_1 \cup A_2, f) \;\to\; (\,S^{2(k_1+k_2)-1},\, e^{2\pi i m_1 m_2\alpha}\,).$$

Hence, $g\,(A_1 \cup A_2, f) \;\leq\; k_1 + k_2 \;=\; g\,(A_1, f) \;+\; g\,(A_2, f)$.

(d) By hypothesis, we have $A \in \Sigma(X, f)$. Let $g\,(A, f) = \infty$, and let $V \subset X$ be an arbitrary invariant open neighborhood of A. Then $V \in \Sigma(X, f)$, and hence, by (b), $g\,(A, f) \leq g\,(V, f)$; therefore $g\,(A, f) = g\,(V, f)$. Now let $g\,(A, f) = k < \infty$. Then there are an $m \in \mathbb{N}$ and a continuous map $\varphi : (A, f) \to (\,S^{2k-1},\, e^{2\pi i m\alpha}\,)$. By Theorem 10, there are a neighborhood $U \in \overset{\circ}{\mathcal{P}}(X, f)$ of A and a continuous map

$$\Phi :\; (\,U,\, f\,) \;\to\; (\,S^{2k-1}, e^{2\pi i m\alpha}\,).$$

Hence $g\,(U,f)\;\leq\;k\;=\;g\,(A,f)\;\leq\;g\,(U,f)$ by (b); therefore $g\,(A,f)\;=\;g\,(U,f)$. This completes the proof. □

In this paper, we have observed almost periodic orbits whose closure is homeomorphic to S^1, in order to obtain an index theory for almost periodic actions. Motivated by the approximation theorem for almost periodic functions, we can consider almost periodic orbits whose closure is homeomorphic to the finite product $(S^1)^n$, where the rational independence plays a fundamental role. The investigation of the possible behaviour of almost periodic orbits and its complexity is of importance in a discrete dynamical system. We can suggest that, for instance, in view of Theorem 8, new observations and ideas are necessary for the study of almost periodic orbits in the case of $(S^1)^n, n \geq 2$. For further information on periodic orbits, see [17].

Acknowledgement

This paper is based on my doctoral thesis [11]. I would like to thank Prof. Steinlein, who posed many questions with helpful discussions. This work was partially supported by the Ministry of Education, Project Number BSRI-97-1413.

References

[1] T. Bartsch, *Topological methods for variational problems with symmetries*, Lecture Notes in Mathematics 1560, Springer, Berlin, 1993 MR **96a:**58078

[2] V. Benci, *A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations*, Comm. Pure Appl. Math. 34 (1981), 393-432 MR **82k:**58040

[3] H. Berestycki, J.M. Lasry, G. Mancini, B. Ruf, *Existence of multiple periodic orbits on star-shaped Hamiltonian surfaces*, Comm. Pure Appl. Math. 38 (1985), 253-289 MR **86j:**58039

[4] G.E. Bredon, *Topology and geometry*, Springer, New York, 1993 MR **94d:**55001

[5] R. Brown, *Topology: a geometric account of general topology, homotopy types and the fundamental groupoid*, Ellis Horwood Ltd., Chichester, 1988 MR **90k:**54001

[6] C. Corduneanu, *Almost periodic functions*, Chelsea, New York, 1989 MR **58:**2006 (first ed.)

[7] A. Dold, *Simple proofs of some Borsuk-Ulam results*, Contemp. Math. 19 (1983), 65-69 MR **85e:**55003

[8] E.R. Fadell, *The relationship between Ljusternik-Schnirelman category and the concept of genus*, Pacific J. Math. 89 (1980), 33-42 MR **82d:**55002

[9] E.R. Fadell, P.H. Rabinowitz, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. 45 (1978), 139-174 MR **57:**17677

[10] A.M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics 377, Springer, Berlin, 1974 MR **57:**792

[11] I.S. Kim, *Zu einer Indextheorie für fastperiodische Aktionen*, Doctoral Thesis, LMU München, 1995

[12] M.A. Krasnosel'skij, *On the estimation of the number of critical points of functionals*, Uspehi Mat. Nauk 7 (1952), no. 2 (48), 157-164 (Russian) MR **14:**55f

[13] M.A. Krasnosel'skij, *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, Oxford London New York Paris, 1964 MR **28:**2414

[14] N.G. Markley, *Transitive homeomorphisms of the circle*, Math. Systems Theory 2 (1968), 247-249 MR **38:**5198

[15] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Springer, New York, 1989 MR **90e:**58016

[16] R.S. Palais, *The classification of G-spaces*, Memoirs Amer. Math. Soc. 36, Providence, Rhode Island, 1960 MR **37:**4119

[17] H. Steinlein, *On the index of approximating sets of periodic points*, Manuscripta Math. 89 (1996), 15-33. MR **97a:**58150

[18] J. de Vries, *Elements of topological dynamics*, Kluwer Academic Publishers, Dordrecht Boston London, 1993 MR **94m:**54098

Department of Mathematics, Sung Kyun Kwan University, Suwon 440-746, Korea

E-mail address: `iskim@math.skku.ac.kr`