PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 8, August 1998, Pages 2433–2441
S 0002-9939(98)04454-2

INTEGRATION ON A CONVEX POLYTOPE

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(Communicated by David Sharp)

Abstract. We present an exact formula for integrating a (positively) homogeneous function f on a convex polytope $\Omega \subset R^n$. We show that it suffices to integrate the function on the $(n-1)$-dimensional faces of Ω, thus reducing the computational burden. Further properties are derived when f has continuous higher order derivatives. This result can be used to integrate a continuous function after approximation via a polynomial.

1. Introduction

We consider the integration of a continuous (positively) homogeneous function $f: R^n \to R$ on a convex polytope $\Omega \subset R^n$. We prove that if f is continuously differentiable, it suffices to integrate the function on the $(n-1)$-dimensional faces of Ω. As a continuous function on a compact set in R^n can be uniformly approximated by a polynomial (a sum of homogeneous functions), this result provides an alternative method for integrating continuous functions on a convex polytope.

A similar result also holds for an exponential $e^{\langle c,x\rangle}$. In fact, it has even been shown in [1], [2] that it suffices to evaluate that function at the *vertices* of Ω. This result was then used for computing the volume and counting integral points in Ω.

When f is twice continuously differentiable, one may proceed further, and we show that it suffices to integrate f on the $(n-2)$-dimensional faces and its derivatives on the $(n-1)$-dimensional faces. One may iterate the process when f has higher order continuous derivatives, etc.

2. Integration of a homogeneous function

Let A be an (m,n)-real matrix, $f: R^n \to R$ a real continuous (positively) *homogeneous* function of degree q, i.e. $f(\lambda x) = \lambda^q f(x)$ for all $\lambda > 0,\ x \in R^n$. For a (positively) homogeneous function of degree q that is continuously differentiable, Euler's formula holds (cf. [5]), i.e.:

$$qf(x) = \langle \nabla f(x), x\rangle \quad \text{for all } x. \tag{2.1}$$

Let

$$h(b) := \int_\Omega f(x)dx \quad \text{with } \Omega := \{x \in R^n \mid Ax \le b\}. \tag{2.2}$$

Received by the editors August 5, 1996 and, in revised form, January 6, 1997.
1991 *Mathematics Subject Classification.* Primary 65D30.
Key words and phrases. Numerical integration in R^n, homogeneous functions, convex polytopes.

We assume that Ω is a convex **polytope**. The following fact is straightforward:

Proposition 2.1. *If f is (positively) homogeneous of degree q, then h is (positively) homogeneous of degree $n+q$.*

Proof. We have

$$h(\lambda b) := \int_{Ax\leq \lambda b} f(x)dx = \int_{A(x/\lambda)\leq b} \lambda^q f(x/\lambda)\lambda^n d(x/\lambda) = \lambda^{n+q}\int_\Omega f(x)dx,$$

which yields the desired result. □

Let $\Omega_i := \{x \in R^n |\ Ax \leq b,\ A_i^T x = b_i\}$, i.e. Ω_i is the $(n-1)$-dimensional face of Ω determined by the hyperplane $A_i^T x = b_i$, where A_i^T is the *ith* row of the matrix A. Let $\mathcal{H}_i$ denote the $(n-1)$-dimensional affine variety that contains Ω_i. The algebraic distance from the point x_0 to $\mathcal{H}_i$ is denoted $d(x_0, \mathcal{H}_i)$, and $d(x_0, \mathcal{H}_i) = (b_i - A_i^T x_0)/||A_i||$ (with $||.||$ the usual Euclidean norm). Let μ be the Lebesgue measure on $\mathcal{H}_i$. The n-dimensional (resp. $(n-1)$-dimensional) volume of Ω (resp. Ω_i) is denoted by $\mathcal{V}_n(\Omega)$ (resp. $\mathcal{V}_{n-1}(\Omega_i)$).

Lemma 2.2. *Assume that f is continuously differentiable, $\mathcal{V}_n(\Omega) \neq 0$, and $\mathcal{V}_{n-1}(\Omega_i) \neq 0$. Then, h is continuously differentiable at b and*

$$\frac{\partial h}{\partial b_i} = \frac{1}{||A_i||}\int_{\Omega_i} f d\mu, \tag{2.3}$$

where μ is the Lebesgue measure on $\mathcal{H}_i$, the $(n-1)$-dimensional affine variety that contains Ω_i.

Proof. The proof is similar to the proof in [4] for the volume of Ω, i.e. when $f(x) \equiv 1$. For $\delta b_i > 0$, let $\Delta(\delta b_i)$ be the set

$$\Delta(\delta b_i) := \{x \in R^n |\ b_i \leq A_i^T x \leq b_i + \delta b_i,\ A_j^T x \leq b_j,\ j \neq i\}.$$

Since $\mathcal{V}_{n-1}(\Omega_i) \neq 0$, $\Delta(\delta b_i) \neq \emptyset$ for δb_i sufficiently small. Consider the change of variables $x = x_0 + zA_i/||A_i|| + \sum_{j=1}^{n-1} y_j v_j$, where $A_i^T x_0 = b_i$ and the v_j form an orthonormal basis of the $(n-1)$-dimensional subspace $A_i^T x = 0$. Equivalently, $\Delta(\delta b_i)$ can be written

$$\begin{aligned} 0 \leq z||A_i|| &\leq \delta b_i, \\ \sum_{k=1}^{n-1} y_k A_j^T v_k &\leq b_j - A_j^T x_0 - zA_j^T A_i/||A_i||,\ j \neq i. \end{aligned}$$

Let

$$s_j := \max[0, \frac{\delta b_i}{||A_i||^2} A_j^T A_i],\ s'_j := \max[0, \frac{-\delta b_i}{||A_i||^2} A_j^T A_i],\ j \neq i,$$

and let $\Delta^1(\delta b_i)$ and $\Delta^2(\delta b_i)$ be the domains in R^n, defined respectively by

$$0 \leq z \leq \frac{\delta b_i}{||A_i||},\ \sum_{k=1}^{n-1} y_k A_j^T v_k \leq b_j - A_j^T x_0 + s'_j,\ j \neq i,$$

and

$$0 \leq z \leq \frac{\delta b_i}{||A_i||},\ \sum_{k=1}^{n-1} y_k A_j^T v_k \leq b_j - A_j^T x_0 - s_j,\ j \neq i.$$

Obviously, $\Delta^2(\delta b_i) \subseteq \Delta(\delta b_i) \subseteq \Delta^1(\delta b_i)$. Define also

$$\Delta'^1(\delta b_i) := \{y \in R^{n-1} | \sum_{k=1}^{n-1} y_k A_j^T v_k \leq b_j - A_j^T x_0 + s'_j,\ j \neq i\}, \tag{2.4}$$

and

$$\Delta'^2(\delta b_i) := \{y \in R^{n-1} | \sum_{k=1}^{n-1} y_k A_j^T v_k \leq b_j - A_j^T x_0 - s_j,\ j \neq i\}. \tag{2.5}$$

Note that $\Delta'^1(0) = \Delta'^2(0) = \Omega_i$.

Assume first that f is nonnegative. From $h(b + \delta b_i e_i) - h(b) = \int_{\Delta(\delta b_i)} f dx$, we have

$$\begin{aligned}
\int_0^{\delta b_i/||A_i||} \int_{\Delta'^2(\delta b_i)} f(x_0 + z\frac{A_i}{||A_i||} + \sum_k y_k v_k) dy dz &\leq h(b + \delta b_i) - h(b),\\
\int_0^{\delta b_i/||A_i||} \int_{\Delta'^1(\delta b_i)} f(x_0 + z\frac{A_i}{||A_i||} + \sum_k y_k v_k) dy dz &\geq h(b + \delta b_i) - h(b).
\end{aligned}$$

f being continuously differentiable, one may write

$$\begin{aligned}
&f(x_0 + \sum_k y_k v_k + z\frac{A_i}{||A_i||})\\
&\quad = f(x_0 + \sum_k y_k v_k) + z\langle \nabla f(x_0 + \sum_k y_k v_k + \theta \frac{A_i}{||A_i||}), \frac{A_i}{||A_i||}\rangle
\end{aligned}$$

for some $0 < \theta < z$. Therefore, ∇f being bounded on a compact set, with a simple continuity argument we get

$$\lim_{\delta b_i \to 0} \frac{h(b + \delta b_i e_i) - h(b)}{\delta b_i} = \frac{\int_{\Delta'^1(0)} f(x_0 + \sum_k y_k v_k) dy}{||A_i||} = \frac{\int_{\Omega_i} f d\mu}{||A_i||}.$$

For f not necessarily nonnegative, simply use the same argument with $(f+M) - M$, where $\sup_{x\in\Omega} |f(x)| \leq M$ (as f is continuous and Ω is compact).

Finally, the same argument also holds if $\delta b_i < 0$, and the continuity of the partial derivatives is immediate from (2.3). □

Remark 2.3. We have not used that f is (positively) homogeneous, so Lemma 2.2 is valid for any continuously differentiable function f. In addition, note that if $\Omega_i = \emptyset$, then $\partial h(b)/\partial b_i = 0$, in accordance with $0 = \int_{\Omega_i} f d\mu$. Indeed, the constraint $A_i^T x \leq b_i$ is strictly redundant and remains strictly redundant with a slight perturbation of b_i.

Theorem 2.4. *Assume that f is continuously differentiable, $\mathcal{V}_n(\Omega) \neq 0$, and, for all $i = 1, ..., m$, $\mathcal{V}_{n-1}(\Omega_i) \neq 0$. Then*

$$\int_\Omega f(x) dx = \frac{1}{n+q} \sum_{i=1}^m \frac{b_i}{||A_i||} \int_{\Omega_i} f d\mu = \sum_{i=1}^m \frac{d(o, \mathcal{H}_i)}{n+q} \int_{\Omega_i} f d\mu, \tag{2.6}$$

where μ is the Lebesgue measure on the $(n-1)$-dimensional affine variety $\mathcal{H}_i$ that contains Ω_i.

Proof. Since $h(b)$ is an homogeneous continuously differentiable function at b, by Euler's formula (2.1), one gets

$$(n+q)h(b) = \langle \nabla h(b), b\rangle, \tag{2.7}$$

which, using Proposition 2.1 and Lemma 2.2 for $\nabla h(b)$, yields (2.6). □

Remark 2.5. (a) Formula (2.6) also holds if $\Omega_i = \emptyset$ for some i's. For such i's, $\int_{\Omega_i} f d\mu = 0$, in accordance with $\partial h(b)/\partial b_i = 0$ (cf. Remark 2.3).

(b) Note that the proof of Theorem 2.4 only uses Euler's formula. An alternative proof is to use Green's formula, i.e., with notation as in Prop. 2.3 , p. 128 in [6],

$$\int_\Omega \mathrm{div}(X) f\, d\omega \,+\, \int_\Omega X f\, d\omega \,=\, \int_{\partial\Omega} \langle X, \vec{n}\rangle\, f d\sigma,$$

where $\vec{n}$ is the unit outward-pointing normal to $\partial\Omega$, and with the vector field $X := \sum_{i=1}^n x_i \partial/\partial x_i$.

Hence, the integration of f on Ω reduces to a weighted integration of f on the $(n-1)$-dimensional faces of Ω (and in fact, only on those faces that do not contain the origin). A similar formula has already been given for $f := e^{\langle c,x\rangle}$, using Stokes' formula (see [1], [2]).

For instance, if P (resp. Q) is an homogeneous polynomial of degree p (resp. q), then

$$\int_\Omega (P+Q)dx = \sum_i d(o, \mathcal{H}_i) \int_{\Omega_i} (\frac{P}{n+p} + \frac{Q}{n+q}) d\mu.$$

With $f \equiv 1$, one retrieves the volume formula given in [4] that is interpreted as a standard result in geometry. Indeed, in (2.6) $\int_{\Omega_i} f d\mu$ reduces to $\mathcal{V}_{n-1}(\Omega_i)$, the $(n-1)$-dimensional volume of Ω_i, so that $b_i/(n||A_i||) \times \mathcal{V}_{n-1}(\Omega_i)$ is simply the n-dimensional version of the standard formula for the area of a triangle (base × height/2) and (2.6) reads

$$\mathcal{V}_n(\Omega) = n^{-1} \sum_{i=1}^m \frac{b_i}{||A_i||} \mathcal{V}_{n-1}(\Omega_i). \tag{2.8}$$

In [4], an algorithm based on (2.8) has been proposed, and the interested reader is referred to [3] for a numerical comparison of several algorithms for exact volume computation, including that one.

Remark 2.6. In fact Theorem 2.4 is also valid at points b where $\mathcal{V}_{n-1}(\Omega_i) = 0$ for some $i \in I \subset \{1, ..., m\}$. Indeed, one may prove that the constraint $A_i^T x \le b_i,\ i \in I$, is redundant and therefore can be removed, i.e. $\Omega \equiv \{x |\ A_i^T x \le b_i,\ i \notin I\}$. After having removed all the redundant constraints, (2.6) is valid, with the summation being now over all $i \notin I$. But (2.6) is also valid if we maintain those $i \in I$, since

$$\mathcal{V}_{n-1}(\Omega_i) = 0 \Rightarrow \mu(\Omega_i) = 0 \Rightarrow \int_{\Omega_i} f d\mu = 0.$$

2.1. **Further results.** We now would like to apply the same technique to $\int_{\Omega_i} f d\mu$ so as to consider integration on faces of lower dimensions. Indeed, we can do so provided f has continuous second derivatives.

Let b^i be the $(m-1)$-vector obtained from b by deleting its *ith* entry, and let $A^{(i)}$ be the matrix obtained from A by deleting its ith row. Let $\{v_k\}$ be $n-1$ orthonormal vectors in the vector space associated with $\mathcal{H}_i$. For every $j \neq i$, let

B_j be the $(n-1)$-vector $\{B_{jk}\}$ with $B_{jk} := A_j^T v_k,\ k = 1, ..., n-1$, and with x_0 arbitrary, define

$$\Gamma_i := \{y \in R^{n-1} |\ B_j^T y \leq b_j - A_j^T x_0,\ j \neq i\} = \{y \in R^{n-1} |\ By \leq b^i - A^{(i)} x_0\}$$

and

$$h(b^i, x_0) := \int_{By \leq b^i - A^{(i)} x_0} f(x_0 + \sum_{k=1}^{n-1} y_k v_k) dy. \tag{2.9}$$

If $x_0 \in \mathcal{H}_i$, then Γ_i is the representation of Ω_i in an orthonormal basis of $\mathcal{H}_i$, and $h(b^i, x_0) = \int_{\Omega_i} f d\mu$, with μ the Lebesgue measure on $\mathcal{H}_i$. Finally, let

$$\Omega_{ij} := \{x \in \Omega |\ A_i^T x = b_i,\ A_j^T x = b_j\}$$

be the (i, j) ($(n-2)$-dimensional) face of Ω and $\mathcal{H}_{ij}$ the $(n-2)$-dimensional affine variety that contains Ω_{ij}.

Theorem 2.7. *Let f be twice continuously differentiable. Assume also that for every $i = 1, ..., m$, either $\Omega_i = \emptyset$ or $\mathcal{V}_{n-1}(\Omega_i) \neq 0$, and for every $j = 1, ..., m$ with $j \neq i$, either $\Omega_{ij} = \emptyset$ or $\mathcal{V}_{n-2}(\Omega_{ij}) \neq 0$. Then:*

(a) $h(b^i, x_0)$ is positively homogeneous of degree $n + q - 1$.

(b) With $x_0 \in \mathcal{H}_i$ fixed, arbitrary,

$$\frac{\partial h(b^i, x_0)}{\partial b_j} = \frac{1}{||B_j||} \int_{\Omega_{ij}} f d\nu,\ j \neq i, \tag{2.10}$$

$$\frac{\partial h(b^i, x_0)}{\partial x_{0k}} = \sum_{j \neq i} \frac{-A_{jk}}{||B_j||} \int_{\Omega_{ij}} f d\nu + \int_{\Omega_i} \frac{\partial f}{\partial x_k} d\mu, \tag{2.11}$$

with μ (resp. ν) the Lebesgue measure on $\mathcal{H}_i$ (resp. $\mathcal{H}_{ij}$).

(c) With $x_0 \in \mathcal{H}_i$ fixed, arbitrary,

$$\int_{\Omega_i} f d\mu = \frac{1}{n+q-1} [\sum_{j \neq i} d_i(x_0, \mathcal{H}_{ij}) \int_{\Omega_{ij}} f d\nu + \int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu], \tag{2.12}$$

with d_i the algebraic (Euclidean) distance in $\mathcal{H}_i$.

Proof. (a) From the definition of $h(b^i, x_0)$ in (2.9), we get

$$\begin{aligned} h(\lambda b^i, \lambda x_0) &= \int_{By \leq \lambda(b^i - A^{(i)} x_0)} f(\lambda x_0 + \sum_{k=1}^{n-1} y_k v_k) dy \\ &= \int_{B(y/\lambda) \leq b^i - A^{(i)} x_0} \lambda^q f(x_0 + \sum_{k=1}^{n-1} (y_k/\lambda) v_k) \lambda^{n-1} d(y/\lambda) \\ &= \lambda^{n+q-1} \int_{By \leq b^i - A^{(i)} x_0} f(x_0 + \sum_{k=1}^{n-1} y_k v_k) dy \\ &= \lambda^{n+q-1} h(b^i, x_0). \end{aligned}$$

(b) If $\Omega_i = \emptyset$, then $\Omega_{ij} = \emptyset$ as well, and $\int_{\Omega_{ij}} f d\nu = 0$. Any slight perturbation of $b_j,\ j \neq i$, leaves Ω_i empty, so that $\partial h(b^i, x_0)/\partial b_j = 0$, and thus (2.10) holds.

Assume now that $\mathcal{V}_{n-1}(\Omega_i) \neq 0$. If $\Omega_{ij} = \emptyset$, it remains empty for every sufficiently small perturbation of b_j, and therefore, Ω_i remains unchanged. Hence, $\partial h(b^i, x_0)/\partial b_j) = 0$, in accordance with $\int_{\Omega_{ij}} f d\nu = 0$, i.e. (2.10) holds.

Consider now the case where $\Omega_{ij} \neq \emptyset$ and write $h(b^i, x_0)$ as $G(\hat{b}) = \int_{By \leq \hat{b}} g(y)dy$, with

$$\hat{b} := b^i - A^{(i)}x_0 \text{ and } g(y) := f(x_0 + \sum_{k=1}^{n-1} y_k v_k).$$

We can also write $By \leq \hat{b}$ as

$$B_j^T y \leq \hat{b}_j := b_j - A_j^T x_0 \quad \text{for all } j \neq i.$$

Applying Lemma 2.2 to G (in Lemma 2.2, we did not use that f was positively homogeneous, cf. Remark 2.3), we see that G is continuously differentiable, and

$$\frac{\partial h(b^i, x_0)}{\partial b_j} = \frac{\partial G(\hat{b})}{\partial \hat{b}_j} = \frac{1}{||B_i||} \int_{By \leq \hat{b},\ B_j^T y = \hat{b}_j} g d\nu,$$

where ν is now the Lebesgue measure on the $(n-2)$-dimensional affine variety $\mathcal{H}_{ij} \subset \mathcal{H}_i$, that contains the polytope

$$\{y \in R^{n-1} | By \leq \hat{b},\ B_j^T y = \hat{b}_j\} = \Omega_{ij}.$$

This yields (2.10). To get (2.11), let $x_0 := x_0 + \lambda e_k$ with e_k the n-vector $\{\delta_{kj}\}$ (and δ_{kj} the Kronecker symbol). Then

$$h(b^i, x_0 + \lambda e_k) = \int_{By \leq b^i - A^{(i)}(x_0 + \lambda e_k)} f(x_0 + \lambda e_k + \sum_{s=1}^{n-1} y_s v_s) dy.$$

Define

$$\Omega_i(\lambda) := \{y \in R^{n-1} | By \leq b^i - A^{(i)}x_0 - \lambda A^{(i)} e_k\} \text{ and } \Omega_i(0) = \Omega_i.$$

Now, writing $x' := x_0 + \sum_{s=1}^{n-1} y_s v_s$, with f twice continuously differentiable, we get

$$f(x_0 + \lambda e_k + \sum_{s=1}^{n-1} y_s v_s) = f(x') + \lambda \frac{\partial f(x')}{\partial x_k} + \lambda^2 \frac{\partial^2 f(x' + \theta e_k)}{\partial x_k^2}$$

for some $0 < \theta < \lambda$. Hence,

$$\begin{aligned} \lambda^{-1}(h(b^i, x_0 + \lambda e_k) - h(b^i, x_0)) \quad &= \quad \lambda^{-1}[\int_{\Omega_i(\lambda)} f(x')dy - \int_{\Omega_i} f(x')dy] \\ &+ \quad \int_{\Omega_i(\lambda)} \frac{\partial f(x')}{\partial x_k} + \lambda \frac{\partial^2 f(x' + \theta e_k)}{\partial x_k^2} dy. \end{aligned}$$

As f is twice continuously differentiable, $(\partial^2 f(x')/\partial x_k^2)$ is bounded on a compact set. In addition, for λ sufficiently small, $\Omega_i(\lambda)$ is contained in some compact set. Therefore, in the above equation, the term $\lambda \int_{\Omega_i(\lambda)} (\partial^2 f(x' + \theta e_k)/\partial x_k^2) dy$ vanishes as $\lambda \to 0$.

In addition, by a simple continuity argument,

$$\text{(2.13)} \qquad \lambda \to 0 \Rightarrow \int_{\Omega_i(\lambda)} \frac{\partial f(x')}{\partial x_k} dy \to \int_{\Omega_i(0)} \frac{\partial f(x')}{\partial x_k} dy = \int_{\Omega_i} \frac{\partial f}{\partial x_k} d\mu,$$

with μ the Lebesgue measure on $\mathcal{H}_i$.

Finally, write

$$g(y) := f(x_0 + \sum_{s=1}^{n-1} y_s v_s) \text{ and } \hat{b}_j(\lambda) := b_j - A_j^T x_0 - \lambda A_{jk},\ j \neq i.$$

Denote

$$G(\hat{b}(\lambda)) := \int_{\Omega_i(\lambda)} f(x')dy = \int_{By\leq \hat{b}(\lambda)} g(y)dy.$$

Assume first that $\mathcal{V}_{n-2}(\Omega_{ij}) \neq 0$. Again, we can apply Lemma 2.2 to G, since g is continuously differentiable and $\mathcal{V}_{n-2}(\Omega_{ij}) \neq 0$. Therefore, one gets

$$\frac{\partial G(\hat{b})}{\partial \hat{b}_j} = \frac{1}{||B_j||}\int_{By\leq \hat{b}, B_j^T y = \hat{b}_j} g d\nu = \frac{1}{||B_j||}\int_{By\leq \hat{b}, B_j^T y = \hat{b}_j} f d\nu,$$

with ν the Lebesgue measure on the $(n-2)$-dimensional affine variety $\mathcal{H}_{ij} \subset \mathcal{H}_i$ that contains the convex polytope $\{y \in R^{n-1} | By \leq \hat{b}, B_j^T y = \hat{b}_j\} = \Omega_{ij}$. Hence, from

$$\lim_{\lambda\to 0} \lambda^{-1}(\int_{\Omega_i(\lambda)} f(x')dy - \int_{\Omega_i} f(x')dy) = \sum_{j\neq i} \frac{\partial G(\hat{b}(0))}{\partial \hat{b}_j}\frac{d\hat{b}_j(0)}{d\lambda}$$

and $d\hat{b}_j/d\lambda = -A_{jk}$, one gets

$$\lim_{\lambda\to 0} \lambda^{-1}(\int_{\Omega_i(\lambda)} f(x')dy - \int_{\Omega_i} f(x')dy) = \sum_{j\neq i} \frac{-A_{jk}}{||B_j||}\int_{\Omega_{ij}} f d\nu. \tag{2.14}$$

If $\Omega_{ij} = \emptyset$, then $\Omega_i(\lambda) = \Omega_i$ for λ sufficiently small, and therefore,

$$\lim_{\lambda\to 0} \lambda^{-1}[\int_{\Omega_i(\lambda)} g(y)dy - \int_{\Omega_i} g(y)dy] = 0,$$

in accordance with $\int_{\Omega_{ij}} f d\nu = 0$. Finally, combining (2.13) and (2.14) yields (2.11).

(c) To get (2.12), we just apply Euler's formula (2.1) to $h(b^i, x_0)$, which is positively homogeneous of degree $n+q-1$, and continuously differentiable. This yields

$$\begin{aligned}\int_{\Omega_i} f d\mu &= h(b^i, x_0) = \frac{1}{n+q-1}\langle \nabla h(b^i, x_0), (b^i, x_0)\rangle \\ &= \frac{1}{n+q-1}[\langle \nabla_{b^i} h(b^i, x_0), b^i\rangle + \langle \nabla_{x_0} h(b^i, x_0), x_0\rangle].\end{aligned}$$

Using (2.10)-(2.11) for $\nabla h(b^i, x_0)$ in the above expression, one gets

$$\int_{\Omega_i} f d\mu = \frac{1}{n+q-1}[\sum_{j\neq i} \frac{b_j - A_j^T x_0}{||B_j||}\int_{\Omega_{ij}} f d\nu + \int_{\Omega_i} \langle \nabla f, x_0\rangle d\mu].$$

Noting that $(b_j - A_j^T x_0)/||B_j||$ is just $d_i(x_0, \mathcal{H}_{ij})$ (the algebraic distance in $\mathcal{H}_i$ from the origin x_0 to $\mathcal{H}_{ij}$), one gets (2.12). □

Hence, integrating f on Ω reduces to

- either integrating f on the $(n-1)$-dimensional faces of Ω (cf. Theorem 2.4),
- or integrating f on the $(n-2)$-dimensional faces of Ω and its derivatives on the $(n-1)$-dimensional faces of Ω (cf. Theorem 2.7).

Provided f has continuous partial derivatives of order $p+1$, one may iterate the above procedure and show that it suffices to evaluate f and its first, second, $\ldots, p$th derivatives at the vertices of Ω, the (1)-dimensional faces, etc.

For instance, consider the term $\int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu$. Let $z_0 \in \mathcal{H}_i$ be arbitrary, and with the same notation as in the proof of Theorem 2.7, write

$$g(b^i, z_0) := \int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu = \int_{By \leq b^i - A^{(i)} z_0} \langle \nabla f(z_0 + \sum_{k=1}^{n-1} y_k v_k), x_0 \rangle dy.$$

Again, g is (positively) homogeneous of degree $(n + q - 2)$ since ∇f is positively homogeneous of degree $q - 1$. Therefore, if f has continuous third derivatives, proceeding with similar arguments as in the proof of Theorem 2.7, one gets:

$$\int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu = \frac{1}{n+q-2} [\sum_{j \neq i} \frac{d_i(z_0, \mathcal{H}_{ij})}{||B_j||} \int_{\Omega_{ij}} \langle \nabla f, x_0 \rangle d\nu + \int_{\Omega_i} \langle z_0, (\partial^2 f) x_0 \rangle d\mu],$$

with $\partial^2 f$ the Hessian matrix of f.

An interesting case is when f is an homogeneous polynomial of degree q. Then the $(q+1)$th derivatives vanish, and integrating that polynomial on Ω requires only knowledge of the polynomial and all its partial derivatives at the vertices of Ω, i.e. at a finite number of points. As a continuous function on a compact set can be approximated by polynomials (a sum of homogeneous polynomials), one may compute a good approximation of the integral by considering only the vertices of Ω.

Finally, one may notice that integration on a *nonconvex* polytope reduces to the above case after a partition of the original polytope into convex polytopes.

2.2. **Illustrative example.** In R^2, consider $J := \int_\Omega xydxdy$ with

$$\Omega := \{(x, y) \in R^2 | \; x + y \leq 1, \; x \geq a, \; y \geq b\},$$

i.e. $n = q = 2$. A direct integration yields

$$J = \frac{1}{8}[(1-b)^4 - a^4] - \frac{1}{3}[(1-b)^3 - a^3] + \frac{1}{4}(1-b^2)[(1-b)^2 - a^2].$$

Now, with $\Omega_1 := \Omega \cap \{x = a\}$, we get

$$d(o, \mathcal{H}_1) \int_{\Omega_1} f d\mu = -a \int_b^{1-a} avdv = -a^2[(1-a)^2 - b^2]/2.$$

With $\Omega_2 := \Omega \cap \{y = b\}$, we get

$$d(o, \mathcal{H}_2) \int_{\Omega_2} f d\mu = -b \int_a^{1-b} bvdv = -b^2[(1-b)^2 - a^2]/2.$$

With $\Omega_3 := \Omega \cap \{x + y = 1\}$, we get

$$d(o, \mathcal{H}_3) \int_{\Omega_3} f d\mu = \frac{1}{\sqrt{2}} \int_a^{1-b} \sqrt{2} v(1-v) dv = \frac{1}{2}[(1-b)^2 - a^2] - \frac{1}{3}[(1-b)^3 - a^3]$$

and one may check that

$$J = \frac{1}{4}[-a^2 \int_b^{1-a} vdv - b^2 \int_a^{1-b} vdv + \int_a^{1-b} v(1-v)dv],$$

i.e.,

$$J = \frac{1}{4} \sum_{i=1}^{3} d(0, \mathcal{H}_i) \int_{\Omega_i} f d\mu,$$

or equivalently, (2.6) is satisfied.

Similarly, take $x_0 := (1-b, b) \in \mathcal{H}_3$. Then

$$d_3(x_0, \mathcal{H}_2) = 0,\ d_3(x_0, \mathcal{H}_1) = \sqrt{2}(1-a-b),\ f((a, 1-a)) = a(1-a).$$

In addition,

$$\int_{\Omega_3} \langle \nabla f(x), x_0 \rangle \mu(dx) = \sqrt{2} \int_b^{1-a} [v(1-b) + (1-v)b] dv,$$

and one may check that

$$\frac{\sqrt{2}}{3}[(1-a-b)a(1-a) + \int_b^{1-a} [v(1-b) + b(1-v)] dv] = \int_{\Omega_3} f d\mu,$$

i.e. (2.12) is satisfied.

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