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THE EXISTENCE AND UNIQUENESS OF INJECTORS FOR FITTING SETS OF SOLVABLE GROUPS

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Abstract. A short alternative proof is given of the existence and uniqueness of injectors in a Fitting set which avoids use of the Carter subgroup, a concept from the dual theory of projectors.

A Fitting set $\mathcal{X}$ of a group G is a non-empty set of subgroups which is closed under normal product, subnormal inheritance, and conjugation in G. If $H \leq G$, then the groups from $\mathcal{X}$ which are subgroups of H form a Fitting set of H which, since there is seldom danger in confusion, we also call $\mathcal{X}$. An $\mathcal{X}$-injector S of G is a subgroup with the property that $S \cap H$ is $\mathcal{X}$-maximal in H for all $H \trianglelefteq G$. If G is solvable, then $\mathcal{X}$-injectors exist uniquely up to conjugacy.

This theorem was first proved by Fischer, Gaschütz and Hartley in [3]. Their proof can also be found in [2], which also provides a comprehensive introduction to solvable group theory. The core of the proof is a clever lemma of Hartley which uses Carter subgroups (projectors for the class of nilpotent groups).

It is unsatisfying to have the proof of this fundamental result dependent on the corresponding result from the dual theory of projectors. We would prefer a proof based on more elementary ideas. Unfortunately alternative proofs [1] are often long, and our preference is not strong grounds for discarding a short proof in favor of a longer one. Here, however, is a short and simple proof of this type.

Recall that a p-subgroup $P_0 \leq P \in \mathrm{Syl}_p(G)$ is *strongly closed* in P if $P_0^g \cap P \leq P_0$ for all $g \in G$. The proof uses two simple lemmas. The first lemma states that strongly closed p-subgroups of solvable groups are normally embedded. This is well known (for example, see [4]). The second lemma is a special case of Corollary I(7.11) in [2], originally due to Schaller. Neither lemma is particularly difficult to prove.

Lemma 1. *Let G be a solvable group. Let P_0 be strongly closed in $P \in \mathrm{Syl}_p(G)$. Then there exists $N \trianglelefteq G$ with $N \cap P = P_0$.*

Proof. Induct on $|G|$. Let $1 \neq M \trianglelefteq G$ and let $\overline{H}$ denote HM/M for $H \leq G$. Since $\overline{P_0}$ is strongly closed in $\overline{P}$ with respect to $\overline{G}$, by hypothesis $\overline{P_0} = \overline{P} \cap \overline{N}$ for $\overline{N} \trianglelefteq \overline{G}$. Hence there exists $N \trianglelefteq G$ with $N \cap P = P_0(M \cap P)$. If $O_{p'}(G) \neq 1$ we can set $M = O_{p'}(G)$, and we are done. If $O_p(G) \cap P_0 \neq 1$ we can set $M = O_p(G) \cap P_0$, and we are done. But now $P_0 \leq C_G(O_p(G)) \leq O_p(G)$ by Phillip Hall's theorem (6.1.3 in [5]), and so $P_0 = P_0 \cap O_p(G) = 1$. □

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Lemma 2. *Let G be a solvable group. Let P_0 be strongly closed in $P \in \mathrm{Syl}_p(G)$ and Q_0 be strongly closed in $Q \in \mathrm{Syl}_q(G)$. Then there exist conjugates of P_0 and Q_0 whose product is a group. Equivalently there exist $s, t \in G$ with $P_0^s Q_0^t = Q_0^t P_0^s$.*

Proof. Choose conjugates to ensure that P and Q are from a Sylow system, and hence $PQ = QP$ is a group. Choose (last lemma) normal subgroups N and M with $N \cap P = P_0$ and $M \cap Q = Q_0$. But now $P_0 Q_0 = PM \cap PQ \cap NQ$, and this is a subgroup as claimed. □

Theorem 3 (Fischer, Gaschütz, Hartley)**.** *If G is a solvable group and $\mathcal{X}$ is a Fitting set of G, then G has a unique conjugacy class of $\mathcal{X}$-injectors.*

Proof. Induct on $|G|$. Assume the result for all groups of smaller order. Choose $O^p(G) < G$. Let S be an $\mathcal{X}$-injector of $O^p(G)$, and let $P \in \mathrm{Syl}_p(N_G(S))$. By the Frattini argument, $G = N_G(S).O^p(G)$, hence $PO^p(G) = G$.

If $S \leq T \in \mathcal{X}$ then $S \leq T \cap O^p(G) \in \mathcal{X}$; hence $T \cap O^p(G) = S$ and $T \leq N_G(S)$. A Sylow p-subgroup of T is conjugate in $N_G(S)$ to a subgroup of P. Hence T is conjugate in $N_G(S)$ to a group of the form $P_0 S \in \mathcal{X}$, where $P_0 \leq P$. All such groups are subnormal in PS, so the group R that they generate is in $\mathcal{X}$, and is the unique maximal $\mathcal{X}$-subgroup of PS. Furthermore, all extensions of S which are elements of $\mathcal{X}$ are conjugate in $N_G(S)$ to subgroups of R. In particular, if G has an $\mathcal{X}$-injector, then it is conjugate to R.

It remains only to show that R is an $\mathcal{X}$-injector. Since R is $\mathcal{X}$-maximal in G, it is enough to prove that R contains (and hence $R \cap M$ is equal to) an $\mathcal{X}$-injector of M for every maximal normal subgroup M of G.

Suppose $[G : M] = q$, and let T be an $\mathcal{X}$-injector of M. Since $T \cap O^p(G)$ and $S \cap M$ are $\mathcal{X}$-injectors of $M \cap O^p(G)$, we may choose T, so that $T \cap O^p(G) = S \cap M = U$. Let $P_1 \in \mathrm{Syl}_p(T)$ and $Q_1 \in \mathrm{Syl}_q(S)$, so that $T = P_1 U$ and $S = Q_1 U$. Choose $P_1 \leq P \in \mathrm{Syl}_p(N_G(U))$ and $Q_1 \leq Q \in \mathrm{Syl}_q(N_G(U))$. If $g \in N_G(U)$, then $(P_1^g \cap P)U \leq T^g \in \mathcal{X}$ and $\langle P_1^g \cap P, P_1 \rangle U \in \mathcal{X}$. Then $P_1^g \cap P \leq P_1$ and P_1 is strongly closed in P with respect to $N_G(U)$. Similarly we can show that Q_1 is strongly closed in Q with respect to $N_G(U)$. So by the last lemma $P_1^g Q_1$ is a group for some $g \in N_G(U)$.

Let $K = P_1^g Q_1 U = (P_1 U)^g (Q_1 U) = T^g S$. Since $K \cap O^p(G) = S$ and $K \cap M = T^g$, we see that $K = T^g S$ is a normal product, and thus $S \leq K \in \mathcal{X}$. Hence R contains a conjugate of $K \geq T$, as claimed. □

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