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VECTOR BUNDLES ON LOG TERMINAL VARIETIES

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ABSTRACT. Let X be an n-dimensional variety and E an ample vector bundle on X of rank e. We give a complete classification of pairs (X,E), with X log terminal and $e \geq n$ such that $K_X + detE$ is not ample. The results we obtain were conjectured by Fujita, and recently by Zhang.

INTRODUCTION

Let X be a log terminal variety and E an ample vector bundle of rank e. We are interested in studying those pairs (X,E) for which the adjoint $\mathbf{Q}$-Cartier divisor $K_X + detE$ is not ample—this is what is usually called generalized adjunction.

In the attempt to prove the Mukai conjectures, [Mu], such pairs with X smooth have been studied, mainly with the help of Mori theory, in a series of papers [YZ], [Fu3], [Pe], and then further investigations took place in [ABW], [PSW], [Zh1], [Ma], [AM]. On the contrary, only recently the full power of minimal model theory was used to extend this study to the case of mildly singular varieties [Zh2], [Zh3], [AM]. In this context pairs (X,E) with $e \geq n+1$ and X log terminal were classified independently in [Zh2] and [AM].

The following theorem characterizes the pairs (X,E), with $e \geq n$; this result was conjectured first by Fujita [Fu3] and then by Zhang [Zh3].

Theorem 1. *Let X be a log terminal variety of dimension n and E an ample vector bundle of rank $e \geq dimX$. Assume that $K_X + detE$ is nef but not ample.*

If $e \geq n+1$ then $(X,E) \simeq (\mathbf{P}^n, \mathcal{O}(1)^{\oplus n+1})$, [Zh1], [AM].
If $e = n$ then (X,E) is one of the following:
- $(\mathbf{P}^n, \mathcal{O}(1)^{\oplus n-1} \oplus \mathcal{O}(2))$,
- $(\mathbf{Q}^n, \mathcal{O}(1)^{\oplus n})$*, where $\mathbf{Q}^n$ is a, maybe singular, hyperquadric,*
- $(\mathbf{P}^n, T_{\mathbf{P}^n})$,
- *X is a smooth scroll over a smooth curve and $E_{|F} \simeq \mathcal{O}(1)^{\oplus n}$, where $F \simeq \mathbf{P}^{n-1}$.*

The idea of the proof is to use the variety $Y = \mathbf{P}(E)$ and the extremal contraction naturally associated to it, to study the possible singularities of X. To do this we mainly use the slicing technique of [AW] and some byproducts of it [An]. We complete the study of the positivity of $K_X + detE$ for a pair (X,E) with $rkE \geq$

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$dimX$, giving a different proof of the main theorem in [Zh3], and start the case $rkE = dimX - 1$, with the following.

Theorem 2. *Let X be a log terminal variety of dimension n and E an ample vector bundle of rank $e \geq dimX - 1$. Then $K_X + detE$ is nef unless*

$e \geq n$ and $(X,E) \simeq (\mathbf{P}^n, \mathcal{O}(1)^{\oplus n})$, [Zh3], or
$e = n-1$ and (X,E) is one of the following:

- $(\mathbf{P}^n, \mathcal{O}(1)^{\oplus n-1})$,
- $(\mathbf{P}^n, \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}(2))$,
- $(\mathbf{Q}^n, \mathcal{O}(1)^{\oplus n-1})$, *where $\mathbf{Q}^n$ is a (possibly singular) hyperquadric,*
- *X is a smooth scroll over a smooth curve and $E_{|F} \simeq \mathcal{O}(1)^{\oplus n-1}$, where $F \simeq \mathbf{P}^{n-1}$.*

In this case the main part is to prove that X is Gorenstein, so as to be able to apply a trick, borrowed from [AM], that reduces the pair (X,E) to a pair (X,E_1) in the list of Theorem 1. We believe that, with similar arguments, it is possible to describe those pairs (X,E) with nef but not ample adjoint $\mathbf{Q}$-divisor and $rkE = n-1$; this should be the borderline of our techniques.

1. Preliminaries

We use the standard notation from algebraic geometry. In particular it is compatible with that of [KMM], to which we refer constantly. Everything is defined over $\mathbf{C}$. A $\mathbf{Q}$-Cartier divisor is an element in $Div(X)_{Q} := Div(X) \otimes \mathbf{Q}$, where $Div(X)$ is the group of Cartier divisors. A contraction $f : Y \to X$ is a birational morphism of normal varieties with connected fibers. Given a contraction $f : X \to Y$ and $A, B \in Div(X)_{Q}$, we say that A is f-numerically equivalent to B ($A \equiv_f B$) if $A \cdot C = B \cdot C$ for any curve contracted by f; and A is f-linearly equivalent to B ($A \sim_f B$) if $A - B \sim f^*(M)$, for some Cartier divisor $M \in Div(Y)$.

A contraction $f : X \to Y$ is extremal if $-K_X$ is f-ample; if $dimX > dimY$ then f is said to be of fiber type, otherwise it is birational. We will say that $K_X + rL$ supports the contraction f, for some $L \in Div(X)$, if $K_X + rL \equiv_f \mathcal{O}_X$. In many occasions we are interested only in a neighborhood of a special fiber; to study this situation we will use a local set-up as described in [AW]; that is, we will shrink Y to an affine scheme in such a way that any line bundle on Y is a unit.

1.1. Let us recall various properties of extremal contractions. Let $\varphi : X \to Y$ be an extremal contraction supported by $K_X + rL$ and F an irreducible component of a positive dimensional fiber of φ; then

i) $dimF \geq r-1$, and if φ is birational then $dimF \geq r$, [Fu1],[AW];
ii) if $dimF < r+1$ (if φ is birational $dimF \leq r+1$), then L is φ-spanned, [AW];
iii) if φ is equidimensional of dimension $r-1$ and there are no fibers contained in $Sing(X)$, then $(X,L) \simeq (\mathbf{P}(\mathcal{F}), \mathcal{O}(1))$, for some ample vector bundle $\mathcal{F}$ on a smooth variety, [BS], [An];
iv) if φ is birational and $dimF = r$, then $(F,L) \simeq (\mathbf{P}^r, \mathcal{O}(1))$, [An].

The following lemma is just a restatement of [Me, Lemma 1.9].

Lemma 1.2. *Let $\varphi : X \to Y$ be a local extremal contraction and T an integral Weil divisor with $T \equiv_\varphi \mathcal{O}_X$. Let $H \in |\varphi^* M|$, for some spanned line bundle M on Y, a generic section. Assume that $T_{|H} \sim_\varphi \mathcal{O}_H$. Then $T \sim_\varphi \mathcal{O}_X$.*

Proof. Since T is a Weil divisor and f^*M is Cartier, the following sequence is exact:

$$0 \to \mathcal{O}(T-H) \to \mathcal{O}_X(T) \to \mathcal{O}_H(T) \to 0.$$

By hypothesis $H^0(H,T)$ has a nonvanishing section σ; therefore, using the above sequences and the Kawamata–Viehweg vanishing theorem, we can lift σ to a nonvanishing section of $H^0(X,T)$. ☐

2. Proof of the theorems

Proof of Theorem 1. If X is smooth then the theorem follows by [Fu3], [Pe]. By hypothesis there exists an extremal contraction $\pi : X \to W$ supported by $K_X + detE$. Let $Y = \mathbf{P}(E)$; then $dimY = n+e-1$, Y is log terminal, in particular CM, and $Sing(Y) = p^{-1}Sing(X)$; furthermore $K_Y + e\xi_E \sim p^*(K_X + detE)$, where ξ_E is the tautological line bundle on Y. Therefore $-K_Y$ is $(\pi \circ p)$-ample and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{P}(E)=Y & \xrightarrow{\varphi} & V \\ \downarrow p & & \downarrow \psi \\ X & \xrightarrow{\pi} & W, \end{array}$$

where φ is an extremal contraction supported by $K_Y + e\xi_E$. Since p is an elementary contraction then $dimF_\varphi \le dimF_\pi$, where F_f is an irreducible component of a fiber of the contraction f. If $e \ge n+1$ then, by (1.1 i)), $e-1 \le dimF_\varphi \le n$; therefore $e = n+1$ and φ is equidimensional of dimension $e-1$. Since $\varphi_{|F_\varphi} : F_\varphi \to X$ is surjective, F_φ is not contained in $Sing(Y)$. Therefore by (1.1 iii)) Y is smooth and consequently X is smooth.

Assume now that $e = n$. Then φ is supported by $K_Y + nH$; hence $dimF_\varphi \ge n-1$.

Case 1. All fibers of φ have dimension $n-1$. Thus φ is of fiber type and Y and X are smooth as above; see also [AM, Theorem 4.2].

Case 2. The generic fiber has dimension $n-1$ but there are special fibers of dimension n. Let S the locus of fibers of dimension n and $G = \varphi(S)$; then $dimS < dimY = 2n-1$, that is, $codG \ge 2$. Slice V with $(n-1)$ generic hyperplane sections A_i, in such a way that $C = \bigcap A_i$ is outside of G and $C \not\subset \varphi(F_p)$, for any fiber F_p. Let $T := \varphi^*(C)$ and $f_T := \varphi_{|T}$; then T is lt, and f_T is an equidimensional extremal contraction, supported by $K_T + n\xi_{E|T}$. Since $p_{|T} : T \to X$ is surjective and X is normal, there are no fibers of f_T contained in $Sing(T)$ and T is smooth by (1.1 iii)). Since T is obtained by Y slicing with Cartier divisors, then Y is smooth along T. Furthermore $p_{|T}$ is surjective and $Sing(Y) = p^{-1}Sing(X)$; thus X is smooth as well.

Case 3. The generic fiber F has dimension n and φ is of fiber type. By the Kobayashi–Ochiai criterion the generic fiber $F \simeq \mathbf{Q}^n$, maybe singular, and $p_{|F} : F \to X$ is finite and surjective. Let $C \subset F$ be a rational curve such that $p_{|C} : C \to B \subset X$ is birational onto the image. Such a curve always exists, for instance $H_1 \cap \ldots \cap H_{n-1}$, where $H_i \in |\mathcal{O}_{\mathbf{Q}^n}(1)|$ are generic elements. Let $\nu : \mathbf{P}^1 \to B$ be the normalization and $Y_C = \mathbf{P}(\nu^*\mathcal{E})$; then we have a morphism $h : Y_C \to V$, induced by the normalization and φ. Let

$$Y_C \xrightarrow{g} U \xrightarrow{i} V$$

be the Stein factorization of h. Since $n = dimY_C = dimV + 1$, then g is not birational, that is, $\nu^*\mathcal{E} \simeq \mathcal{O}(a)^{\oplus n}$ and $U \simeq \mathbf{P}^{n-1}$. On the other hand, i is birational and V is normal; therefore $V \simeq \mathbf{P}^{n-1}$. By the universal property of a product and our construction we have a morphism α

$$Y \overset{\alpha}{\rightarrow} X \times \mathbf{P}^{n-1} \begin{array}{cc} \overset{p_1}{\rightarrow} & X \\ \overset{p_2}{\rightarrow} & \mathbf{P}^{n-1} \end{array},$$

compatible with the projections, that is, $p_1 \circ \alpha = p$ and $p_2 \circ \alpha = \varphi$. In other words, $Y \simeq \mathbf{Q}^n \times \mathbf{P}^{n-1}$ and $E \simeq \mathcal{O}(1)^{\oplus n}$.

Case 4. φ is birational. In this case all fibers have dimension n, $(F, \xi_{E|F}) \simeq (\mathbf{P}^n, \mathcal{O}(1))$ and ξ_E is φ-spanned (1.1 iv), ii)); let us consider a local set-up and slice with $(n-1)$ sections $A_i \in |\xi_E|$, to get a log terminal n-fold T with a birational contraction $f := \varphi_{|T} : T \to Z$, supported by $K_X + L$, where $L = \xi_{E|T}$, with one-dimensional fibers. X is normal; therefore the generic fiber $F \simeq \mathbf{P}^1$ is out of $Sing(T)$, and so, by the Ionescu–Wisniewski inequality, [Wi], the exceptional locus of f, hence of φ, is a divisor. In particular we can slice vertically on Y with $(n-2)$ generic hyperplane sections φ^*H_i, to get $S = Y \cap (\bigcap_i f^*H_i)$ and a birational extremal contraction, $f_S : S \to U$ supported by $K_Y + n\xi_{E|S}$, whose general fiber F has dimension n. By [An] we know that under this hypothesis either $dim(F \cap Sing(S)) = n-1$ or F is on the smooth locus. The former is impossible since X is normal; therefore the latter is the case and X is smooth. □

Lemma 2.1. *Let X be a log terminal variety of dimension n and E an ample vector bundle on X of rank $e \geq n-1$. Assume that $K_X + detE$ is not nef. Then X is Gorenstein.*

Proof. Let $Y = \mathbf{P}(E)$, and let $\varphi : Y \to V$, as in the above commutative diagram, be the extremal contraction supported by $K_Y + r\xi_E$ for some $r > e$. Let F be a generic fiber of φ. Since $Sing(Y) = Sing(X) \times \mathbf{P}^{e-1}$, it is enough to prove that for the generic point $y \in Y$ of any fiber F_p, K_Y is Cartier at y.

Case 1. $dimF = n$ and φ of fiber type. Consider a local set-up and slice on V with $(e-1)$ general hyperplane sections A_i. Let $F = \varphi^*(\bigcap A_i)$, with $(K_X + r\xi_E)_{|F} \equiv O_F$, where $r > e \geq n-1$. Since $p_{|F} : F \to X$ is surjective, X is Gorenstein if Y is Gorenstein along F. On the other hand F is obtained from Y by vertical slicing; therefore by Lemma 1.2, Y is Gorenstein along F if F is Gorenstein and r is an integer.

Claim. F is either a hyperquadric or the projective space, and r is an integer.

Proof of the claim. This is a standard application of the Kawamata–Viehweg vanishing theorem. F is log terminal. Let $p(t) = \chi(F, t\xi_{E|F}) = \sum a_i t^i$. Since $K_F + r\xi_{E|F} \sim O_F$ with $r > n-1$, then $p(t) = 0$ for $0 > t \geq n-1$ and $p(0) = 1$. Furthermore, by the Riemann–Roch formula we know that $-K_F \cdot \xi_{E|F} = 2(n-1)!a_{n-1}$, and we can now immediately write down the possible polynomials of degree n satisfying those prescriptions and check that they are the Hilbert polynomials of projective spaces and hyperquadrics. □

Case 2. $dimF \leq n-1$. Since $r > e \geq n-1$, then $dimF = n-1$ and φ is of fiber type. Slice on V with $(e-1)$ general hyperplane sections A_i. Let $T = \varphi^*(\bigcap A_i)$, with $dimT = n$ and $\varphi_T = \varphi_{|T}$ an extremal contraction supported by $K_T + r\xi_{E|T}$, with $r > n-1$; by an easy dimension count, as in the proof of Theorem 1, φ_T is equidimensional of dimension $n-1$. As in the above claim the general fiber of f_T is $\mathbf{P}^{n-1}$ and $r = n$; therefore by (1.1 iii)) T is smooth. Since $p_{|T} : T \to X$ is surjective and T is obtained from Y by vertical slicing, then X is smooth.

Case 3. $dimF = n$ and φ is birational. With the same argument as in Case 4 of Theorem 1, we can prove that the exceptional locus of φ is a divisor. Slice vertically on Y with $(e-2)$ generic hyperplane sections, to get a birational extremal contraction $f_S : S \to U$, from an $n+1$ log terminal variety S with fibers F of dimension n. Again we conclude that $F \simeq \mathbf{P}^n$ is on the smooth locus and $r = n$. Therefore X is smooth. □

Proof of Theorem 2. Let $\pi : X \to W$ be an extremal contraction such that the divisor $-(K_X + detE)$ is π-ample. By Lemma 2.1 we know that X is Gorenstein. Let $E_1 = E \oplus (-(K_X + detE))$ then $rkE_1 = e + 1$ and $K_X + detE_1$ is π-nef. Therefore the relative version of Theorem 1 and easy computations to recover E from E_1 allow us to draw the desired conclusion. □

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