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LAPLACE TRANSFORMS AND GENERATORS OF SEMIGROUPS OF OPERATORS

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Abstract. In this paper, a characterization for continuous functions on $(0,\infty)$ to be the Laplace transforms of $f \in L^\infty(0,\infty)$ is obtained. It is also shown that the vector-valued version of this characterization holds if and only if the underlying Banach space has the Radon-Nikodým property. Using these characterizations, some results, different from that of the Hille-Yosida theorem, on generators of semigroups of operators are obtained.

1. Introduction

The theory of Laplace transforms plays an important role in the theory of semigroups of operators. Given a function F on $(0,\infty)$, under what conditions is F the Laplace transform of a certain function f? This problem has been investigated extensively. In [7], Widder obtained the following characterization of Laplace transforms of scalar-valued functions:

> A function F on $(0,\infty)$ is the Laplace transform of $f \in L^\infty(0,\infty)$ if and only if F is infinitely differentiable and satisfies

$$(\mathrm{W}_\infty) \qquad \sup\{|\frac{1}{n!}\lambda^{n+1}F^{(n)}(\lambda)| : \lambda > 0, n \in \mathbf{N}\cup\{0\}\} < \infty.$$

The vector-valued version of Widder's theorem has been investigated by Arendt among others. In [1], Arendt obtained an "integrated version of Widder's theorem" (see [1, Theorem 1.1]), and from this generalization, the relation between the Hille-Yosida theorem and Widder's theorem is revealed.

It is worth noting that in Widder's characterization of Laplace transforms, condition (W_∞) involves not only the original function, but also its higher derivatives, and so in certain practical problems it may be difficult to verify condition (W_∞). In Section 2, we give a characterization of Laplace transforms which involves only the original function but not its derivatives. Applications of this characterization can be found in [6].

In the theory of semigroups of operators, it is known that whether a linear operator A is the generator of a certain semigroup (C_0-semigroup or integrated semigroup) is related to the Laplace representation of its resolvent $R(\lambda, A)$ (see [1], [5], [3]). In Section 3, using the results in Section 2, we obtain some characterization

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results for generators of semigroups of operators. These results are different from those given by the Hille-Yosida theorem.

2. Characterizations of Laplace transforms

Let $f \in L^\infty(0,\infty)$. The Laplace transform F of f is given by

$$F(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt \quad (\lambda > 0).$$

The following result gives a characterization of those $F \in C(0,\infty)$ that are Laplace transform of an element f in $L^\infty(0,\infty)$. This characterization involves only the original function F, not its higher derivatives.

Theorem 2.1. *Let $F \in C(0,\infty)$. The following assertions are equivalent.*

1. *F is the Laplace transform of some $f \in L^\infty(0,\infty)$.*
2. *There exists a constant M such that $|\lambda F(\lambda)| \leq M$ for a.e. $\lambda > 0$ and $|\sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jn} \lambda F(j\lambda)| \leq M$ for a.e. $\lambda > 0$ for infinitely many $n \in \mathbf{N}$.*
3. *Same as* (2)*, with the inequalities holding for all $\lambda > 0$ and all $n \in \mathbf{N}$.*

Proof. (1 implies 3) Put $M = \operatorname{ess\,sup}_{0<t<\infty} |f(t)|$. It is clear that $|\lambda F(\lambda)| \leq M$ for all $\lambda > 0$. Let $\lambda > 0$ and $n \in \mathbf{N}$. Then

$$\begin{aligned} |\sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jn} \lambda F(j\lambda)| &= |\int_0^\infty \lambda \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jn} e^{-j\lambda t} f(t)\, dt| \\ &= |\int_0^\infty \lambda e^{-e^{n-\lambda t}} e^{n-\lambda t} f(t)\, dt| \\ &\leq M. \end{aligned}$$

(3 implies 2) Obvious.

(2 implies 1) Let $f_n(t) = \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jn} \frac{n}{t} F(\frac{jn}{t})$. Then the given condition on F implies that there exist $n_1 < n_2 < \cdots$ such that (f_{n_i}) is a bounded sequence in $L^\infty(0,\infty)$. Since $L^\infty(0,\infty)$ is the dual of the separable space $L^1(0,\infty)$, (f_{n_i}) has a subsequence $(f_{n_{i_k}})$ which converges in the weak*-topology to $f \in L^\infty(0,\infty)$. In particular, for every $\lambda > 0$,

$$\lim_{k\to\infty} \int_0^\infty e^{-\lambda t} f_{n_{i_k}}(t)\, dt = \int_0^\infty e^{-\lambda t} f(t)\, dt.$$

On the other hand, since

$$\int_0^\infty \sum_{j=1}^\infty \frac{e^{jn}}{(j-1)!} \frac{n}{t} |F(\frac{jn}{t})| e^{-\lambda t}\, dt < \infty$$

and

$$\int_0^\infty \sum_{j=1}^\infty \frac{e^{jn}}{(j-1)!} \frac{n}{s} |F(\frac{1}{s})| e^{-\lambda jns}\, ds < \infty,$$

we have

$$\begin{aligned} \int_0^\infty f_n(t) e^{-\lambda t}\, dt &= \int_0^\infty \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jn} \frac{n}{t} F(\frac{jn}{t}) e^{-\lambda t}\, dt \\ &= \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jn} \int_0^\infty \frac{n}{s} F(\frac{1}{s}) e^{-\lambda jns}\, ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-e^{n(1-\lambda s)}} e^{n(1-\lambda s)} \frac{n}{s} F(\frac{1}{s})\, ds \\
&= \int_{-n}^\infty e^{-e^{-u}} e^{-u} \frac{n}{n+u} F(\frac{\lambda n}{n+u})\, du \\
&= \int_{-\infty}^\infty \chi_{(-n,\infty)} e^{-e^{-u}} e^{-u} \frac{n}{n+u} F(\frac{\lambda n}{n+u})\, du,
\end{aligned}
$$

so by the dominated convergence theorem (using the condition that $|\lambda F(\lambda)| \leq M$ a.e. $\lambda > 0$),

$$
\lim_{n\to\infty} \int_0^\infty f_n(t) e^{-\lambda t}\, dt = \int_{-\infty}^\infty e^{-e^{-u}} e^{-u} F(\lambda)\, du = F(\lambda).
$$

Hence F is the Laplace transform of f. □

In the proof of the above theorem, we use the following version of the dominated convergence theorem: if $\int_X \sum_{j=1}^\infty |g_j| < \infty$, then $\int_X \sum_{j=1}^\infty g_j = \sum_{j=1}^\infty \int_X g_j$. This kind of argument will be used in later proofs and will not be mentioned explicitly.

Corollary 2.2. *Suppose a continuous function F on $(0,\infty)$ satisfies*

$$
\sup_{\lambda>0} |\lambda F(\lambda)| < \infty
$$

and

$$
\sup_{\lambda>0, n\in\mathbf{N}} \Big| \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jn} \lambda F(j\lambda) \Big| < \infty.
$$

Then F is infinitely differentiable and can be extended to an analytic function on the right half-plane $\{z \in \mathbf{C} : \operatorname{Re} z > 0\}$.

Note that unlike Bernstein's theorem on completely monotone functions (see [7]), the condition given in the above corollary does not involve higher derivatives of F.

Next we want to consider Laplace transforms of vector-valued functions. Given $f \in L^\infty((0,\infty), E)$, where E is a Banach space, using the same argument as in the proof of Theorem 2.1, we see that the Laplace transform F of f satisfies

$$
(\mathrm{P}_\infty) \qquad \sup_{\lambda>0} \|\lambda F(\lambda)\| < \infty \quad \text{and} \quad \sup_{\lambda>0, n\in\mathbf{N}} \Big\| \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jn} \lambda F(j\lambda) \Big\| < \infty.
$$

We will show that the converse holds if E has the Radon-Nikodým property. In fact, this gives a characterization for Banach spaces with the Radon-Nikodým property. The idea is to show that condition (P_∞) is equivalent to Widder's condition.

Theorem 2.3. *Let E be a Banach space and let $F \in C((0,\infty), E)$. The following assertions are equivalent.*

1. *There exists a Lipschitz continuous function $\alpha : [0,\infty) \longrightarrow E$ with $\alpha(0) = 0$ such that*
$$
F(\lambda) = \int_0^\infty \lambda e^{-\lambda t} \alpha(t)\, dt \quad \forall\, \lambda > 0.
$$
2. *F satisfies condition (P_∞).*
3. *F is infinitely differentiable and* $\sup\{\|\frac{1}{n!}\lambda^{n+1} F^{(n)}(\lambda)\| : \lambda > 0, n \in \mathbf{N}\cup\{0\}\} < \infty$.

Proof. (1 implies 2) Let $x^* \in E^*$. Consider the scalar-valued function $g(t) = \langle \alpha(t), x^* \rangle$. The conditions on α imply that there exists $f \in L^\infty(0,\infty)$ such that $g(t) = \int_0^t f(s)\,ds$ for all $t \geq 0$. So for every $\lambda > 0$, we have (using Fubini's theorem)

$$\langle F(\lambda), x^* \rangle = \int_0^\infty \left(\lambda e^{-\lambda t} \int_0^t f(s)\,ds \right) dt = \int_0^\infty e^{-\lambda t} f(t)\,dt.$$

Using the proof of Theorem 2.1 together with the uniform boundedness principle, we see that F satisfies condition (P_∞).

(2 implies 1) For every $x^* \in E^*$, we consider the function $\lambda \mapsto \langle F(\lambda), x^* \rangle$. By Theorem 2.1, there exists $\tilde{f}_{x^*} \in L^\infty(0,\infty)$ such that

$$\langle F(\lambda), x^* \rangle = \int_0^\infty e^{-\lambda t} \tilde{f}_{x^*}(t)\,dt \quad \forall\, \lambda > 0.$$

It follows from the proof of [1, Theorem 1.1] that there exists a function α which satisfies the requirements.

The equivalence of 1 and 3 is just [1, Theorem 1.1]. $\square$

Theorem 2.4. *A Banach space E has the Radon-Nikodým property if and only if every $F \in C((0,\infty), E)$ satisfying condition* (P_∞) *is the Laplace transform of some $f \in L^\infty((0,\infty), E)$.*

Proof. This is an immediate consequence of Theorem 2.3 and [1, Theorem 1.4]. $\square$

Remark 2.1. If E is a dual space and has the Radon-Nikodým property, then $L^\infty((0,\infty), E)$ is a dual space (see [4]). So given $F \in C((0,\infty), E)$ satisfying condition (P_∞), the bounded sequence (f_n) constructed in the proof of Theorem 2.1 has a weak* limit f which is the inverse Laplace transform of F.

For continuous $f \in L^\infty((0,\infty), E)$, where E is a Banach space not necessarily possessing the Radon-Nikodým property, we have the following inversion formula.

Theorem 2.5. *Let E be a Banach space. Let $f : (0,\infty) \longrightarrow E$ be a bounded continuous function and F its Laplace transform. Then*

$$f(t) = \lim_{n\to\infty} \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jnt} nF(jn) \quad \forall\, t > 0\ ,$$

the convergence is uniform on compact subsets of $(0,\infty)$, and uniform on bounded subsets of $(0,\infty)$ if $f(0+) = \lim_{t\to 0+} f(t)$ exists, and in this case,

$$f(0+) = (1-e^{-1})^{-1} \lim_{n\to\infty} n \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} F(jn)\ .$$

Proof. Let $t \geq 0$ and $n \in \mathbf{N}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jnt} nF(jn) &= \lim_{n\to\infty} \int_0^\infty n \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{jnt} e^{-jnr} f(r)\,dr \\
&= \lim_{n\to\infty} \int_0^\infty n e^{-e^{n(t-r)}} e^{n(t-r)} f(r)\,dr \\
&= \lim_{n\to\infty} \int_{-nt}^\infty e^{-e^{-u}} e^{-u} f\Big(\frac{nt+u}{n}\Big)\,du
\end{aligned}$$

$$= \begin{cases} \int_{-\infty}^{\infty} e^{-e^{-u}} e^{-u} f(t)\, du & \text{if } t > 0, \\ \int_{0}^{\infty} e^{-e^{-u}} e^{-u} f(0+)\, du & \text{if } t = 0 \text{ and } f(0+) \text{ exists,} \end{cases}$$

where the last equality follows from the dominated convergence theorem and the condition that f is continuous. Since f is uniformly continuous on $[a, b]$ for $0 < a < b < \infty$ (on $(0, b]$ if $f(0+)$ exists), the convergence given in the last equality is uniform on $[a, b]$ (on $(0, b]$ if $f(0+)$ exists). □

Remark 2.2. Using the same idea as in the above proof, we see that the sequence (f_n) constructed in the proof of Theorem 2.1 converges to f for all $t > 0$ if f is continuous. However, we cannot consider the convergence at $t = 0$ for this sequence.

3. Semigroups of operators

Let E be a Banach space. The space of all bounded linear operators from E into itself is denoted by $\mathcal{B}(E)$. A family $(S(t))_{t>0} \subset \mathcal{B}(E)$ is said to be a semigroup if $S(s+t) = S(s)S(t)$ for all $s, t > 0$. If $(S(t))_{t>0}$ is a strongly continuous semigroup and SOT-$\lim_{t\to 0+} S(t) = I := S(0)$, $(S(t))_{t\geq 0}$ is called a C_0-semigroup.

Proposition 3.1. *Let E be a Banach space. Let $A : \mathcal{D}(A) \subset E \longrightarrow E$ be a closed linear operator and let $w \in \mathbf{R}$. If there exists a strongly continuous semigroup $(S(t))_{t>0} \subset \mathcal{B}(E)$ satisfying $\|S(t)\| \leq Me^{wt}$ for all $t > 0$, where M is a constant, such that for all $x \in E$,*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x\, dt \quad \forall\, \lambda > w,$$

then $(w, \infty) \subset \rho(A)$ and the function $F : (0, \infty) \longrightarrow \mathcal{B}(E)$ defined by

$$F(\lambda) = R(w + \lambda, A)$$

satisfies condition (P_∞)*. The converse is true if E has the Radon-Nikodým property.*

Proof. The condition on $(S(t))_{t>0}$ implies that F is the Laplace transform (in the strong operator topology) of the bounded function $t \mapsto e^{-wt} S(t)$. Hence F satisfies condition (P_∞).

Conversely, if F satisfies condition (P_∞), by Theorem 2.3, it satisfies the Hille-Yosida condition, namely,

$$\sup_{\lambda>0, m\in \mathbf{N}\cup\{0\}} \|(\lambda R(\lambda, A-w))^m\| < \infty.$$

Hence by [1, Theorem 6.2], there exists a strongly continuous semigroup $(T(t))_{t>0}$ satisfying $\sup_{t>0} \|T(t)\| < \infty$ such that $R(\lambda, A-w)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt$ for all $\lambda > 0$, $x \in E$. Hence $(S(t) = e^{wt} T(t))_{t>0}$ is the required semigroup. □

Remark 3.1. The converse is also true if A is densely defined. In this case, the strongly continuous semigroup $(S(t))_{t>0}$ can be extended to a C_0 semigroup $(S(t))_{t\geq 0}$ (see Corollary 3.7).

Proposition 3.2. *Let E be a Banach space. Let $w \in \mathbf{R}$. Suppose $A : \mathcal{D}(A) \subset E \longrightarrow E$ is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ with $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$, where M is a constant. Then for every $x \in E$, we have*

$$\begin{aligned} S(t)x &= e^{wt} \lim_{n\to\infty} n \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{jnt} R(jn+w, A)x \quad \text{for } t > 0, \\ (1-e^{-1})x &= \lim_{n\to\infty} n \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} R(jn+w, A)x, \end{aligned}$$

and the convergence is uniform on $(0, b]$ for $0 < b < \infty$.

Proof. This is an immediate consequence of Theorem 2.5 □

Let $n \in \mathbf{N}$. A strongly continuous family $(S(t))_{t\geq 0} \subset \mathcal{B}(E)$ is called an n-times integrated semigroup if $S(0) = 0$ and, for all $x \in E$,

$$S(t)S(s)x = \frac{1}{(n-1)!} \left[\int_t^{s+t} (s+t-r)^{n-1} S(r)x\, dr - \int_0^s (s+t-r)^{n-1} S(r)x\, dr \right]$$

$\forall\, s, t \geq 0$. For convenience, a C_0-semigroup is also called a 0-times integrated semigroup.

An n-times integrated semigroup $(S(t))_{t\geq 0}$ (where $n \in \mathbf{N}$) is said to be

1. exponentially bounded if there exist constants M, w such that $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$;
2. non-degenerate if $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$;
3. locally Lipschitz if there exist constants M, w such that $\|S(t+h) - S(t)\| \leq Me^{w(t+h)}h$ for all $t, h \geq 0$.

Given a non-degenerate, exponentially bounded n-times integrated semigroup $(S(t))_{t\geq 0}$ (where $n \in \mathbf{N}$), there exists a unique operator A and there exists $a \in \mathbf{R}$ with $(a, \infty) \subset \rho(A)$ such that $R(\lambda, A)x = \int_0^\infty \lambda^n e^{-\lambda t} S(t)x\, dt$ for all $\lambda > a, x \in E$. This unique operator is called the generator of $(S(t))_{t\geq 0}$. Since we are mainly interested in generators, for $n \in \mathbf{N}$, a non-degenerate, exponentially bounded n-times integrated semigroup will be called an n-times integrated semigroup for simplicity.

It should be pointed out that for an n-times integrated semigroup $(S(t))_{t\geq 0}$ ($n \in \mathbf{N}$) with $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$, the constant w must be non-negative. This follows from the equality

$$S(t)x = \frac{t^n}{n!}x + \int_0^t S(s)Ax\, ds,$$

which holds for all $x \in \mathcal{D}(A)$ and $t \geq 0$. Similarly, if $(S(t))_{t\geq 0}$ is locally Lipschitz with $\|S(t+h) - S(t)\| \leq Me^{w(t+h)}h$ for all $t, h \geq 0$, the constant w must be non-negative.

If A generates an n-times integrated semigroup $(S(t))_{t\geq 0}$, then for every $\lambda \in \mathbf{C}$, $A - \lambda$ generates an n-times integrated semigroup $(\widetilde{S}(t))_{t\geq 0}$, where

$$\widetilde{S}(t)x = e^{-\lambda t}S(t)x + \sum_{k=1}^{n} \lambda^k \binom{n}{k} \int_0^t \int_0^{u_k} \cdots \int_0^{u_2} e^{-\lambda u_1} S(u_1)x\, du_1 \cdots du_k \quad \forall\, x \in E\,.$$

(To see this, it suffices to check that $\int_0^\infty e^{-\mu t}\widetilde{S}(t)x\, dt = \frac{1}{\mu^n}R(\mu, A-\lambda)x$.) The following two lemmas give the relation between the locally Lipschitz constants of $(S(t))_{t\geq 0}$ and $(\widetilde{S}(t))_{t\geq 0}$.

Lemma 3.3. *Let $n \in \mathbf{N}$. Suppose A generates an n-times integrated semigroup $(S(t))_{t\geq 0}$ satisfying*

$$\|S(t+h) - S(t)\| \leq Mh \quad \forall\, t, h \geq 0,$$

where M is a constant. Then for every $\lambda > 0$, $A+\lambda$ generates an n-times integrated semigroup $(\widetilde{S}(t))_{t\geq 0}$ with the property that given any $\epsilon > 0$, there exists a constant $\widetilde{M}$ such that

$$\|\widetilde{S}(t+h) - \widetilde{S}(t)\| \leq \widetilde{M} e^{(\lambda+\epsilon)(t+h)} h \quad \forall\, t, h \geq 0.$$

Proof. Let $\lambda, \epsilon > 0$. Take $M_1 > 0$ such that $\|S(t)\| \leq M_1 e^{\epsilon t}$ for all $t \geq 0$. Then for every $t, h \geq 0$, we have

$$\begin{aligned}
&\|\widetilde{S}(t+h) - \widetilde{S}(t)\| \leq \|e^{\lambda(t+h)} S(t+h) - e^{\lambda t} S(t)\| \\
&\quad + \sum_{k=1}^{n} \lambda^k \binom{n}{k} \int_t^{t+h} \int_0^{u_k} \cdots \int_0^{u_2} e^{\lambda u_1} \|S(u_1)\|\, du_1 \cdots du_k \\
&\leq e^{\lambda(t+h)} \|S(t+h) - S(t)\| + (e^{\lambda(t+h)} - e^{\lambda t}) \|S(t)\| \\
&\quad + \sum_{k=1}^{n} \lambda^k \binom{n}{k} \int_t^{t+h} \int_0^{u_k} \cdots \int_0^{u_2} M_1 e^{(\lambda+\epsilon) u_1}\, du_1 \cdots du_k \\
&\leq M e^{\lambda(t+h)} h + e^{\lambda(t+h)} \lambda h M_1 e^{\epsilon t} + \sum_{k=1}^{n} \lambda^k \binom{n}{k} M_1 (\lambda+\epsilon)^{1-k} e^{(\lambda+\epsilon)(t+h)} h \\
&\leq \left[M + \lambda M_1 + M_1 \sum_{k=1}^{n} \lambda^k \binom{n}{k} (\lambda+\epsilon)^{1-k} \right] e^{(\lambda+\epsilon)(t+h)} h. \qquad \square
\end{aligned}$$

Lemma 3.4. *Let $n = 1$ or 2. Suppose A generates an n-times integrated semigroup $(S(t))_{t\geq 0}$ satisfying*

$$\|S(t+h) - S(t)\| \leq M e^{w(t+h)} h \quad \forall\, t, h \geq 0,$$

where M, w are constants. Then for every $\lambda > w$, $A - \lambda$ generates an n-times integrated semigroup $(\widetilde{S}(t))_{t\geq 0}$ satisfying

$$\|\widetilde{S}(t+h) - \widetilde{S}(t)\| \leq \widetilde{M} h \quad \forall\, t, h \geq 0,$$

where $\widetilde{M}$ is a constant.

Proof. Let $\lambda > w$. Take $\epsilon > 0$ such that $\lambda > w + \epsilon$. It follows from the condition on $(S(t))_{t\geq 0}$ that there exists $M_1 > 0$ such that $\|S(t)\| \leq M_1 e^{(w+\epsilon)t}$ for all $t \geq 0$. So for every $t, h \geq 0$, we have

$$\begin{aligned}
&\|\widetilde{S}(t+h) - \widetilde{S}(t)\| \leq \|e^{-\lambda(t+h)} S(t+h) - e^{-\lambda t} S(t)\| + 2\lambda \int_t^{t+h} e^{-\lambda r} \|S(r)\|\, dr \\
&\quad + \lambda^2 \int_t^{t+h} \int_0^s e^{-\lambda r} \|S(r)\|\, dr\, ds \\
&\leq e^{-\lambda(t+h)} \|S(t+h) - S(t)\| + |e^{-\lambda(t+h)} - e^{-\lambda t}| M_1 e^{(w+\epsilon)t} \\
&\quad + 2\lambda \int_t^{t+h} M_1 e^{(w+\epsilon-\lambda) r}\, dr + \lambda^2 \int_t^{t+h} \int_0^s M_1 e^{(w+\epsilon-\lambda) r}\, dr\, ds \\
&\leq e^{(w-\lambda)(t+h)} M h + \lambda e^{-\lambda t} h M_1 e^{(w+\epsilon)t} + 2\lambda M_1 h + \lambda^2 M_1 (\lambda - w - \epsilon)^{-1} h \\
&\leq [M + 3\lambda M_1 + \lambda^2 M_1 (\lambda - w - \epsilon)^{-1}] h. \qquad \square
\end{aligned}$$

Lemma 3.5. *Let E be a Banach space and let $w, M \geq 0$. Suppose $F : [0,\infty) \longrightarrow E$ satisfies $\limsup_{h\to 0+} h^{-1}\|F(t+h)-F(t)\| \leq Me^{wt}$ for all $t \geq 0$. Then*

$$\|F(t+h)-F(t)\| \leq Me^{w(t+h)}h \quad for\ all\ t,h \geq 0.$$

Proof. It suffices to prove the result for the case where $E = \mathbf{R}$. First, we note that F is Lipschitz continuous on every bounded interval in $[0,\infty)$. Indeed, for every $\eta > 0$, take $M_1 > Me^{w\eta}$; then we have $\limsup_{h\to 0+} h^{-1}|F(t+h)-F(t)| < M_1$ for all $t \in [0,\eta)$. From this it follows that $|F(t+h)-F(t)| \leq M_1 h$ whenever $0 \leq t < t+h \leq \eta$.

Next, since F is absolutely continuous on bounded intervals in $[0,\infty)$,

$$\int_0^t F'(s)\,ds = F(t) - F(0) \quad for\ all\ t \geq 0.$$

Hence for $t, h \geq 0$,

$$|F(t+h)-F(t)| = |\int_t^{t+h} F'(s)\,ds| \leq \int_t^{t+h} Me^{ws}\,ds \leq Me^{w(t+h)}h.$$

□

Theorem 3.6. *Let E be a Banach space and let $A : \mathcal{D}(A) \subset E \longrightarrow E$ be a linear operator.*

1. *Let $n \in \mathbf{N} \cup \{0\}$. Suppose there exists $w \geq 0$ such that $(w,\infty) \subset \rho(A)$ and the function $F : (0,\infty) \longrightarrow \mathcal{B}(E)$ defined by*
$$F(\lambda) = \frac{1}{\lambda^n} R(w+\lambda, A)$$
satisfies condition (P_∞). *Then A generates an $(n+1)$-times integrated semigroup $(S(t))_{t\geq 0}$ with the property that, given any $w_1 > w$, there exists $M_1 > 0$ such that*
$$\limsup_{h\to 0+} h^{-1}\|S(t+h)-S(t)\| \leq M_1 e^{w_1 t} \quad \forall\, t \geq 0.$$
2. *Let $n = 0$ or 1. Suppose A generates an $(n+1)$-times integrated semigroup $(S(t))_{t\geq 0}$ satisfying*
$$\limsup_{h\to 0+} h^{-1}\|S(t+h)-S(t)\| \leq M_1 e^{w_1 t} \quad \forall\, t \geq 0,$$
where M_1, w_1 are constants. Then $(w_1,\infty) \subset \rho(A)$ and for every $w > w_1$, the function $F_w : (0,\infty) \longrightarrow \mathcal{B}(E)$ defined by
$$F_w(\lambda) = \frac{1}{\lambda^n} R(w+\lambda, A)$$
satisfies condition (P_∞).

Proof. (1) By Theorem 2.3, there exists a constant $M > 0$ and a function $T : [0,\infty) \longrightarrow \mathcal{B}(E)$ satisfying $T(0) = 0$ and $\|T(t+h)-T(t)\| \leq Mh$ for all $t, h \geq 0$ such that for all $x \in E$,

$$R(\lambda, A-w)x = \int_0^\infty \lambda^{n+1} e^{-\lambda t} T(t)x\,dt \quad \forall\, \lambda > 0.$$

By [1, Theorem 3.1], $(T(t))_{t\geq 0}$ is an $(n+1)$-times integrated semigroup with generator $A-w$. Hence by Lemma 3.3, A generates an $(n+1)$-times integrated semigroup $(S(t))_{t\geq 0}$ with the required property.

(2) For every $w > w_1$, by Lemma 3.5 and Lemma 3.4, $A-w$ generates a Lipschitz continuous $(n+1)$-times integrated semigroup. Hence by Theorem 2.3, F_w satisfies condition (P_∞). $\square$

Remark 3.2. The second assertion in the above theorem does not hold if $n \geq 2$. For example, in $\mathbf{R}$ or $\mathbf{C}$, $A = -1$ generates a 3-times integrated semigroup $(S(t) = -e^{-t} + \frac{t^2}{2} - t + 1)_{t\geq 0}$ satisfying $\limsup_{h\to 0+} h^{-1}\|S(t+h) - S(t)\| \leq 2e^t$ for all $t \geq 0$. However, $F_w(\lambda) = \frac{1}{\lambda^2(\lambda+w+1)}$ does not satisfy condition (P_∞) for any w.

Corollary 3.7. *Let $A : \mathcal{D}(A) \subset E \longrightarrow E$ be closed and densely defined and let $n \in \mathbf{N} \cup \{0\}$. If there exists $w > 0$ such that $(w, \infty) \subset \rho(A)$ and the function $F : (0, \infty) \longrightarrow \mathcal{B}(E)$ defined by*

$$F(\lambda) = \frac{1}{\lambda^n} R(w + \lambda, A)$$

satisfies condition (P_∞), then A generates an n-times integrated semigroup. The converse is true for $n = 0, 1$.

Proof. If A satisfies the given condition, then by Theorem 3.6, A generates a locally Lipschitz $(n + 1)$-times integrated semigroup. Hence by [1, Corollary 4.2], A generates an n-times integrated semigroup.

Conversely, for $n = 0$ or 1, if A generates an n-times integrated semigroup $(S(t))_{t\geq 0}$ with $\|S(t)\| \leq Me^{w_1 t}$ for all $t \geq 0$, where M, w_1 are constants, then A generates an $(n + 1)$-times integrated semigroup $(\widetilde{S}(t) = \int_0^t S(r)\,dr)_{t\geq 0}$ (in the strong operator topology) satisfying $\|\widetilde{S}(t+h) - \widetilde{S}(t)\| \leq Me^{w_1(t+h)}h$ for all $t, h \geq 0$. Hence the required result follows from Theorem 3.6. $\square$

To close our disscussion, we give the following example studied in [6].

Example 3.1. Let $E = L^1[0, R) \times L^1[0, R)$, where R is a positive constant (larger than the life span of human beings). Let $A : \mathcal{D}(A) \subset E \longrightarrow E$ be given by

$$A\varphi = (-\varphi_1' - (\mu + \delta)\varphi_1 + \sigma\varphi_2\ ,\ -\varphi_2' - (\tilde{\mu} + \sigma)\varphi_2 + \sigma\varphi_1),$$

where $\mathcal{D}(A)$ consists of all $\varphi = (\varphi_1, \varphi_2) \in E$ with φ_1, φ_2 absolutely continuous and satisfying

$$\varphi_1(0) = \beta \int_0^R h(r)k(r)\varphi_1(r)\,dr + \tilde{\beta} \int_0^R \tilde{h}(r)\tilde{k}(r)\varphi_2(r)\,dr,$$
$$\varphi_2(0) = \alpha \int_0^R h(r)k(r)\varphi_1(r)\,dr + \tilde{\alpha} \int_0^R \tilde{h}(r)\tilde{k}(r)\varphi_2(r)\,dr\ ,$$

and $\mu, \tilde{\mu}, \sigma, \delta, k, \tilde{k}, h, \tilde{h}$ are nonnegative measurable functions on $[0, R)$ ($\mu, \tilde{\mu}$ are the age specific mortality moduli of normal and disabled people; $0 \leq \sigma(r), \delta(r) \leq 1$ represent the recover rate and disabled rate at age r; $0 < k(r), \tilde{k}(r) < 1$ represent the proportion of the female population and that of the female disabled population of age r; $h, \tilde{h}$ with L^1-norm equal to 1 are the birth modes of females and disabled females respectively) and $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ are constants (which, in fact, depend on government population policy). Then A satisfies the conditions given in Corollary 3.7 for $n = 0$ (for details, see [6]) and thus generates a C_0-semigroup.

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