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A LERAY-SCHAUDER TYPE THEOREM FOR APPROXIMABLE MAPS: A SIMPLE PROOF

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Abstract. We present a simple and direct proof for a Leray-Schauder type alternative for a large class of condensing or compact set-valued maps containing convex as well as nonconvex maps.

The aim of this note is to extend the Leray-Schauder type nonlinear alternative presented in [BI] to a condensing upper semicontinuous approximable set-valued map $F\colon\ X \to E$ when X is a closed subset with nonempty interior of a locally convex topological vector space E. The proof presented here is even shorter and simpler than the one given in [BI].

In what follows E stands for a Hausdorff locally convex topological vector space with a fundamental basis $\mathcal{N}$ of convex, symmetric neighborhoods of the origin; if X, Y are nonempty subsets of E, then $F\colon\ X \to Y$ is a set-valued map with nonempty values (simply called *map*). The boundary, the interior, the closure, and the convex hull of a subset A in E are denoted by ∂A, $\operatorname{int} A$, $\overline{A}$, and $\operatorname{co} A$ respectively.

Definition 1. F is said to be *upper semicontinuous* (*u.s.c.*) on X if and only if for any open subset V of Y, the set $\{x \in X\colon\ F(x) \subset V\}$ is open in X.

Definitions 2 ([BD], [BI], see also [GGK] for metric spaces)**.** (1) Given $U, V \in \mathcal{N}$, a function $s\colon\ X \to Y$ is said to be a (U, V)-*approximative selection* of F if for any $x \in X$, $s(x) \in (F[(x + U) \cap X] + V) \cap Y$.

(2) $F\colon\ X \to Y$ is said to be *approachable* if it has a continuous (U, V)-approximative selection for any $(U, V) \in \mathcal{N} \times \mathcal{N}$. $\mathcal{A}(X, Y)$ denotes the class of such maps. We write $\mathcal{A}(X)$ for $\mathcal{A}(X, X)$.

(3) F is said to be *approximable* if its restriction $F|_K$ to any compact subset K of X is approachable.

Note that an approachable map is approximable (cf. [B]).

Examples. It is well-known that if F is *u.s.c.* with nonempty convex values, then F is approachable provided X is paracompact and Y is convex (cf. [DG]). Obviously, F is approximable without conditions on X (see [C]).

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In the absence of convexity, we have:
Assume that X is dominated by finite polyhedra (e.g. X is a compact *ANR*) and that $F\colon\ X \to Y$ is a *u.s.c.* map with nonempty compact values. Then F is in $\mathcal{A}(X, Y)$ if either one of the following situations holds:

(a) Y is an *ANR* and the values of F are contractible ([BD]).

(b) The values of F are ∞-proximally connected in Y ([GGK]), see also [BD]).

Definitions 3 ([PF], see also [CF])**.** (1) If C is a lattice with a minimal element, denoted by 0, a function $\Phi\colon\ 2^E \to C$ is called a *measure of noncompactness* provided that the following conditions hold for any $A, B \in 2^E$:

(i) $\Phi(\overline{\mathrm{co}}(A)) = \Phi(A)$;
(ii) $\Phi(A) = 0$ if and only if A is precompact;
(iii) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

(2) $F\colon\ X \to Y$ is said to be Φ-*condensing* provided that if $A \subset X$ with $\Phi(F(A)) \geq \Phi(A)$, then A is relatively compact.

It should be noticed that there exist Φ-condensing maps $F\colon\ X \to E$ only if, for the subsets of X, precompactness coincides with relative compactness. On the other hand, a compact map $F\colon\ X \to E$ is Φ-condensing if either the domain X is complete or if E is quasicomplete. Every map defined on a compact set is necessarily Φ-condensing.

The following finite-type approximation property of compact approximable maps plays a crucial role in the proof of the main theorem of this note.

Lemma 1. *Let $F\colon\ X \to E$ be a compact approximable map. Given any $V \in \mathcal{N}$, there exist a finite subset N_V of $\overline{F(X)}$ and an approximable map F_V with values in $\mathrm{co}(N_V)$ such that $F_V(x) \subset F(x) + V$, for every $x \in X$. Moreover, F_V is u.s.c. with nonempty closed values whenever F has the same properties.*

If $F \in \mathcal{A}(X, E)$ takes its values in a convex compact subset K of E, then $F \in \mathcal{A}(X, K)$.

Proof. Let $V \in \mathcal{N}$ be arbitrary, and let $N_V = \{y_1, \dots, y_n\}$ be a finite subset of $\overline{F(X)}$ such that the collection $\{y_i + \frac{1}{6}V\colon\ i = 1, \dots, n\}$ forms an open cover of the compact set $\overline{F(X)}$. Consider the Schauder projection (cf. [DG]) $\pi_V\colon \bigcup_{i=1}^n (y_i + \frac{1}{3}V) \to \mathrm{co}(N_V)$ defined by:

$$\pi_V(y) := \frac{1}{\sum_{i=1}^n \mu_i(y)} \sum_{i=1}^n \mu_i(y) y_i, \quad \text{for all } y \in \bigcup_{i=1}^n \left(y_i + \frac{1}{3}V\right),$$

where $\mu_i(y) = \max\{0, 1 - p_{\frac{1}{3}V}(y - y_i)\}$ and $p_{\frac{1}{3}V}$ is the Minkowski functional of $\frac{1}{3}V$. One readily verifies that:

$$\pi_V(y) - y \in \frac{1}{3}V, \quad \text{for all } y \in \bigcup_{i=1}^n \left(y_i + \frac{1}{3}V\right).$$

Define the map $F_V\colon\ X \to \mathrm{co}(N_V)$ as the composition product $F_V := \pi_V \circ F$. If F is approximable, F_V is approximable since its restriction to any compact subset of X is approachable as the composition product of two approachable maps (see [B, Proposition 2.5]). Moreover, $F_V(x) \subset F(x) + V$ for all $x \in X$, and F_V is *u.s.c.* and it has nonempty compact values whenever F has the same properties.

If $F \in \mathcal{A}(X, E)$ has values in a convex compact subset K of E, for a given $U \in \mathcal{N}$, let $s\colon\ X \to E$ be a continuous $(U, \frac{1}{6}V)$-approximative selection of F. Then one has for all $x \in X$,

$$s(x) \in F((x+U) \cap X) + \frac{1}{6}V \subset \bigcup_{i=1}^{n} \left(y_i + \frac{1}{6}V \right) + \frac{1}{6}V = \bigcup_{i=1}^{n} \left(y_i + \frac{1}{3}V \right),$$

$$\pi_V(s(x)) \in s(x) + \frac{1}{3}V \subset F((x+U) \cap X) + \frac{1}{6}V + \frac{1}{3}V \subset F((x+U) \cap X) + V.$$

That is, $\pi_V \circ s$ is a continuous (U, V)-approximative selection of F with values in $\mathrm{co}(N_V) \subset K$. □

We shall also need the following generalization of the Fan-Kakutani fixed point theorem.

Lemma 2 ([BD]). *Assume that X is convex and compact in E and that $F \in \mathcal{A}(X)$ is u.s.c. with nonempty closed values. Then F has a fixed point, that is, a point $x_0 \in X$ with $x_0 \in F(x_0)$.*

Finally, we prove for condensing maps the following useful result (see [PF]).

Lemma 3. *Assume that X is a nonempty subset of E and that $F\colon\ X \to E$ is a Φ-condensing map. Then there exists a nonempty compact and convex subset K of E such that $F(K \cap X) \subset K$.*

Proof. Let $x_0 \in X$ be fixed. Let us consider the family $\mathcal{F}$ of all closed convex subsets C of E such that $x_0 \in C$ and $F(C \cap X) \subset C$. Clearly $\mathcal{F} \neq \varnothing$, since $\overline{\mathrm{co}}(F(X) \cup \{x_0\}) \in \mathcal{F}$. Let $K = \bigcap_{C \in \mathcal{F}} C$. K is convex and closed and $x_0 \in K$. If $x \in K \cap X$, $F(x) \subset C$ for all $C \in \mathcal{F}$, so that $F(K \cap X) \subset K$ and thus $K \in \mathcal{F}$. It remains to prove that K is compact. If K is not compact, then $\Phi(F(K)) \not\geq \Phi(K)$, since F is Φ-condensing. Let $K' = \overline{\mathrm{co}}(\{x_0\} \cup F(K \cap X))$. Then $K' \subset K$ which implies that $F(K' \cap X) \subset F(K \cap X) \subset K'$. Hence $K' \in \mathcal{F}$ and $K \subset K'$. Therefore $K = K'$, $\Phi(K) = \Phi(K') = \Phi(F(K \cap X)) \leq \Phi(F(K))$ which contradicts $\Phi(F(K)) \not\geq \Phi(K)$. □

We are ready now to present the main result of this note.

Theorem. *Assume that X is a closed subset of E with boundary ∂X and that 0 is an interior point of X. Let $F\colon\ X \to E$ be a Φ-condensing or compact u.s.c. approximable map with nonempty closed values. Then one of the following properties holds:*

(1) $\exists x_0 \in X$, *with* $x_0 \in F(x_0)$*;*
(2) $\exists \hat{x} \in \partial X$, $\exists \lambda \in (0,1)$, *with* $\hat{x} \in \lambda F(\hat{x})$*.*

Proof. **Case 1.** F is Φ-condensing.

Suppose for each $x \in X$, $x \notin F(x)$ and for each $(\lambda, x) \in (0,1) \times \partial X$, $x \notin \lambda F(x)$. By Lemma 3, there exists a nonempty convex and compact subset K of E such that $F(K \cap X) \subset K$. Without loss of generality we can assume that $0 \in K$. Since $K \cap X$ is compact, $F|_{K \cap X} \in \mathcal{A}(K \cap X, E)$ and, by Lemma 1, $F|_{K \cap X} \in \mathcal{A}(K \cap X, K)$. Let $F'\colon\ K \to K$ be the map defined by:

$$F'(x) := \begin{cases} F(x) & \text{if } x \in \mathrm{int}\, X, \\ K & \text{if } x \notin \mathrm{int}\, X; \end{cases}$$

F' is *u.s.c.* with nonempty closed values. We first claim that $F' \in \mathcal{A}(K)$. Indeed, let $(U,V) \in \mathcal{N} \times \mathcal{N}$ be arbitrary and $s\colon\ K \cap X \to K$ be a continuous $(U, \frac{1}{2}V)$-approximative selection of $F|_{K\cap X}$. By Proposition 1.6 of [BD], there exists a continuous function $s'\colon\ K \to K$ such that s and $s'|_{K\cap X}$ are $\frac{1}{2}V$-near. Therefore s' is a continuous (U,V)-approximative selection of F'.

Consider now the set $C = \{x \in X \cap K | x \in \lambda F(x) \text{ for some } 0 \leq \lambda \leq 1\}$. C is nonempty ($0 \in C$) and C is closed (F is *u.s.c.* and $F(X \cap K) \subset K$), hence compact. Since E is Hausdorff, it is in fact a uniformizable space, hence completely regular (see [K], p. 47). Since $C \cap (E \setminus \operatorname{int} X) = \varnothing$, there is a continuous function $a\colon\ E \to [0,1]$ such that $a(x) = 1$ for $x \in C$ and $a(x) = 0$ for $x \in E \setminus \operatorname{int} X$. Let $G\colon\ K \to K$ be the map defined by:

$$G(x) := a(x)F'(x).$$

G is *u.s.c.* with nonempty closed values and by Proposition 2.4 and Proposition 2.5 of [B], $G \in \mathcal{A}(K)$. By Lemma 2, G has a fixed point $x_0 \in K$, $x_0 \in a(x_0)F'(x_0)$. If $x_0 \notin \operatorname{int} X$, $a(x_0) = 0$ and $x_0 = 0$, which contradicts the hypothesis $0 \in \operatorname{int} X$. If $x_0 \in \operatorname{int} X$, $x_0 \in a(x_0)F(x_0)$, hence $x_0 \in C$, $a(x_0) = 1$ and x_0 is a fixed point of F, another contradiction.

Case 2. F is compact.

Let $V \in \mathcal{N}$ be arbitrary but fixed. Consider the finite subset N_V of $\overline{F(X)}$ and the approximable map $F_V\colon\ X \to \operatorname{co}(N_V)$ verifying $F_V(x) \subset F(x) + V$, for all $x \in X$, both provided by Lemma 1. Without loss of generality, we can assume that $0 \in \operatorname{co}(N_V)$ (otherwise, replace $\operatorname{co}(N_V)$ by $\operatorname{co}(0 \cup N_V)$) and note that $F_V|_{X\cap \operatorname{co}(N_V)} \in \mathcal{A}(X \cap \operatorname{co}(N_V), \operatorname{co}(N_V))$.

The same proof as in Case 1 applied to $F_V|_{X\cap\operatorname{co}(N_V)}$ in place of $F|_{X\cap K}$ leads to the following alternative:

$(1)_V$ $\exists x_V \in X$, with $x_V \in F_V(x_V)$; or

$(2)_V$ $\exists x_V \in \partial X$, $\exists \lambda_V \in (0,1)$ with $x_V \in \lambda_V F_V(x_V)$.

A straightforward argument (see [BI]) based on the compactness of F, its upper semicontinuity and the closedness of its values ends the proof. □

We conclude the note with some remarks.

Remarks. (i) Obviously, if X is complete or E is quasicomplete, the result in the compact case follows from the result in the condensing case.

(ii) The previous result extends the Theorem in [R3] proved for convex-valued maps. Our proof is an adaptation of its proof. For Φ-condensing convex-valued maps, the result was obtained in [PF] using a topological degree argument.

(iii) Note that for F compact, the preceding theorem reduces to the Theorem in [BI]. However, the proof provided here is much shorter and simpler.

(iv) The single-valued condensing case was treated in [R1], [R2]. We also refer to [DG] for a treatment of the single valued compact case based on the theory of transversality and to [BI] for other references and comments.

(v) After acceptance of this note, reference [P] came to the authors' attention. There, Case 2 of our main theorem is treated, independently, with a similar argument.

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