

## A SHORT PROOF OF A CHARACTERIZATION OF REFLEXIVITY OF JAMES

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ABSTRACT. A short direct proof is given to a well-known intrinsic characterization of reflexivity due to R. C. James.

The following famous intrinsic geometric characterization of reflexivity is due to R. C. James [4] (cf. also e.g. [1, p. 51], [2, p. 58] or [3]).

**Theorem.** *A Banach space  $X$  is reflexive if and only if there is a  $\theta \in (0, 1)$  such that if  $(x_n)_{n=1}^\infty$  is a sequence of elements of the unit sphere of  $X$ ,  $S_X$ , with  $\|u\| > \theta$  for all  $u \in \text{conv}\{x_1, x_2, \dots\}$ , then there are  $n_0 \in \mathbb{N}$ ,  $u \in \text{conv}\{x_1, x_2, \dots, x_{n_0}\}$  and  $v \in \text{conv}\{x_{n_0+1}, x_{n_0+2}, \dots\}$  such that  $\|u - v\| \leq \theta$ .*

This note provides a short and easy direct proof of the James theorem. It does not rely on Helly's theorem (like the proof in [4]) nor the Šmulian-Eberlein theorem (like the proof in [5, pp. 95–99]).

*Proof of the Theorem. Necessity.* The following proof is traditional. We present it here for the sake of completeness. Let  $X$  be reflexive. Fix any  $\theta > 0$ , consider any  $(x_n)_{n=1}^\infty \subset S_X$ , and denote  $K_n = \overline{\text{conv}}\{x_{n+1}, x_{n+2}, \dots\}$ . Since  $(K_n)_{n=0}^\infty$  is a nested sequence of weakly compact sets, there is  $x \in \bigcap_{n=0}^\infty K_n$ . Since  $x \in K_0$ , there are  $n_0 \in \mathbb{N}$  and  $u \in \text{conv}\{x_1, \dots, x_{n_0}\}$  such that  $\|x - u\| < \theta/2$ . Since  $x \in K_{n_0}$ , there is  $v \in \text{conv}\{x_{n_0+1}, x_{n_0+2}, \dots\}$  such that  $\|x - v\| < \theta/2$ . Hence  $\|v - u\| < \theta$ .

*Sufficiency.* Denote  $B_\theta = \{F \in X^{**} : \|F\| \leq \theta\}$ . This is a weak\* closed set. If  $X \neq X^{**}$ , then by Riesz's lemma there is  $F_\theta \in S_{X^{**}} \setminus (B_\theta + \{x\})$  for all  $x \in X$ . Note that  $F_\theta$  is in the weak\* closure of  $S_X$  (by Goldstine's theorem and weak\* lower-semicontinuity of the norm). Pick any  $x_0 \in S_X$ . Since the weak\* open set  $X^{**} \setminus (B_\theta + \{x_0\})$  contains  $F_\theta$ , it also contains a convex weak\* neighbourhood  $V_1$  of  $F_\theta$ , which means

$$\|v - x_0\| > \theta \quad \forall v \in V_1.$$

Since  $F_\theta \in X^{**} \setminus B_\theta$ , we can assume that  $V_1 \subset X^{**} \setminus B_\theta$ . Pick any  $x_1 \in V_1 \cap S_X$ . Since  $F_\theta \in X^{**} \setminus (B_\theta + \text{conv}\{x_0, x_1\})$ , there is a convex weak\* neighbourhood  $V_2 \subset V_1$  of  $F_\theta$  such that

$$\|v - u\| > \theta \quad \forall u \in \text{conv}\{x_0, x_1\}, \quad \forall v \in V_2.$$

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Pick any  $x_2 \in V_2 \cap S_X$  and continue as above. The sequences of convex sets  $V_1 \supset V_2 \supset \dots$  and elements  $(x_n)_{n=1}^\infty \subset S_X$  satisfy  $x_n \in V_n$  and

$$\|v - u\| > \theta \quad \forall u \in \text{conv}\{x_1, \dots, x_n\}, \quad \forall v \in V_{n+1}.$$

Since  $\text{conv}\{x_n, x_{n+1}, \dots\} \subset V_n \subset X^{**} \setminus B_\theta$ , this contradicts the assumption.  $\square$

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