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HYPERSURFACES WITH POSITIVE PRINCIPAL CURVATURES IN SYMMETRIC SPACES

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A classical result in differential geometry, known as Hadamard's theorem, establishes that a *compact connected surface in Euclidean space whose principal curvatures are everywhere positive is the boundary of a convex body. In particular, the surface is diffeomorphic to a sphere* ([H]). We present here, using a simple observation as proof, a partial extension of this theorem to immersions of arbitrary codimension and to other spaces than the Euclidean one, as symmetric spaces of noncompact type.

Let M^n and N^{n+k} be Riemannian manifolds of dimensions n and $n+k$, $n \geq 2$, $k \geq 1$, M compact, connected, and let $\phi : M \to N$ be an isometric immersion. Denote by $N(M)$ the unit normal bundle of ϕ, namely

$$N(M) = \{(p,\eta) \,|\, p \in M, \eta \in T_{\phi(p)}N, \eta \perp \phi_*(T_pM), \|\eta\| = 1\}.$$

We denote by $N^*(M)$ the subbundle of $N(M)$ consisting of the pairs (p,η) such that the 2nd fundamental form A of ϕ with respect to η at p, that is, $A_\eta(v) = (\nabla_{\phi_* v}\eta)^{\mathrm{T}}$, $v \in T_pM$, has positive eigenvalues, ∇ being the Riemannian connection of N and $(\ )^{\mathrm{T}}$ the orthogonal projection to $\phi_*(T_pM)$. We prove:

Proposition. *According to the above notation, assume the existence of $n+1$ Killing fields $X_1, \ldots, X_{n+1}$ in N which are linearly independent on $\phi(M)$ and a global section $\eta : M \to N^*(M)$ such that $\eta(p) \in span\{X_1, \ldots, X_{n+1}\}$ for all $p \in M$. Then M is diffeomorphic to an n-dimensional sphere.*

We have the following immediate consequences:

Corollary 1. *Let N be a homogeneous manifold with an invariant metric. Then there is an open dense subset U of N such that any immersed compact, connected hypersurface of U whose principal curvatures are positive is diffeomorphic to a sphere.*

Corollary 2. *Let G be a Lie group with a left invariant metric. Then, any compact connected immersed hypersurface of G whose principal curvatures are positive is diffeomorphic to a sphere.*

We remark that Corollary 2 applies, for example, to a symmetric space G/K of noncompact type since, by the Iwasawa decomposition $G = KAN$, G/K is isometric to the solvable Lie group $S = AN$ with a certain left invariant metric.

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Proof of the proposition. Given $p \in M$, set

$$E_p = \{a_1 X_1(\phi(p)) + \cdots + a_{n+1} X_{n+1}(\phi(p)) \,|\, a_1^2 + \cdots + a_{n+1}^2 = 1\}.$$

From the hypothesis, there is a unit normal vector field η to ϕ such that the 2nd fundamental form of M with respect to η is positive definite and $\eta(p) \in span\{X_1(\phi(p)), \ldots, X_{n+1}(\phi(p))\}$, for all $p \in M$. There exists a differentiable map $f : M \to \mathbb{R}^+$ such that $f(p)\eta(p) \in E_p$, for all $p \in M$. We define a map $\gamma : M \to S^n$, where S^n is the unit sphere centered at the origin in $\mathbb{R}^{n+1}$, by setting

$$\gamma(p) = (a_1(p), \ldots, a_{n+1}(p)) \tag{1}$$

if

$$f(p)\eta(p) = a_1(p) X_1(\phi(p)) + \cdots + a_{n+1}(p) X_{n+1}(\phi(p)), \qquad p \in M. \tag{2}$$

We claim that γ is a diffeomorphism. Clearly γ is a differentiable map. Choose $p \in M$ and let $v \in T_pM$ such that $d\gamma_p(v) = 0$. From (1), we have $d(a_j)_p(v) = 0,\ j = 1, \ldots, n+1$.

Taking the covariant derivative of (2) with respect to v we therefore obtain

$$df_p(v)\eta(p) + f(p)\nabla_v \eta(p) = a_1(p)\nabla_v X_1 + \cdots + a_{n+1}(p)\nabla_v X_{n+1}.$$

Taking the inner product of both sides with v, since the X_j are Killing fields, we obtain $f(p)\langle \nabla_v \eta, v\rangle = 0$, which implies $v = 0$, since $\eta \in N^*(M)$. It follows that γ is a local diffeomorphism so that, since it goes into the sphere and M is compact, γ is a global diffeomorphism. □

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References

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