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AN APPLICATION OF SCHAUDER'S FIXED POINT THEOREM WITH RESPECT TO HIGHER ORDER BVPS

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ABSTRACT. We shall provide conditions on the function $f(t, u_1, \cdots, u_{n-1})$. The higher order boundary value problem

(BVP)

$$\begin{cases} (E)\ u^{(n)}(t) + f(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) = 0 \ \text{ for } \ t \in (0,1) \text{ and } n \geq 2, \\ (BC)\ \begin{cases} u^{(i)}(0) = 0,\ 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0 \end{cases} \end{cases}$$

has at least one solution.

1. INTRODUCTION

In this article, we shall attempt to construct some existence criteria for the following $n-$th order boundary value problem:

(BVP)

$$\begin{cases} (E)\ u^{(n)}(t) + f(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) = 0 \ \text{ for } \ t \in (0,1) \ \text{ and } \ n \geq 2, \\ (BC)\ \begin{cases} u^{(i)}(0) = 0,\ 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0. \end{cases} \end{cases}$$

The motivation for the present work stems from many recent investigations in [1]–[3], [8], [13], [23]–[24]. In fact, particular cases of the boundary value problem (BVP) occur in various physical phenomena [4]–[7], [9]–[10], [13], especially such as gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel [7], catalysis theory [9], chemically reacting systems, as well as adiabatic tubular reactor processes. For other related works, we refer to recent contributions of Agarwal and Wong [1]–[3], Anuradaha, Hai and Shivaji [4], Bailey, Shampine and Waltman [5], Erbe and Wang [13], Granas, Guenther and Lee [16], Lee and O'Regan [21], Chyan and Henderson [8], Henderson [17], Vasilev and Klokov [23] and Kelevedjiev [18]–[19] and the references therein.

Here, we shall remark that there are four main techniques to treat the existence of (BVP) as follows:

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Method 01. Shooting method ([25])**.** *This method has been used successfully in the study of some special boundary value problems, if one can guarantee the uniqueness of related initial value problems.*

Method 02. Fixed point index method ([1]–[2], [11], [20])**.** *This method has many advantages in treating non-singular boundary value problems and relies on the following lemma:*

Let E be a Banach space, and let $C \subseteq E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let

$$T : C \cap (\overline{\Omega_2} \backslash \Omega_1) \longrightarrow C$$

be a completely continuous operator such that either

(i) $||Tu|| \leq ||u||,\ u \in C \cap \partial\Omega_1$, *and* $||Tu|| \geq ||u||,\ u \in C \cap \partial\Omega_2$; *or*
(ii) $||Tu|| \geq ||u||,\ u \in C \cap \partial\Omega_1$, *and* $||Tu|| \leq ||u||,\ u \in C \cap \partial\Omega_2$.

Then T has a fixed point in $C \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Method 03. Nonlinear alternative or topological method ([6], [15]–[16], [22])*. This method was initiated in Granas, Guenther and Lee* [15]–[16]*:*

Let X, Z be real vector normed spaces, $L : domL \subset X \rightarrow Z$ a linear Fredholm mapping of index zero, $\Omega \subset X$ an open bounded subset, and $N : \overline{\Omega} \rightarrow Z$ an L-compact mapping. If $kerL = \{0\},\ 0 \in \Omega$ and

$$Lx - \mu Nx \neq 0$$

for every $(x, \mu) \in (domL \cap \partial\Omega) \times (0, 1)$, then the equation

$$Lx = Nx$$

has at least one solution in $domL \cap \overline{\Omega}$.

Method 04. Schauder's or Barrier's method ([12])**.** *In the next section, we attempt to establish a general existence principle for (BVP), which relies on* **Schauder's fixed point theorem:**

Let C be a convex subset of a normed linear space $\mathbb{E}$. Then every compact continuous function $T : C \longrightarrow C$ has at least one fixed point.

2. Main results

Let $\alpha, \gamma, \beta, \delta \geq 0$, $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$ and $\mathbb{B}$ be the Banach space

$$\{u \in C^{(n)}(0, 1) \cap C^{(n-1)}[0, 1] \mid u^{(i)}(0) = 0,\ 0 \leq i \leq n - 3\}$$

with norm $||u|| \equiv \sup_{t \in [0,1]} |u^{(n-2)}(t)|$.

In order to abbreviate our discussion, we suppose throughout this paper that the following assumptions hold:

(C_1)**:** $K(t, s)$ is the Green's function of the differential equation

$$-u^{(n)}(t) = 0 \text{ in } (0, 1)$$

subject to the boundary conditions (BC).

(C_2)**:** $k(t, s)$ is the Green's function of the differential equation

$$-u''(t) = 0 \text{ in } (0, 1)$$

subject to the boundary conditions

$$\text{(BC}^*\text{)}\qquad \begin{cases}\alpha u(0)-\beta u'(0)=0,\\ \gamma u(1)+\delta u'(1)=0.\end{cases}$$

(C_3)**:** $f\in C([0,1]\times\mathbb{R}^{n-1};\mathbb{R})$.

(C_4)**:** $v,\ w\in\mathbb{B}$ are lower solutions and upper solutions of (BVP) in the sense:

$$\begin{cases}(1^0) & v^{(n)}(t)+f(t,v(t),v^{(1)}(t),\cdots,v^{(n-2)}(t))\geq 0 \ \text{ for } \ t\in(0,1),\\ (2^0) & w^{(n)}(t)+f(t,w(t),w^{(1)}(t),\cdots,w^{(n-2)}(t))\leq 0 \ \text{ for } \ t\in(0,1),\\ (3^0) & \begin{cases} v^{(i)}(0)=0,\ 0\leq i\leq n-3,\\ \alpha v^{(n-2)}(0)-\beta v^{(n-1)}(0)\leq 0,\\ \gamma v^{(n-2)}(1)+\delta v^{(n-1)}(1)\leq 0,\end{cases}\\ (4^0) & \begin{cases} w^{(i)}(0)=0,\ 0\leq i\leq n-3,\\ \alpha w^{(n-2)}(0)-\beta w^{(n-1)}(0)\geq 0,\\ \gamma w^{(n-2)}(1)+\delta w^{(n-1)}(1)\geq 0,\end{cases}\end{cases}$$

respectively.

(C_5)**:** $f(t,u_1,\cdots,u_{n-2},u_{n-1}),\ v(t)$ and $w(t)$ satisfy

$$\begin{aligned}&v^{(n-2)}(t)\leq w^{(n-2)}(t) \quad\text{on}\quad [0,1],\quad\text{and}\\ &f(t,v(t),\cdots,v^{(n-3)}(t),u_{n-1})\\ &\leq f(t,u_1,\cdots,u_{n-2},u_{n-1})\\ &\leq f(t,w(t),\cdots,w^{(n-3)}(t),u_{n-1})\\ &\text{for}\ \ t\in[0,1],\ (v(t),\cdots,v^{(n-3)}(t))\leq(u_1,\cdots,u_{n-2})\leq(w(t),\cdots,w^{(n-3)}(t)),\end{aligned}$$

in which

$$(x_1,\cdots,x_{n-2})\leq(y_1,\cdots,y_{n-2})\Longleftrightarrow x_i\leq y_i \ \text{ for } \ i=1,\cdots,n-2.$$

Remark 2.1. It is clear that

(a) if $f(t,u_1,\cdots,u_{n-2},u_{n-1})$ is increasing with respect to $(u_1,\cdots,u_{n-2})$ on $\mathbb{R}^{n-2}$ for each fixed $(t,u_{n-1})\in[0,1]\times\mathbb{R}$, then (C_5) holds;

(b) a simple calculation can show that

$$\begin{aligned}&\frac{\partial^{n-2}}{\partial t^{n-2}}K(t,s)=k(t,s)\\ &=\begin{cases}\frac{1}{\rho}(\beta+\alpha s)\{\delta+\gamma(1-t)\},\ 0\leq s\leq t\leq 1,\\ \frac{1}{\rho}(\beta+\alpha t)\{\delta+\gamma(1-s)\},\ 0\leq t\leq s\leq 1;\end{cases}\end{aligned}$$

(c) there exists an $M\in(0,1)$ such that

$$\begin{cases}(R_1)\ \frac{k(t,s)}{k(s,s)}\leq 1, \quad \textit{for } t\in[0,1] \textit{ and } s\in[0,1],\\ (R_2)\ \frac{k(t,s)}{k(s,s)}\geq M, \quad \textit{for } t\in[\frac{1}{4},\frac{3}{4}] \textit{ and } s\in[0,1].\end{cases}$$

Now, we can state and prove our main result:

Theorem 2.2 (Main result)**.** *Boundary value problem* (BVP) *has at least one solution* $u\in\mathbb{B}$ *such that*

$$v^{(i)}(t)\leq u^{(i)}(t)\leq w^{(i)}(t) \quad \textit{on} \quad [0,1] \quad \textit{for} \quad i=0,1,\cdots,n-2.$$

Proof. We separate the proof into the following steps:

Step (1). Consider the modified problem

$$\text{(BVP}^*\text{)}\quad \begin{cases} (E^*)\ u^{(n)}(t) + f^*(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) = 0 \ \text{ for } \ t \in (0,1), \\ (BC)\ \begin{cases} u^{(i)}(0) = 0,\ 0 \le i \le n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{cases} \end{cases}$$

where

$$f^*(t, u_1, \cdots, u_{n-1}) := f(t, \eta_1, \cdots, \eta_{n-1}) + \rho(\eta_{n-1} - u_{n-1}),$$

$$\eta_i := \begin{cases} w^{(i-1)}(t) \ \text{ if } \ u_i > w^{(i-1)}(t), \\ u_i \ \text{ if } \ v^{(i-1)}(t) \le u_i \le w^{(i-1)}(t), \\ v^{(i-1)}(t) \ \text{ if } \ u_i < v^{(i-1)}(t), \end{cases}$$

for all $i = 1, 2, \cdots, n-1$, $t \in [0,1]$ and $\rho : \mathbb{R} \longrightarrow [-1,1]$ is the radial retraction defined by

$$\rho(r) := \begin{cases} r \ \text{ for } \ |r| \le 1, \\ \frac{r}{|r|} \ \text{ for } \ |r| > 1. \end{cases}$$

It is clear that (BVP*) has a solution $u = u(t)$ if, and only if, u is the solution of the operator equation

$$u(t) = \int_0^1 K(t,s) f^*(s, u(s), u^{(1)}(s), \cdots, u^{(n-2)}(s)) ds := (Tu)(t),\ u \in \mathbb{B},$$

or

$$u^{(n-2)}(t) = \int_0^1 k(t,s) f^*(s, u(s), u^{(1)}(s), \cdots, u^{(n-2)}(s)) ds := (Tu)^{(n-2)}(t),\ u \in \mathbb{B}.$$

Since f^* is continuous and bounded on $[0,1] \times \mathbb{R}^{n-1}$, $T : \mathbb{B} \longrightarrow \mathbb{B}$ is continuous and compact. Therefore, it follows from Schauder's fixed point theorem (cf. Method 04) that T has a fixed point $u \in \mathbb{B}$, i.e. (BVP*) has a solution $u \in \mathbb{B}$.

Step (2). Let

$$H(t) := u^{(n-2)}(t) - w^{(n-2)}(t) \ \text{ on } \ [0,1].$$

Then we have

$$\begin{aligned} &H''(\theta) \\ &\ge -f^*(\theta, u(\theta), \cdots, u^{(n-3)}(\theta), u^{(n-2)}(\theta)) \\ &+ f(\theta, w(\theta), \cdots, w^{(n-3)}(\theta), w^{(n-2)}(\theta)) \\ &= -f(\theta, \eta_1, \cdots, \eta_{n-2}, w^{(n-2)}(\theta)) - \rho(w^{(n-2)}(\theta) - u^{(n-2)}(\theta)) \\ &+ f(\theta, w(\theta), \cdots, w^{(n-3)}(\theta), w^{(n-2)}(\theta)) \ (\eta_i \ \text{ is defined as in Step (1) }) \\ &\ge -\rho(w^{(n-2)}(\theta) - u^{(n-2)}(\theta)) > 0 \ (\text{ by using } \ (C_5)\ ) \end{aligned}$$

if $\theta \in (0,1)$ such that $H(\theta) > 0$.

Therefore, we see that there is no $\theta \in (0,1)$ such that $H(\theta) > 0$ and $H''(\theta) \le 0$.

Step (3). Now, we claim that $H(t) \le 0$ on $[0,1]$. Suppose to the contrary that there exists a $t_0 \in [0,1]$ such that $H(t_0) > 0$. Then there is a $\theta \in [0,1]$ such that

$$H(\theta) := \max_{t\in[0,1]} H(t) > 0.$$

Case(1). Suppose that $\beta = 0$, which implies $H(0) \le 0$ and $\theta \in (0,1]$.

(1^0) Suppose that $\delta = 0$, which implies $H(1) \le 0$ and $\theta \in [0,1)$. Thus, we have $\theta \in (0,1)$ and $H''(\theta) \le 0$. This contradicts the conclusion of Step (2).

(2^0) Suppose that $\gamma = 0$, which implies $H'(1) \le 0$. It is clear that $\theta = 1$. In fact, if $\theta \in (0,1)$, then $H''(\theta) \le 0$. This gives a contradiction.

Suppose that $H'(\theta) = H'(1) < 0$; then $H(t)$ is strictly decreasing near $t = \theta = 1$. This implies $H(\theta) = H(1)$ cannot be the maximum of $H(t)$, thus we obtain $H'(1) = 0$.

Since $H(\theta) = H(1) > 0$, there exists an $\epsilon > 0$ such that $H(t) > 0$ in $(1-\epsilon, 1]$. Thus $H''(t) > 0$ in $(1-\epsilon, 1)$, which implies $H'(t)$ is strictly increasing on $[1-\epsilon, 1]$. It follows from $H'(t) < H'(1) = H'(\theta) = 0$ on $[1-\epsilon, 1)$ and we see that $H(1) = H(\theta)$ cannot be the maximum of $H(t)$. This gives a contradiction.

(3^0) Suppose that $\gamma\delta > 0$, which implies

$$w^{(n-1)}(1) \ge \frac{-w^{(n-2)}(1)\gamma}{\delta} \quad \text{and} \quad u^{(n-1)}(1) = \frac{-u^{(n-2)}(1)\gamma}{\delta}.$$

By Case(1)-(1^0), we see that

$$u^{(n-2)}(1) > w^{(n-2)}(1).$$

Hence, we have

$$w^{(n-1)}(1) \ge \frac{-w^{(n-2)}(1)\gamma}{\delta} > \frac{-u^{(n-2)}(1)\gamma}{\delta} = u^{(n-1)}(1).$$

It follows from Case(1)-(2^0) that we obtain a contradiction.

Case(2). Suppose that $\alpha = 0$, which implies $H'(0) \ge 0$.

(4^0) Suppose that $\delta = 0$, which implies $H(1) \le 0$ and $\theta \in [0,1)$. It is clear that $\theta = 0$. In fact, if $\theta \in (0,1)$, then $H''(\theta) \le 0$. This gives a contradiction.

Suppose that $H'(\theta) = H'(0) > 0$; then $H(t)$ is strictly increasing near $t = \theta = 0$. This implies $H(\theta) = H(0)$ cannot be the maximum of $H(t)$, thus we obtain $H'(0) = 0$.

Since $H(\theta) = H(0) > 0$, there exists an $\epsilon > 0$ such that $H(t) > 0$ in $[0, \epsilon)$. Thus $H''(t) > 0$ in $(0, \epsilon)$, which implies $H'(t)$ is strictly increasing on $[0, \epsilon]$. It follows from $H'(t) > H'(0) = H'(\theta) = 0$ on $(0, \epsilon]$ and we see that $H(0) = H(\theta)$ cannot be the maximum of $H(t)$. This gives a contradiction.

(5^0) Suppose that $\gamma = 0$, which implies $H'(1) \le 0$. By Case(1)-(2^0) and Case(2)-(4^0), we see that

$$H(0) > 0 \quad \text{and} \quad H(1) > 0.$$

By continuity and Step (2), we see that there exist $t_1,\ t_2 \in (0,1)$ such that

$$t_1 < t_2,\ H(t) > 0,\ H''(t) > 0 \quad \text{on} \quad (0, t_1) \cup (t_2, 1).$$

It follows from $H'(0) \ge 0$ and $H'(1) \le 0$ that

$$H'(t) > 0 \quad \text{on} \quad (0, t_1) \quad \text{and} \quad H'(t) < 0 \quad \text{on} \quad (t_2, 1).$$

This implies that there exists a $\theta \in (t_1, t_2)$ such that

$$H(\theta) > 0,\ H'(\theta) = 0 \ \text{ and } \ H''(\theta) \leq 0,$$

which contradicts the conclusion of Step (2).

(6^0) Suppose that $\gamma\delta > 0$, which implies

$$w^{(n-2)}(1) \geq \frac{-w^{(n-1)}(1)\delta}{\gamma} \ \text{ and } \ u^{(n-2)}(1) = \frac{-u^{(n-1)}(1)\delta}{\gamma}.$$

By Case(2)-(5^0), we see that

$$u^{(n-1)}(1) > w^{(n-1)}(1).$$

Hence, we have

$$w^{(n-2)}(1) \geq \frac{-w^{(n-1)}(1)\delta}{\gamma} > \frac{-u^{(n-1)}(1)\delta}{\gamma} = u^{(n-2)}(1).$$

It follows from Case(2)-(4^0) that we obtain a contradiction.

Case(3). Suppose that $\delta = 0$, which implies $H(1) \leq 0$. By Case(1)-(1^0) and Case(2)-(4^0), we see that

$$\alpha\beta > 0 \ \text{ and } \ u^{(n-2)}(0) > w^{(n-2)}(0).$$

Thus, we have

$$w^{(n-1)}(0) \leq \frac{w^{(n-2)}(0)\alpha}{\beta} < \frac{u^{(n-2)}(0)\alpha}{\beta} = u^{(n-1)}(0).$$

It follows from Case(2)-(4^0) that we obtain a contradiction.

Case(4). Suppose that $\gamma = 0$, which implies $H'(1) \leq 0$. By Case(1)-(2^0) and Case(2)-(5^0), we see that

$$\alpha\beta > 0 \ \text{ and } \ u^{(n-1)}(0) < w^{(n-1)}(0).$$

Thus, we have

$$w^{(n-2)}(0) \geq \frac{w^{(n-1)}(0)\beta}{\alpha} > \frac{u^{(n-1)}(0)\beta}{\alpha} = u^{(n-2)}(0).$$

It follows from Case(1)-(2^0) that we obtain a contradiction.

Case(5). Suppose that $\alpha\beta\gamma\delta > 0$. By Case(1)-($3^0$), Case(2)-($6^0$), Case(3) and Case(4), we see that

$$u^{(n-2)}(0) > w^{(n-2)}(0), u^{(n-1)}(0) < w^{(n-1)}(0),$$
$$u^{(n-2)}(1) > w^{(n-2)}(1), u^{(n-1)}(1) > w^{(n-1)}(1).$$

Thus, we have

$$w^{(n-1)}(0) \leq \frac{w^{(n-2)}(0)\alpha}{\beta} < \frac{u^{(n-2)}(0)\alpha}{\beta} = u^{(n-1)}(0) < w^{(n-1)}(0),$$

which gives a contradiction.

From Cases(1)-(5), we see that

$$u^{(n-2)}(t) \leq w^{(n-2)}(t) \ \text{ on } \ [0,1].$$

Similarly, we may show that

$$v^{(n-2)}(t) \leq u^{(n-2)}(t) \ \text{ on } \ [0,1].$$

Since $u,\ v,\ w \in \mathbb{B}$ and satisfy

$$v^{(n-2)}(t) \leq u^{(n-2)}(t) \leq w^{(n-2)}(t) \ \text{ on } \ [0,1],$$

we obtain

$$v^{(i)}(t) \leq u^{(i)}(t) \leq w^{(i)}(t) \ \text{ on } \ [0,1] \ \text{ for } \ i = 0, 1, \cdots, n-2.$$

Therefore,

$$f^*(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) = f(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) \ \text{ on } \ [0,1].$$

That is, $u(t)$ is a solution of (BVP) and satisfies

$$v^{(i)}(t) \leq u^{(i)}(t) \leq w^{(i)}(t) \ \text{ on } \ [0,1] \ \text{ for } \ i = 0, 1, \cdots, n-2. \qquad \square$$

Theorem 2.3 (Main result)**.** *Suppose that*

(H) there exists a function $g \in ([0,1] \times [0,\infty)^{n-1}; [0,\infty))$ *which satisfies*

$$f(t, 0, 0, \cdots, 0) \geq 0 \ \ on \ \ [0,1] \ (f \ \ maybe\ has\ negative\ value\ for \ \ u_i \neq 0),$$

$$g(t, |u_1|, |u_2|, \cdots, |u_{n-1}|) \geq f(t, u_1, u_2, \cdots, u_{n-1}) \ \ on \ \ [0,1] \times \mathbb{R}^{n-1}$$

and one of the following:

(2.1) $\max g_0 = A_1 \in [0, D_1)$ *and* $\min g_\infty = A_2 \in (\frac{D_2}{M}, \infty]$,

(2.2) $\min g_0 = A_3 \in (\frac{D_2}{M}, \infty]$ *and* $\max g_\infty = A_4 \in [0, D_1)$,

(2.3) there exist two non-negative functions $h \in C([0,\infty)^{n-1}; [0,\infty))$, *increasing with respect to* $u_{n-1} \in [0,\infty)$, *and* $q \in C([0,1]; [0,\infty))$ *such that*

$$\begin{cases} g(t, u_1, u_2, \cdots, u_{n-1}) := q(t) h(u_1, u_2, \cdots, u_{n-1}) \ \ on \ \ [0,1] \times [0,\infty)^{n-1}, \\ \sup\limits_{u_{n-1} \in (0,\infty)} \min\limits_{(u_1, \cdots, u_{n-2}) \in [0,\infty)} \frac{u_{n-1}}{Q h(u_1, \cdots, u_{n-1})} > 1, \end{cases}$$

where

$$\max g_0 := \lim_{u_1, u_2, \cdots, u_{n-1} \to 0^+} \max_{t \in [0,1]} \frac{g(t, u_1, u_2, \cdots, u_{n-1})}{u_{n-1}},$$

$$\min g_0 := \lim_{u_1, u_2, \cdots, u_{n-1} \to 0^+} \min_{t \in [\frac{1}{2}, \frac{3}{4}]} \frac{g(t, u_1, u_2, \cdots, u_{n-1})}{u_{n-1}},$$

$$\max g_\infty := \lim_{u_1, u_2, \cdots, u_{n-1} \to \infty} \max_{t \in [0,1]} \frac{g(t, u_1, u_2, \cdots, u_{n-1})}{u_{n-1}},$$

$$\min g_\infty := \lim_{u_1, u_2, \cdots, u_{n-1} \to \infty} \min_{t \in [\frac{1}{2}, \frac{3}{4}]} \frac{g(t, u_1, u_2, \cdots, u_{n-1})}{u_{n-1}},$$

$$\left(\int_0^1 k(s,s) ds\right)^{-1} := D_1 = \frac{6\rho}{6\delta\beta + 3\gamma\beta + \alpha\gamma + 3\alpha\delta},$$

$$\left(\int_{\frac{1}{2}}^{\frac{3}{4}} k\left(\frac{1}{2}, s\right) ds\right)^{-1} := D_2 = \frac{64\rho}{16\beta\delta + 6\beta\gamma + 3\alpha\gamma + 8\alpha\delta}$$

and

$$Q := \max_{t \in [0,1]} \int_0^1 k(t,s) q(s) ds.$$

Then (BVP) *has at least one non-negative solution.*

Proof. From the results of Agarwal and Wong [1], [2], [3], we can see that (BVP**)

$$\begin{cases} (E^{**})\ w^{(n)}(t) + g(t, u(t), w^{(1)}(t), \cdots, w^{(n-2)}(t)) = 0 \quad \text{for} \quad t \in (0,1) \text{ and } n \geq 2, \\ (BC^{*}) \begin{cases} w^{(i)}(0) = 0,\ 0 \leq i \leq n-3, \\ \alpha w^{(n-2)}(0) - \beta w^{(n-1)}(0) = 0, \\ \gamma w^{(n-2)}(1) + \delta w^{(n-1)}(1) = 0 \end{cases} \end{cases}$$

has at least one non-negative solution $w(t)$. It is clear that $w(t)$ and $v(t) := 0$ are the upper-solution and lower-solution of (BVP), respectively. From Theorem 2.2, we obtain the desired results. ☐

Remark 2.4. For $n = 2$, there are many functions $g(t, u)$ that do not satisfy

$$\max g_0,\ \min g_0, \max g_\infty, \min g_\infty \in \{0, \infty\},$$

for example, $g(t, u) := \frac{e^u - 1}{1+t^2}$ ($\max g_0 = 1$ and $\min g_0 = \frac{16}{25}$), $g(t, u) := (t+1) sinhu$ ($\max g_0 = 2$ and $\min g_0 = \frac{3}{2}$), $g(t, u) := u + t^2 e^{-u}$ ($\max g_0 = \infty, \max g_\infty = \min g_\infty = 1$).

Therefore, our main result generalizes all the recent investigations about the existence of non-negative solutions of (BVP).

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