

PERIODIC SOLUTIONS OF THE SECOND ORDER
DIFFERENTIAL EQUATIONS
WITH LIPSCHITZIAN CONDITION

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ABSTRACT. In this paper, we prove an infinity of periodic solutions to the periodically forced nonlinear Duffing equation $\ddot{x} + g(x) = p(t)$.

1. INTRODUCTION

Consider the Duffing equation

$$(1.1) \quad \ddot{x} + g(x) = p(t),$$

where $g(x)$, $p(t) \in C(\mathbb{R}, \mathbb{R})$ and $p(t)$ is periodic, whose least period is 2π . When $g(x)$ satisfies the condition

$$(1.2) \quad 0 < g_* = \liminf_{|x| \rightarrow \infty} \frac{g(x)}{x} \leq \limsup_{|x| \rightarrow \infty} \frac{g(x)}{x} = g^* < \infty,$$

Eq.(1.1) is called the semi-linear Duffing equation. It is well-known that the existence problem of periodic solutions for semi-linear Eq.(1.1) is restricted by the resonant point set $\Sigma = \{i^2 | i = 0, 1, 2, \dots\}$. In [1], D.E. Leach proved the existence and uniqueness of the 2π -periodic solution of (1.1) under Loud's condition

$$(1.3) \quad m^2 < \lambda \leq g'(x) \leq \mu < (m+1)^2,$$

with a given integer $m \geq 0$ and two constants λ and μ . R. Reissig [2] proved the existence of periodic solutions of (1.1) under a weaker condition:

$$(1.4) \quad m^2 < \lambda \leq \frac{g(x)}{x} \leq \mu < (m+1)^2, \quad |x| \gg 1.$$

J. Mawhin [3] also obtained the same result. The main point of the conditions above is to exclude the resonance case. From then on, there appeared many works around the results of D.E. Leach, R. Reissig and J. Mawhin (see [4], [5], [6]).

It was T. Ding who first studied the existence problem of periodic solutions for Eq.(1.1) when $g(x)$ crosses the resonant points. Assume the following conditions hold:

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(g_1): Let $g(x) \in C^1(R, R)$ and let K be a positive constant such that

$$|g'(x)| \leq K, \quad x \in R.$$

(g_2): There exist two constants $A_0 > 0$ and $M_0 > 0$ such that

$$\frac{g(x)}{x} \geq A_0, \quad |x| \geq M_0.$$

(τ_0): There exist a constant $\sigma > 0$, an integer $m > 0$, and two sequences $\{a_k\}$ and $\{b_k\}$ ($k \in N$), such that $a_k \rightarrow \infty$ and $b_k \rightarrow \infty$ as $k \rightarrow \infty$; and moreover

$$\tau(a_k) < \frac{2\pi}{m} - \sigma, \quad \tau(b_k) > \frac{2\pi}{m} + \sigma,$$

where $\tau(e)$ denotes the least positive period of the solution $x(t)$ ($x(0) = 0, \dot{x}(0) = \sqrt{2e}$) for the equation $\ddot{x} + g(x) = 0$.

By using a generalized form of the Poincaré-Birkhoff twist theorem, T. Ding [6] proved that Eq.(1.1) has infinitely many 2π -periodic solutions. Lately, T. Ding, R. Iannaci and F. Zanolin [7] further generalized the condition (τ_0) to the following condition:

$$(\tau_1) \quad \Delta\tau = \limsup_{e \rightarrow \infty} \tau(e) - \liminf_{e \rightarrow \infty} \tau(e) > 0.$$

D. Qian [8] improved the results in [6], [7] by dropping the condition (g_1) and assuming, for $G(x) = \int_0^x g(s)ds$,

$$\limsup_{|x| \rightarrow \infty} \frac{G(x)}{g^2(x)} < \infty.$$

This hypothesis is always satisfied if condition (1.2) holds. Hao Dunyuan and Ma Shiwang [9] extended the results in [6], [7] by improving conditions on the time-mapping.

In the present paper, we study the existence of 2π -periodic solutions for Eq.(1.1) under conditions (τ_0) and

$$(H_1): \lim_{|x| \rightarrow \infty} \operatorname{sgn}(x)g(x) = \infty,$$

(H_2): $g(x)$ satisfies the globally Lipschitzian condition. That is, there exists a positive constant a such that $|g(x) - g(y)| \leq a|x - y|$.

The main difference between conditions (g_1), (g_2) and (H_1), (H_2) lies in that the former implies the semi-linear condition (1.2), but the latter doesn't. By developing an idea in [6], we obtain the following

Theorem 1.1. *Assume that conditions (H_1), (H_2) and (τ_0) hold. Then Eq.(1.1) possesses infinitely many 2π -periodic solutions $\{x_k(t)\}_{k=1}^{\infty}$ which satisfy*

$$\lim_{k \rightarrow \infty} \left(\min_{t \in R} (|x_k(t)| + |\dot{x}_k(t)|) \right) = \infty.$$

Theorem 1.2. *Assume that conditions (H_1), (H_2) and (τ_1) hold. Then Eq.(1.1) possesses at least one 2π -periodic solution. Moreover, there exists a positive integer n_0 such that, for any integer $n \geq n_0$, Eq.(1.1) possesses a sequence of infinitely many periodic solutions $\{x_{n,k}(t)\}_{k=1}^{\infty}$ of minimal period $2n\pi$ which satisfy*

$$\lim_{k \rightarrow \infty} \left(\min_{t \in R} (|x_{n,k}(t)| + |\dot{x}_{n,k}(t)|) \right) = \infty.$$

In section 4, we construct an example for an application of the above theorems. This example also shows that Theorem 1.1 and Theorem 1.2 are not contained in the results of the previously quoted articles.

2. SEVERAL LEMMAS

At first, we consider the auxiliary autonomous equation

$$(2.1) \quad \ddot{x} + g(x) = 0,$$

or, its equivalent system

$$(2.2) \quad \dot{u} = w, \quad \dot{w} = -g(u).$$

The orbits Γ_e of the autonomous system (2.2) are curves determined by the equation

$$(2.3) \quad \frac{1}{2}w^2 + G(u) = e,$$

where e is an arbitrary constant.

Then, we can easily prove the following:

Lemma 2.1. *If (H_1) holds, then there is a constant $e_0 > 0$, such that, for each $e \geq e_0$, Γ_e is a closed curve which is star-shaped with respect to the origin O .*

It follows from Lemma 2.1 that each curve Γ_e ($e \geq e_0$) intersects the u -axis at two points $(d(e), 0)$ and $(c(e), 0)$, where $d(e) < 0$ and $c(e) > 0$ are uniquely determined by the formula

$$G(d(e)) = G(c(e)) = e.$$

Let $(u(t), w(t))$ be any solution of (2.2) whose orbit is Γ_e ($e \geq e_0$). Clearly, this solution is periodic. Let $\tau(e)$ denote the least positive period of this solution. It follows from (2.2) and (2.3) that

$$\tau(e) = \sqrt{2} \int_{d(e)}^{c(e)} \frac{du}{\sqrt{e - G(u)}}.$$

By the definition, it follows that $\tau(e)$ is continuous for $e \geq e_0$.

Now we perform some phase-plane analysis for Eq.(1.1). First of all, we write the Duffing equation (1.1) in the equivalent system:

$$(2.4) \quad \dot{x} = y, \quad \dot{y} = -g(x) + p(t).$$

Let $(x(t, x_0, y_0), y(t, x_0, y_0))$ be the solution of the system (2.4) through the initial point $(x(0), y(0)) = (x_0, y_0)$. It is not hard to show that every solution satisfying the initial value problem exists uniquely on the whole t -axis under conditions (H_1) and (H_2) . Then the Poincaré mapping $P : R^2 \mapsto R^2$ is well defined by

$$(x_0, y_0) \mapsto (x(2\pi, x_0, y_0), y(2\pi, x_0, y_0)).$$

It is well known that P is an area-preserving homeomorphism.

By applying the transformation $x(t) = r(t) \cos \theta(t)$, $y(t) = r(t) \sin \theta(t)$ to the system (2.4), we get the equations for $r(t)$ and $\theta(t)$,

$$(2.5) \quad \begin{aligned} \dot{r} &= r \sin \theta \cos \theta - g(r \cos \theta) \sin \theta + p(t) \sin \theta, \\ \dot{\theta} &= -\sin^2 \theta - \frac{1}{r}(g(r \cos \theta) \cos \theta - p(t) \cos \theta). \end{aligned}$$

Let $(r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ be the solution of equations (2.5) through the initial point $(r(0), \theta(0)) = (r_0, \theta_0)$. Then the mapping P can also be written in the polar coordinate form

$$r^* = r(2\pi, r_0, \theta_0), \quad \theta^* = \theta(2\pi, r_0, \theta_0) + 2l\pi,$$

where l is an arbitrary integer. It can be easily seen that if (r_0, θ_0) is such that

$$r(t, r_0, \theta_0) > 0, \quad t \in [0, 2\pi],$$

then $\theta(2\pi, r_0, \theta_0)$ is well defined and continuous in (r_0, θ_0) , and moreover,

$$\theta(2\pi, r_0, \theta_0 + 2\pi) = \theta(2\pi, r_0, \theta_0) + 2\pi.$$

Next, we take the transformation $u(t) = \rho(t) \cos \varphi(t)$, $w(t) = \rho(t) \sin \varphi(t)$ to the system (2.2). The resulting equations for $\rho(t)$ and $\varphi(t)$ are

$$(2.6) \quad \begin{aligned} \dot{\rho} &= \rho \cos \varphi \sin \varphi - g(\rho \cos \varphi) \sin \varphi, \\ \dot{\varphi} &= -\sin^2 \varphi - \frac{1}{\rho} g(\rho \cos \varphi) \cos \varphi. \end{aligned}$$

Let $(\rho(t, \rho_0, \varphi_0), \varphi(t, \rho_0, \varphi_0))$ be the solution of equations (2.6) through the initial point $(\rho(0), \varphi(0)) = (\rho_0, \varphi_0)$. Using (2.5) and (2.6), we can easily prove the following

Lemma 2.2. *If (H_1) and (H_2) hold, then there exist positive constants $\alpha > 1$ and $\Lambda_0 > 1$ such that*

- (1) $\rho_0/\alpha \leq \rho(t) \leq \alpha\rho_0$ and $\dot{\varphi}(t) < 0$, for $t \in [0, 2\pi]$ and $\rho_0 \geq \Lambda_0$.
- (2) $r_0/\alpha \leq r(t) \leq \alpha r_0$ and $\dot{\theta}(t) < 0$, for $t \in [0, 2\pi]$ and $r_0 \geq \Lambda_0$.

Lemma 2.3. *Assume that conditions (H_1) , (H_2) and (τ_0) hold and let $\Phi(\rho_0, \varphi_0) = \varphi(2\pi, \rho_0, \varphi_0) - \varphi_0$. Then, for any constant $c > 0$, there exists a constant $\Lambda > 0$ such that*

$$\Phi(\rho_0, \varphi_0) \leq -2m\pi - \frac{c}{\rho_0}, \quad (\rho_0 \cos \varphi_0, \rho_0 \sin \varphi_0) \in \Gamma_{a_k}, a_k \geq \Lambda,$$

$$\Phi(\rho_0, \varphi_0) \geq -2m\pi + \frac{c}{\rho_0}, \quad (\rho_0 \cos \varphi_0, \rho_0 \sin \varphi_0) \in \Gamma_{b_k}, b_k \geq \Lambda.$$

Proof. It follows from condition (H_1) that there exists a positive constant β such that, if $|x| \geq \beta$, then

$$(2.7) \quad \operatorname{sgn}(x)g(x) \geq \frac{2c\alpha}{m\sigma},$$

where α is given in Lemma 2.2 and σ is given in condition (τ_0) . Set

$$E = \max_{-\beta \leq x \leq \beta} |g(x)|.$$

Assume curve Γ_e intersects the lines $x = \beta$ and $x = -\beta$ at points (β, y_1) , (β, y_2) and $(-\beta, y_3)$, $(-\beta, y_4)$, respectively. Obviously, we can take e_0^* large enough so that, for $e \geq e_0^*$, we have

$$(2.8) \quad \min\{|\arctan \frac{y_i}{\beta}| : i = 1, 2, 3, 4\} \geq \frac{\pi}{3},$$

and

$$\rho_0 \geq \Lambda_0, \quad 0 < \frac{\alpha E}{\rho_0} \leq \frac{1}{4}, \quad \text{for } (\rho_0 \cos \varphi_0, \rho_0 \sin \varphi_0) \in \Gamma_e,$$

where Λ_0 is given in Lemma 2.2.

Set

$$D_1 = \{(x, y) \mid |x| \leq \beta\}, D_2 = \{(x, y) \mid |x| \geq \beta, |y| \geq \sqrt{3}|x|\}, D_3 = R^2 - D_1 \cup D_2.$$

If $(\rho(t), \varphi(t)) \in \Gamma_e$, $e \geq e_0^*$ and $(\rho(t), \varphi(t)) \in D_1$, then

$$(2.9) \quad \dot{\varphi}(t, \rho_0, \varphi_0) \leq -\sin^2\left(\frac{\pi}{3}\right) + \frac{1}{4} = -\frac{1}{2}.$$

If $(\rho(t), \varphi(t)) \in \Gamma_e$, $e \geq e_0^*$ and $(\rho(t), \varphi(t)) \in D_2$, then

$$(2.10) \quad \dot{\varphi}(t, \rho_0, \varphi_0) \leq -\sin^2\left(\frac{\pi}{3}\right) - \frac{1}{\rho(t)}g(\rho(t) \cos \varphi(t)) \cos \varphi(t) \leq -\frac{3}{4}.$$

If $(\rho(t), \varphi(t)) \in \Gamma_e$, $e \geq e_0^*$ and $(\rho(t), \varphi(t)) \in D_3$, then

$$(2.11) \quad \dot{\varphi}(t, \rho_0, \varphi_0) \leq -\frac{1}{\rho(t)}g(\rho(t) \cos \varphi(t)) \cos \varphi(t) \leq -\frac{c}{m\sigma\rho_0}.$$

It follows from (2.9), (2.10) and (2.11) that

$$(2.12) \quad \dot{\varphi}(t, \rho_0, \varphi_0) \leq -\frac{c}{m\sigma\rho_0},$$

for ρ_0 sufficiently large. Assume $(u(t, u_0, w_0), w(t, u_0, w_0))$ is any solution of (2.2) whose orbit is Γ_{a_k} ($a_k \geq e_0$). Since the solution $(u(t, u_0, w_0), w(t, u_0, w_0))$ has the least period $\tau(a_k)$, we see the time in which φ has a decrement 2π is just $\tau(a_k)$. Write

$$\Phi(\rho_0, \varphi_0) = \varphi(2\pi, \rho_0, \varphi_0) - \varphi_0 = -2l\pi - \eta,$$

where $l \geq 0$ is an integer, and $0 \leq \eta < 2\pi$. Let t_η denote the time in which $\varphi(t)$ decreases from $\varphi_0 - 2l\pi$ to $\varphi_0 - 2l\pi - \eta$. Then we have

$$l\tau(a_k) + t_\eta = 2\pi.$$

Since $0 \leq t_\eta < \tau(a_k)$, we obtain

$$(2.13) \quad 2\pi = l\tau(a_k) + t_\eta < (l+1)\tau(a_k) \leq (l+1)\left(\frac{2\pi}{m} - \sigma\right).$$

It follows from (2.13) that $l \geq m$. If $l \geq m+1$, we have

$$(2.14) \quad \Phi(\rho_0, \varphi_0) \leq -2l\pi \leq -2(m+1)\pi.$$

Now, assume $l = m$. Then we have

$$(2.15) \quad t_\eta = 2\pi - m\tau(a_k) \geq 2\pi - m\left(\frac{2\pi}{m} - \sigma\right) = m\sigma.$$

Therefore, we have

$$-\eta = \int_{l\tau(a_k)}^{l\tau(a_k)+t_\eta} \dot{\varphi}(t, \rho_0, \varphi_0) dt \leq -\frac{c}{\rho_0}.$$

Thus,

$$(2.16) \quad \Phi(\rho_0, \varphi_0) \leq -2l\pi - \eta \leq -2m\pi - \frac{c}{\rho_0}.$$

Combining (2.14) and (2.16) yields

$$\Phi(\rho_0, \varphi_0) \leq -2m\pi - \frac{c}{\rho_0}, \quad (\rho_0 \cos \varphi_0, \rho_0 \sin \varphi_0) \in \Gamma_{a_k},$$

with k large enough. The second inequality of Lemma 2.3 can be proved in a similar way. The proof of Lemma 2.3 is then completed. \square

Lemma 2.4. *Assume that conditions (H_1) , (H_2) hold and let*

$$\Phi(r_0, \theta_0) = \varphi(2\pi, r_0, \theta_0) - \theta_0, \quad \Theta(r_0, \theta_0) = \theta(2\pi, r_0, \theta_0) - \theta_0.$$

Then there exist two positive constants c_0 and Γ_0 such that

$$|\Phi(r_0, \theta_0) - \Theta(r_0, \theta_0)| \leq \frac{c_0}{r_0}, \quad \text{for } r_0 \geq \Gamma_0.$$

Proof. The proof follows from arguments in [6]. Let $(u(t, x_0, y_0), w(t, x_0, y_0))$ be the solution of (2.2) through the initial point $(u(0), w(0)) = (x_0, y_0)$. Let

$$\begin{aligned} s(t) &= s(t, x_0, y_0) = x(t, x_0, y_0) - u(t, x_0, y_0), \\ v(t) &= v(t, x_0, y_0) = y(t, x_0, y_0) - w(t, x_0, y_0). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{ds(t)}{dt} &= v(t), \\ \frac{dv(t)}{dt} &= p(t) + g(u(t, x_0, y_0)) - g(x(t, x_0, y_0)). \end{aligned}$$

Let $\eta(t) = (s^2(t) + v^2(t))^{\frac{1}{2}}$. Then we have

$$\eta(t) \frac{d\eta(t)}{dt} = s(t)v(t) + p(t)v(t) + [g(u(t, x_0, y_0)) - g(x(t, x_0, y_0))]v(t).$$

It follows from (H_2) that

$$|\eta(t) \frac{d\eta(t)}{dt}| \leq |p(t)v(t)| + (1+a)|s(t)v(t)|.$$

Furthermore,

$$(2.17) \quad \left| \frac{d\eta(t)}{dt} \right| \leq \frac{1}{2}(1+a)\eta(t) + B,$$

where $B = \max_{t \in [0, 2\pi]} |p(t)|$. The differential inequality (2.17) together with $\eta(0) = 0$ yields

$$\eta(t) \leq \frac{2B}{1+a} [e^{(1+a)\pi} - 1] \equiv H_0,$$

for $t \in [0, 2\pi]$. Write $\psi(t) = \psi(t, r_0, \theta_0) = \varphi(t, r_0, \theta_0) - \theta(t, r_0, \theta_0)$. It is clear that if $|\psi(t)| < \pi$, then $\psi(t)$ is just the angle between the vectors $(x(t), y(t))$ and $(u(t), w(t))$. Therefore, we have

$$\cos \psi(t) = \frac{r^2(t) + \rho^2(t) - \eta^2(t)}{2r(t)\rho(t)} \geq 1 - \frac{H_0^2}{2r(t)\rho(t)}, \quad t \in [0, 2\pi].$$

On the other hand, we have $\rho(t) \geq r(t) - H_0$. It follows from Lemma 2.2 that, for r_0 sufficiently large, we have

$$(2.18) \quad \frac{1}{\alpha^2} - \frac{H_0}{\alpha r_0} > \frac{1}{4\alpha^2},$$

and

$$r(t) - H_0 > 0, \quad \text{for } t \in [0, 2\pi].$$

Therefore,

$$1 - \cos \psi(t) \leq \frac{H_0^2}{2r(t)(r(t) - H_0)}, \quad t \in [0, 2\pi].$$

This results in

$$(2.19) \quad \sin^2\left(\frac{1}{2}\psi(t)\right) \leq \frac{H_0^2}{4r(t)(r(t) - H_0)}, \quad \text{for } t \in [0, 2\pi].$$

It follows from Lemma 2.2 and inequality (2.19) that

$$(2.20) \quad \left| \sin\left(\frac{1}{2}\psi(t)\right) \right| \leq \frac{H_0}{2r_0 \sqrt{\frac{1}{\alpha^2} - \frac{H_0}{\alpha r_0}}}.$$

Since $\psi(0) = 0$ and $\psi(t)$ varies continuously as t increases from 0 to 2π , we can see from (2.20) that

$$(2.21) \quad |\psi(t)| \leq 4 \left| \sin\left(\frac{1}{2}\psi(t)\right) \right|, \quad \text{for } t \in [0, 2\pi],$$

for r_0 large enough.

Combining (2.18), (2.20) and (2.21) yields

$$|\psi(t)| \leq \frac{4\alpha H_0}{r_0}, \quad \text{for } t \in [0, 2\pi].$$

Write $c_0 = 4\alpha H_0$. Then we have, in particular,

$$|\Phi(r_0, \theta_0) - \Theta(r_0, \theta_0)| \leq \frac{c_0}{r_0},$$

for r_0 large enough. □

3. THE PROOF OF THE MAIN RESULTS

At first, we restate a generalized form of the Poincaré-Birkhoff fixed point theorem.

Let D denote an annular region in the (x, y) -plane. The boundary of D consists of two simple closed curves: the inner boundary curve C_1 and the outer boundary curve C_2 . Let D_1 denote the simple connected open set bounded by C_1 . Consider an area-preserving mapping $T : R^2 \mapsto R^2$. Suppose that $T(D) \subset R^2 - \{O\}$, where O is the origin. Let (r_0, θ_0) be the polar coordinate of (x_0, y_0) , that is, $x_0 = r_0 \cos \theta_0, y_0 = r_0 \sin \theta_0$. Write the mapping T in the form

$$r^* = f(r_0, \theta_0), \quad \theta^* = \theta_0 + h(r_0, \theta_0),$$

where f and h are continuous in (r_0, θ_0) and 2π -periodic in θ_0 .

A generalized form of the Poincaré-Birkhoff fixed point theorem ([10]).

Besides the above-mentioned assumptions, we assume that

- (1): C_1 is star-shaped about the origin O ;
- (2): $O \in T(D_1)$;
- (3):

$$h(r_0, \theta_0) > 0 (< 0), \quad (r_0 \cos \theta_0, r_0 \sin \theta_0) \in C_1;$$

$$h(r_0, \theta_0) < 0 (> 0), \quad (r_0 \cos \theta_0, r_0 \sin \theta_0) \in C_2.$$

Then T has at least two fixed points in D .

Now, without loss of generality, we assume $e_0 \leq a_k < b_k$ for all $k \in N$. Then, Γ_{a_k} and Γ_{b_k} bound an annular region A_k . In the following, we will apply the above theorem to show that the Poincaré mapping P induced by equation (1.1) has at least two fixed points in the region A_k (with k large). As a consequence, Eq.(1.1) has infinitely many 2π -periodic solutions.

Proof of Theorem 1.1. Using the notations in Lemma 2.3 and Lemma 2.4, we consider the Poincaré mapping $P|_{A_k} : A_k \mapsto R^2$. Write the mapping P in the form:

$$r^* = r(2\pi, r_0, \theta_0), \quad \theta^* = \theta_0 + \Theta_1(r_0, \theta_0),$$

where $\Theta_1(r_0, \theta_0) = \Theta(r_0, \theta_0) + 2m\pi$. It follows from Lemma 2.3 that, for k large enough,

$$(2.22) \quad \begin{aligned} \Phi(r_0, \theta_0) &\leq -2m\pi - \frac{2c_0}{r_0}, & (r_0 \cos \theta_0, r_0 \sin \theta_0) &\in \Gamma_{a_k}, \\ \Phi(r_0, \theta_0) &\geq -2m\pi + \frac{2c_0}{r_0}, & (r_0 \cos \theta_0, r_0 \sin \theta_0) &\in \Gamma_{b_k}, \end{aligned}$$

where c_0 is a constant given in Lemma 2.4. Applying Lemma 2.4, we have

$$(2.23) \quad |\Theta_1(r_0, \theta_0) - \Phi(r_0, \theta_0) - 2m\pi| \leq \frac{c_0}{r_0}$$

whenever r_0 is sufficiently large. Using (2.22), (2.23), we obtain

$$\begin{aligned} \Theta_1(r_0, \theta_0) &< 0, & (r_0 \cos \theta_0, r_0 \sin \theta_0) &\in \Gamma_{a_k}, \\ \Theta_1(r_0, \theta_0) &> 0, & (r_0 \cos \theta_0, r_0 \sin \theta_0) &\in \Gamma_{b_k}, \end{aligned}$$

whenever k is sufficiently large.

Thus we have proved that the area-preserving homeomorphism P is twisting on the annulus A_k . Moreover, we also have that, for k sufficiently large,

$$O \in P(D_k),$$

where $D_k \subset R^2$ is a open bounded set with boundary Γ_{a_k} . Finally, by Lemma 2.1, Γ_{a_k} is star-shaped with respect to the origin O (for k large), so that all the assumptions of the generalized Poincaré-Birkhoff fixed point theorem are fulfilled. Therefore, we have proved that, for each k large enough, the mapping P has at least two fixed points in A_k . Thus we have obtained the existence of a sequence $\{x_k(t)\}_{k=1}^\infty$ of periodic solutions of Eq.(1.1), having minimal period 2π and such that

$$\lim_{k \rightarrow \infty} \left(\min_{t \in R} (|x_k(t)| + |\dot{x}_k(t)|) \right) = \infty.$$

The proof of Theorem 1.1 is thus completed.

Similarly, applying methods in [7] and methods in proving Lemma 2.3 and Lemma 2.4, we can prove Theorem 1.2. □

From Theorem 1.1 and Theorem 1.2, we know that the existence problem of periodic solutions for Eq.(1.1) has a tight relation with the property of the time mapping $\tau(\cdot)$. In case $g(x)$ is odd, we can easily check conditions (τ_0) and (τ_1) . Set

$$G_*^+ = \liminf_{x \rightarrow \infty} \frac{2G(x)}{x^2}, \quad G_+^* = \limsup_{x \rightarrow \infty} \frac{2G(x)}{x^2}.$$

Then we can prove

Lemma 3.1 ([7]). *Assume $g(x)$ is odd and $G_*^+ < G_+^*$. Then*

$$\left[\frac{2\pi}{\sqrt{G_+^*}}, \frac{2\pi}{\sqrt{G_*^+}} \right] \subset [\tau_*, \tau^*],$$

where $\tau_* = \liminf_{e \rightarrow \infty} \tau(e)$, $\tau^* = \limsup_{e \rightarrow \infty} \tau(e)$.

Applying Lemma 3.1 and Theorem 1.1, we obtain the following

Corollary 3.2. *Assume that (H_1) , (H_2) hold and let $g(x)$ be an odd function. If*

$$\exists m \in N, \quad m^2 \in (G_*^+, G_+^*),$$

then Eq.(1.1) has an infinite sequence $\{x_j(t)\}_{j=1}^\infty$ of 2π -periodic solutions which satisfy

$$\lim_{j \rightarrow \infty} (\min_{t \in R} (|x_j(t)| + |\dot{x}_j(t)|)) = \infty.$$

Remark. All the results of this section are still valid if, in place of (H_2) , the assumption

$$\exists L_0, K_0 \geq 0, \text{ such that } |g(x) - g(y)| \leq L_0|x - y| + K_0, \forall x, y \in R,$$

holds.

4. AN EXAMPLE

In this section, we give an example for the application of Theorem 1.1 obtained before.

Example. Assume $g(x)$ is an odd function and $g(0) = 0$:

$$(4.1) \quad g(x) = (x + 1)^{\frac{1}{2}} + x(1 + \sin \ln x) - 1, \quad \text{for } x > 0.$$

It follows from the definition that $g(x)$ satisfies condition (H_1) . By a direct calculation, we have

$$g'(x) = \frac{1}{2}(x + 1)^{-\frac{1}{2}} + \sin \ln x + \cos \ln x + 1 \quad \text{for } x > 0.$$

Obviously, $g'(x)$ is bounded in interval $(0, \infty)$. Therefore, $g(x)$ satisfies condition (H_2) . On the other hand, it is easy to check that

$$\liminf_{x \rightarrow \infty} \frac{g(x)}{x} = 0, \quad \limsup_{x \rightarrow \infty} \frac{g(x)}{x} = 2$$

and

$$G(x) = \frac{2}{3}(x + 1)^{\frac{3}{2}} + \frac{1}{2}x^2 + \frac{x^2}{5}(2 \sin \ln x - \cos \ln x) - x - \frac{2}{3}, \quad x > 0.$$

Furthermore, we have

$$(4.2) \quad G_*^+ = \liminf_{x \rightarrow \infty} \frac{2G(x)}{x^2} = 1 + \frac{2}{5} \liminf_{x \rightarrow \infty} (2 \sin \ln x - \cos \ln x) = 1 - \frac{2\sqrt{5}}{5}$$

and

$$(4.3) \quad G_+^* = \limsup_{x \rightarrow \infty} \frac{2G(x)}{x^2} = 1 + \frac{2}{5} \limsup_{x \rightarrow \infty} (2 \sin \ln x - \cos \ln x) = 1 + \frac{2\sqrt{5}}{5}.$$

Then, (4.2) and (4.3) imply that

$$(4.4) \quad \frac{2\pi}{\sqrt{G_+^*}} < 2\pi, \quad \frac{2\pi}{\sqrt{G_*^+}} > 4\pi.$$

It follows from (4.4) and Lemma 3.1 that

$$2\pi \in]\tau_*, \tau^*[.$$

Applying Theorem 1.1, we obtain that Eq.(1.1) possesses infinitely many 2π -periodic solutions.

To the best of our knowledge, none of the previously known existence theorems for Eq.(1.1) can be applied for a nonlinearity like $g(x)$ in (4.1).

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