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ROTATION INTERVALS FOR CHAOTIC SETS

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Abstract. Chaotic invariant sets for planar maps typically contain periodic orbits whose stable and unstable manifolds cross in grid-like fashion. Consider the rotation of orbits around a central fixed point. The intersections of the invariant manifolds of two periodic points with distinct rotation numbers can imply complicated rotational behavior. We show, in particular, that when the unstable manifold of one of these periodic points crosses the stable manifold of the other, and, similarly, the unstable manifold of the second crosses the stable manifold of the first, so that the segments of these invariant manifolds form a topological rectangle, then all rotation numbers between those of the two given orbits are represented. The result follows from a horseshoe-like construction.

Let f be an orientation-preserving diffeomorphism of the annulus $A = S^1 \times I$, where we parametrize S^1 by x in the unit interval, identifying endpoints. The average rotation of an orbit under the action of f is given by the rotation number, which is an asymptotic average of the rate of rotation (i.e., angle per iterate) along an orbit of f. See, for example, [4] for background material on rotation numbers. Formally, let $U = R \times I$ be the universal cover of A, let $\tilde{f} : U \to U$ be a lift of f, and let $\pi_x : U \to R$ be the projection onto the first coordinate. Then the forward and backward rotation numbers, ρ^+ and ρ^-, respectively, are defined for a point (x, y) of A as

$$\rho^+(x, y) = \lim_{n\to\infty} \frac{1}{n}[\pi_x \tilde{f}^n(x, y) - x],$$

if this limit exists, and

$$\rho^-(x, y) = \lim_{n\to\infty} \frac{1}{n}[\pi_x \tilde{f}^{-n}(x, y) - x],$$

if this limit exists.

In one dimension the situation is simple: for an orientation-preserving homeomorphism of the circle, the rotation number $\rho^+ = \rho^-$ always exists and is independent of the choice of point on the circle. For annulus maps, however, ρ^+ need not equal ρ^- or even exist, for any given point, and different points can have different

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rotation numbers. We define the *rotation set* J of f as

$$J = \{\rho | \rho = \rho^+(x) = \rho^-(x), \text{ for some } x \in A\}.$$

While one often studies ρ mod(1) in dealing with maps on the circle, here it is important to view J as a subset of all real numbers, not just those between 0 and 1. A different choice of the lift can translate J by an integer.

There are several now classic situations in which J has been shown to equal or to contain a nontrivial interval. For an area-preserving, invertible, monotone twist map of the annulus (see [9] for the definition and a discussion of such maps), the rotation set is an interval whose endpoints are the rotation numbers of the boundary circles [8, 6]. Similarly, the rotation set of a Birkhoff attractor for an area-contracting monotone twist map of the annulus is an interval [7, 3].

We can also describe the rotation of points around a fixed point p of an orientation-preserving planar homeomorphism f. In this case, $R^2 - \{p\} \approx S^1 \times (0,1)$ is invariant under f. Angles are measured in polar coordinates (centered at p). Two theorems are particularly relevant to the result presented here. First is a theorem of M. Barge [2]: Let f be an area-contracting, orientation-preserving homeomorphism of the plane, and let p be a saddle fixed point of f. If the closure of the unstable manifold of p, $\overline{W^u(p)}$, contains periodic points of rotation number a/b and c/d (with respect to the fixed point p), then for each reduced rational r/s in $[a/b, c/d]$, $\overline{W^u(p)}$ contains a periodic orbit of rotation number r/s. The second theorem is originally due to Aronson, Chory, Hall, and McGehee [1] (with a different proof given by Hockett and Holmes [5]). In their theorem, f is an orientation-preserving homeomorphism of the annulus A with a saddle fixed point p. They assume that branches of the stable and unstable manifolds of p cross in such a way so as to encircle the inner boundary A. Then, by choosing $N > 0$ large enough so that f^N forms a hyperbolic horseshoe containing p, it is shown that the rotation set of f contains the interval $[0, 1/N]$. In the result that follows, by comparison, we require a second point with larger (i.e., nonzero) rotation number to provide a minimum upper bound for the rotation interval.

We assume that f is either a homeomorphism of the annulus or the plane. Let p and q be periodic saddles. We say that p and q form a *heteroclinic pair* if (as shown in Figure 1) one branch u_p of the unstable manifold of p crosses (i.e., has a transverse intersection with) a branch s_q of the stable manifold of q, and one branch u_q of the unstable manifold of q crosses a branch s_p of the stable manifold of p so as to form a topological rectangle D; specifically, the segments of u_p and u_q which form opposite sides of the rectangle are on the same side of s_p and are on the same side of s_q. We labal the four sides of D as L (left), R (right), T (top), and B (bottom).

This geometric condition is a natural one for chaotic sets which typically contain many periodic orbits whose stable and unstable manifolds cross in grid-like fashion. When f is a homeomorphism of the plane, we assume that rotation numbers are evaluated with respect to a fixed point which is located outside the rectangle D. (In the application to chaotic attractors, it may be desireable to allow this fixed point to be one point of the heteroclinic pair, i.e., a corner point of D, and look at rotations with respect to that fixed point. This case is addressed in the remarks following the proof of the theorem.) Notice that the rotation number of one point of the pair is defined only up to an integer (which depends on the particular lift

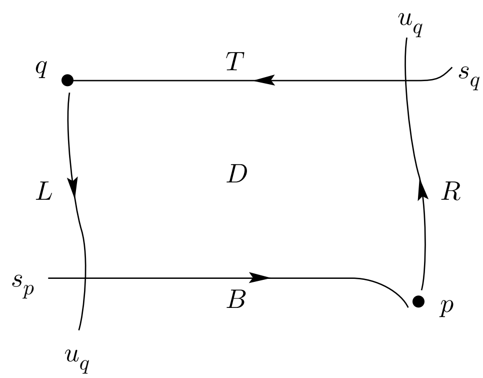

FIGURE 1. Branches of the stable and unstable manifolds of a heteroclinic pair, p and q, form a topological rectangle D.

$\tilde{f}$). Then, however, the rotation number of the other periodic orbit in the pair is determined.

Theorem 0.1. *Let f be a diffeomorphism of the plane or annulus. Assume that p and q form a heteroclinic pair with $\rho(p) = a/b$ and $\rho(q) = c/d$ $(c/d \geq a/b)$. Then for each α, $\alpha \in [a/b, c/d]$, there is a point in D with rotation number α.*

Remark. Specifically, for each α, $\alpha \in [a/b, c/d]$, there is a continuum Γ in D (extending from L to R) such that $\rho^+(x) = \alpha$ for each $x \in \Gamma$. Similarly, there is a continuum Λ in D (extending from B to T) such that $\rho^-(y) = \alpha$ for each $y \in \Lambda$. Points in $\Gamma \cap \Lambda$ have $\rho = \rho^- = \rho^+ = \alpha$.

Proof. We give the proof first in the case that the eigenvalues of both $Df^d(q)$ and $Df^b(p)$ are positive. Let x be the point of intersection of L and B, a corner point of D. Each iterate of f^b will move x closer to p along B. For some iterate, say the kth iterate, $f^{kb}(L)$ will extend through D from B to T, lying roughly parallel to R. This property follows from the λ-lemma (see, for example, [10]) and will also hold for any iterate of f^b greater than k. Similarly, for sufficiently large j, the curve $f^{jd}(R)$ will extend through D from T to B, roughly parallel to L. Let m be the least common multiple of k and j, let V_p be the component of $f^{mb}(D) \cap D$ containing p, and let V_q be the component of $f^{md}(D) \cap D$ containing q. By choosing k and j sufficiently large, we can assume that f^{mb} (resp., f^{md}) stretches V_p (resp., V_q) in a direction nearly parallel to the unstable manifold of p (resp., q).

Notice that $H_p = f^{-mb}(V_p)$ is a topological rectangle which extends from L to R and contains B. (See Figure 2.) The fourth side of the subrectangle H_p is a piece of s_q. Similarly, $H_q = f^{-md}(V_q)$ is a subset of D extending from L to R and containing T. The fourth side of H_q is a piece of s_p. Points in H_p map back into D after mb iterates (advancing ma fundamental domains in the lift), while points in H_q map back into D after md iterates (advancing mc fundamental domains in the lift).

Let r be a rational number between a/b and c/d. Then r can be expressed as the Farey sum of k copies of a/b and j copies of c/d; i.e., $r = \frac{ka}{kb} \oplus \frac{jc}{jd} = \frac{ka+jc}{kb+jd}$. (See, for example, [5].) In this representation, r may not be in reduced form.

Let $\alpha \in [a/b, c/d]$, rational or irrational, be given. Choose a sequence $\{r_n\}$ such that each r_i is a/b or c/d and the sequence S_n of partial Farey sums $\{r_1 \oplus r_2 \oplus \cdots \oplus r_n\}$

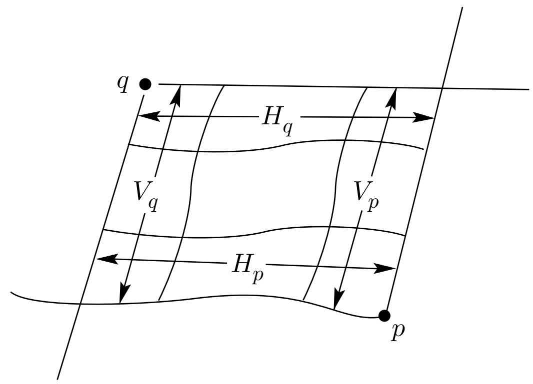

FIGURE 2. Construction of subrectangles.

converges to α. Such a sequence may be constructed by choosing $r_{n+1} = a/b$, if $S_n \geq \alpha$, and $r_{n+1} = c/d$, if $S_n < \alpha$. We define inductively a sequence $\{H_n\}$ of nested topological rectangles corresponding to the sequence $\{S_n\}$. Let h_n (resp., l_n) be defined as the number of times a/b (resp., c/d) occurs in the set $\{r_1, \ldots, r_n\}$. (In particular, $h_n + l_n = n$.)

Let

$$H_1 = \begin{cases} H_p & \text{if } r_1 = a/b, \\ H_q & \text{otherwise.} \end{cases}$$

Let

$$H_{n+1} = \begin{cases} f^{-i}(H_p \cap f^i(H_n)) & \text{if } r_{n+1} = a/b, \\ f^{-i}(H_q \cap f^i(H_n)) & \text{otherwise,} \end{cases} \qquad \text{where } i = h_n mb + l_n md.$$

Since H_n is a compact, connected subset of D and extends from L to R for each n, $n \geq 1$, and the sequence $\{H_n\}$ is nested, $\Gamma = \bigcap_{n\geq 1} H_n$ is connected, containing points in L and R. In the lift, points in H_n move through $h_n ma + l_n mc$ fundamental domains after $h_n mb + l_n md$ iterates. Since the sequence of partial Farey sums

$$\left\{ r_1 \oplus \cdots \oplus r_n = \frac{h_n a + l_n c}{h_n b + l_n d} = \frac{h_n ma + l_n mc}{h_n mb + l_n md} \right\}$$

converges to α, the rotation number ρ^+ of each x in Γ is α.

The proof for the existence of the continuum Λ all of whose points have backward rotation number ρ^- equal to α is the same as above if we apply it to f_{-1} instead of f. In this case, however, the subrectangle H_{n+1} is defined as the appropriate forward iterate of $H_p \cap H_n$ or $H_q \cap H_n$.

If either $Df^b(p)$ or $Df^d(q)$ has negative eigenvalues, replace f by f^2 in the proof above. Then the rotation set of f^2 in D contains $[2a/b, 2c/d]$ (not reduced mod 1), implying that the rotation set of f in D contains $[a/b, c/d]$.

Considering the case where the fixed point of rotational reference is inside D, we note that in some cases additional rotations are acquired through this construction. Let x be a point in $\Gamma \cap \Lambda$, as defined above, and let a_n be the sequence of a/b's and c/d's whose partial Farey sums S_n converge to α. Each time the sequence changes from a/b to c/d (or vice versa) and then back to a/b again, the points in the (lifted) orbit of x will move through one extra fundamental domain. This fact is demonstrated by the following argument:

We assume that H_p is contained in one fundamental domain. Assume further that $r_i = a/b,\ r_{i+1} = r_{i+2} = \cdots = r_{i+k} = c/d$, and $r_{i+k+1} = a/b$. Then $f^{i-1}(x) \in H_p,\ f^i(x) \in V_p \cap H_q,\ f^{i+j}(x) \in V_q,\ 1 \le j \le k$, and $f^{i+k+1}(x) \in H_p$. In the lift, the segment of the (lifted) orbit, $\tilde{f}^i(x)$ to $\tilde{f}^{i+k+1}(x)$, has advanced $kc + a + 1$ fundamental domains. Here, the contribution of the "extra" $+1$ comes from the fact that the orbit gains one additional fundamental domain as it travels from H_p to $V_p \cap H_q$, along H_q (toward q), and through V_q to H_p again.

Taking into account the extra rotations which occur in this configuration, we define the sequence $\{r_n\}$ of rationals converging to α differently in this case. Specifically, let $n(a/b)$ (resp., $n(c/d)$) denote the Farey sum of n copies of a/b (resp., c/d). Then choose the r_n as before until, for k sufficiently large, $|S_k \oplus n(a/b) - S_k| < 1/n$ and $|S_k \oplus n(c/d) - S_k| < 1/n$. From then on in the sequence r_n, add a/b's or c/d's in clusters of at least n. That is, if $S_k \ge \alpha$, let $S_{k+n} = S_k \oplus n(a/b)$; if $S_k < \alpha$, let $S_{k+n} = S_k \oplus n(c/d)$. Then $\{S_n\} \to \alpha$, as $n \to \infty$, and the "extra" rotations per iterate go to zero. □

Remarks. 1. If $a/b = 0 = 0/1$ and the rotation numbers are given with respect to the fixed point p, there are two cases:

(a) If the eigenvalues of $Df(p)$ are positive, then the result as stated above holds.
(b) If the eigenvalues of $Df(p)$ are negative, then, assuming that $c/d > 1/2$, the result holds for each $\alpha,\ \alpha \in (1/2, c/d]$. The rotation rate of points near p is now roughly $1/2$ (although the rotation number $1/2$ is not necessarily obtained). When replacing f by f^2, as in the proof above, the rotation set of f^2 will contain $(1, 2c/d]$.

2. Additional information (beyond the statement of the theorem) is obtained when the fixed point of reference is inside the rectangle—but only in the case that $\rho(p) = \rho(q)$. Then the rotation set of D includes an interval $[\rho(p), \rho(q) + d]$, for some $d > 0$. This reproduces the result in [1, 5].

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