C*-ALGEBRAS THAT ARE ONLY WEAKLY SEMIPROJECTIVE

TERRY A. LORING

(Communicated by Palle E. T. Jorgensen)

Abstract. We show that the C*-algebra of continuous functions on the Cantor set is a weakly semiprojective C*-algebra that is not semiprojective.

The two most basic forms of semiprojectivity, both equivalent to useful notions of stability for C*-algebra relations, are called semiprojectivity and weak semiprojectivity. A C*-algebra \( A \) is \textit{weakly semiprojective} if we can always solve the \( \ast \)-homomorphism lifting problem

\[
\begin{array}{c}
A \xrightarrow{\varphi} \prod_{n=1}^{\infty} B_n / \bigoplus_{n=1}^{\infty} B_n \\
\rho_n \downarrow \varphi
\end{array}
\]

\[
[b_N, b_{N+1}, \ldots]
\]

and it is \textit{semiprojective} if we can always solve the lifting problem

\[
\begin{array}{c}
A \xrightarrow{\varphi} B / J_N \\
\rho_{\varphi} \downarrow \varphi
\end{array}
\]

\[
B / J_N \bigcup_{1}^{\infty} J_n
\]

\[
[0, \ldots, 0, b_N, b_{N+1}, \ldots]
\]

If \( A \) is finitely generated, we have an advantage proving weak semiprojectivity, as it suffices to prove that the lifting can be done only to produce a \( \ast \)-homomorphism \( \overline{\varphi} \) so that

\[
\| \rho_N \circ \overline{\varphi}(g) - \varphi(g) \| \leq \epsilon \quad (\forall g \in G)
\]

for preordained positive \( \epsilon \) and finite subset \( G \). (See [9] or [3].)

What has been lacking is an example of a C*-algebra that is weakly semiprojective but not semiprojective. With the Cantor set

\[
X = \prod_{1}^{\infty} \{0, 1\}, \quad X = \lim \overleftarrow{X_j}, \quad X_j = \prod_{1}^{j} \{0, 1\},
\]

we get such an example, \( C(X) \). It is not semiprojective because its spectrum \( X \) is not an ANR, as is necessary for a commutative C*-algebra to be semiprojective. It

Received by the editors November 19, 1996 and, in revised form, February 4, 1997.

1991 Mathematics Subject Classification. Primary 46L05.

Key words and phrases. Stable relations, semiprojectivity.

The research summarized here was supported, in part, by the National Science Foundation, DMS-9531841.
is weakly semiprojective by the following theorem, the well-known semiprojectivity of finite-dimensional \( C^* \)-algebras, and the existence of the \(*\)-homomorphisms
\[
\theta_j : C(X) \to C(X_j) \quad \gamma_j : C(X_j) \to C(X)
\]
of truncation and zero-padding, whose composition \( \gamma_j \circ \theta_j \) converges pointwise to the identity. Although \( C(X) \) is the direct limit of the \( C(X_j) \), this is not very relevant. The point is that \( C(X) \) is an “approximate retract” of \( C(X_j) \). A retract of a semiprojective is semiprojective ([1]), so it is not surprising that we can conclude that \( C(X) \) is weakly semiprojective.

The proof of Theorem 0.1 was inspired by similar arguments by Lin. See any of the references for interesting results regarding semiprojectivity and stable relations.

**Theorem 0.1.** Suppose that \( A_1, A_2, \ldots \) are semiprojective \( C^* \)-algebras and that \( A \) is a finitely generated \( C^* \)-algebra. If there are \(*\)-homomorphisms \( \theta_j : A \to A_j \) and \( \gamma_j : A_j \to A \) such that
\[
\| \gamma_j \circ \theta_j (a) - a \| \to 0
\]
for all \( a \) in \( A \), then \( A \) is weakly semiprojective.

**Proof.** Let \( \epsilon > 0 \) and a finite set \( G \subset A \) be given, as well as a \(*\)-homomorphism of the form
\[
\varphi : A \to \prod_{i=1}^\infty B_n / \bigoplus_{i=1}^\infty B_n ,
\]
which we now must lift, approximately. Fix \( j \) such that \( \| \gamma_j \circ \theta_j (g) - g \| \leq \epsilon \) for all \( g \) in \( G \). Since \( A_j \) is semiprojective, there exist \( N \) and a \(*\)-homomorphism
\[
\psi : A_j \to \prod_{i=1}^N B_n
\]
such that \( \rho_N \circ \psi = \varphi \circ \gamma_j \). Let \( \bar{\varphi} = \psi \circ \theta_j \), so that, for any \( g \) in \( G \),
\[
\| \rho_K \circ \bar{\varphi}(g) - \varphi(g) \| = \| \varphi \circ \gamma_j \circ \theta_j (g) - \varphi(g) \|
\leq \| \gamma_j \circ \theta_j (g) - g \|
\leq \epsilon .
\]

\[\square\]

**References**

6. H. Lin *Homomorphisms from \( C(X) \) into \( C^* \)-algebras*, preprint.


DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87131

E-mail address: loring@math.unm.edu