CAUCHY-SCHWARZ AND MEANS INEQUALITIES FOR ELEMENTARY OPERATORS INTO NORM IDEALS

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Abstract. The Cauchy-Schwarz norm inequality for normal elementary operators
\[ \left\| \sum_{n=1}^{\infty} A_n X B_n \right\| \leq \left\| \left( \sum_{n=1}^{\infty} A_n^* A_n \right)^{1/2} X \left( \sum_{n=1}^{\infty} B_n^* B_n \right)^{1/2} \right\|, \]
implies a means inequality for generalized normal derivations
\[ \left\| AX + XB \right\| \leq \left\| X \right\|^{1-\frac{1}{r}} \left\| A^r X + X B^r \right\|^{\frac{1}{r}}, \]
for all \( r \geq 2 \), as well as an inequality for normal contractions \( A \) and \( B \)
\[ \left\| (I - A^* A)^{\frac{1}{2}} X (I - B^* B)^{\frac{1}{2}} \right\| \leq \left\| X - AXB \right\|, \]
for all \( X \) in \( B(H) \) and for all unitarily invariant norms \( \left\| \cdot \right\| \).

1. Introduction

Let \( B(H) \) and \( C_\infty \) stand respectively for spaces of all bounded and all compact linear operators acting on a separable, infinite-dimensional, complex Hilbert space \( H \). For an \( X \in B(H) \) let \( \| X \| \) denote its norm, and for an arbitrary \( X \in C_\infty \) let \( s_1(X) \geq s_2(X) \geq \cdots \) denote the singular values of \( X \), i.e., the eigenvalues of \( |X| = (X^* X)^{1/2} \), arranged in a non-increasing order, with their multiplicities counted. Each "symmetric gauge function" \( \Phi \) on sequences gives rise to a unitarily invariant (u.i.) norm on operators defined by \( \| A \|_\Phi = \Phi(\{ s_n(A) \}) \). We will denote by the symbol \( \left\| \cdot \right\| \) any such norm. Any such norm is defined on a natural subclass \( C_{\| \cdot \|} \) of \( C_\infty \) called the norm ideal associated with the norm \( \left\| \cdot \right\| \), and satisfies the invariance property \( \left\| U A V \right\| = \| A \| \) for all \( A \) in this ideal and for all unitary operators \( U, V \). Each norm ideal \( C_{\| \cdot \|} \) is closed in the topology generated by the norm \( \left\| \cdot \right\| \). Particularly well known among unitarily invariant norms are the Schatten \( p \)-norms, defined as \( \| X \|_p = (\sum_{n=1}^{\infty} s_n^p(A))^{1/p} \) for \( 1 \leq p < \infty \) and \( \| X \|_\infty = \| X \| = s_1(X) \), which represent the norms on the Schatten \( p \)-ideals \( C_p \). The Ky Fan norms, defined as \( \| A \|_k = \Phi_k(s_1(A)) = \sum_{i=1}^{k} s_i(A) \) for \( k = 1, 2, \cdots \), represent another interesting family of unitarily invariant norms. The associated ideals \( C_{\infty}^{(k)} \) consist of all compact operators as every Ky Fan \( k \)-norm is equivalent to the norm in \( C_\infty \). The property saying that for all \( X \in C_\infty \) and \( Y \in C_{\| \cdot \|} \) with \( \left\| X \right\|_k \leq \left\| Y \right\|_k \) for all \( k \geq 1 \) we have

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X ∈ C_{∥·∥} with ∥X∥ ≤ ∥Y∥ is known as the Ky Fan dominance property ([GK], ch. 3, §4). For a complete account of the theory of norm ideals, the reader is referred to [GK], [Sch] and [Si].

If $A = (A_1, \ldots, A_N)$ and $B = (B_1, \ldots, B_N)$ are $N$-tuples of bounded Hilbert space operators, then the elementary operator $R = R_{A,B}$ on $B(H)$ is defined by $R(X) = \sum_{n=1}^{N} A_n X B_n$. Elementary operators were introduced by Lumer and Rosenblum in [LR], who studied their spectral properties. In this setting many authors subsequently studied spectral, algebraic, metric and structural properties of elementary operators (see [F83], [F85], [F87], [FL], [McI], and the references therein).

If both $\{A_n\}_{n=1}^{N}$ and $\{B_n\}_{n=1}^{N}$ are families of commuting normal operators, one can easily show that the associated elementary operator $R_{A,B}$ is normal when restricted to the Hilbert space $C_2$. So, in the sequel, by a normal elementary operator we will always mean such an operator, including those cases of $N = \infty$ with the convergence guaranteed. As was shown by G. Weiss in his paper [W83], a famous Fuglede-Putnam type theorem extends to such operators. Very important examples of normal elementary operators are so called “pinching” operators $R_P(X) = \sum_{n=1}^{\infty} P_n X P_n$, generated by a family of mutually orthogonal self-adjoint projections $\{P_n\}_{n=1}^{\infty}$.

2. Main results

We start with the basic Cauchy-Schwarz norm inequality for normal elementary operators. The following theorem extends “pinching” theorems 2.5.1 of [GK] and 1.19 of [Si].

**Theorem 2.1.** If $\sum_{n=1}^{\infty} C_n C_n^* \leq 1$, $\sum_{n=1}^{\infty} C_n^* C_n \leq 1$, $\sum_{n=1}^{\infty} D_n D_n^* \leq 1$ and $\sum_{n=1}^{\infty} D_n^* D_n \leq 1$ for some operator families $\{C_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$, then also $\sum_{n=1}^{\infty} C_n Y D_n \in C_{∥·∥}$ whenever $Y \in C_{∥·∥}$ for some unitarily invariant norm $∥·∥$, and moreover

\[
\sum_{n=1}^{\infty} C_n Y D_n \leq ∥Y∥.
\]

**Proof.** For arbitrary $f$ and $g$ in $H$ a straightforward calculation gives

\[
\left\|\left(\sum_{n=1}^{\infty} C_n Y D_n\right) f, g\right\| \leq \sum_{n=1}^{\infty} ∥Y∥ \left\|D_n f\right\| ∥C_n^* g∥ \leq ∥Y∥ \left\{\sum_{n=1}^{\infty} \left\|D_n f\right\|^2\right\}^{1/2} \left\{\sum_{n=1}^{\infty} ∥C_n^* g∥^2\right\}^{1/2} = ∥Y∥ \left(\sum_{n=1}^{\infty} D_n^* D_n f, f\right)^{1/2} \left(\sum_{n=1}^{\infty} C_n C_n^* g, g\right)^{1/2} = ∥Y∥ \left\|\left(\sum_{n=1}^{\infty} C_n C_n^* \right)^{1/2} g\right\| \left\|\left(\sum_{n=1}^{\infty} D_n^* D_n\right)^{1/2} f\right\| \leq ∥Y∥ \left\|f\right\| \left\|g\right\|,
\]

from which we conclude that

\[
\sum_{n=1}^{\infty} C_n Y D_n \leq ∥Y∥.
\]
Therefore, for all \( N = 1, 2, \cdots \), for \( Y \in C_1 \) and for all \( W \in B(H) \) we have

\[
|\text{tr}(\sum_{n=1}^{N} C_n Y D_n W^*)| = |\text{tr}(\sum_{n=1}^{N} C_n^* W D_n^*)| \\
\leq \|Y\|_1 \|\sum_{n=1}^{N} C_n^* W D_n^*\| \leq \|Y\|_1 \|W\|,
\]

according to (2.2), from which we deduce that

\[
(2.3) \quad \|\sum_{n=1}^{N} C_n Y D_n\|_1 \leq \|Y\|_1.
\]

If \( Y \in C_\infty \), let \( Y = \sum_{n=1}^{\infty} s_n(Y) \langle \cdot, e_n \rangle f_n \) be a singular value decomposition for some orthonormal systems \( \{e_n\} \) and \( \{f_n\} \). For all \( k \geq 2 \) we introduce operators \( Z = \sum_{n=1}^{k-1} (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \langle \cdot, e_j \rangle f_j \) and \( V = s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n + \sum_{n=k+1}^{\infty} s_n(Y) \langle \cdot, e_n \rangle f_n \). We see that

\[
Z = \sum_{n=1}^{k-1} \sum_{j=1}^{n} (s_n(Y) - s_{n+1}(Y)) \langle \cdot, e_j \rangle f_j \\
= \sum_{j=1}^{k} (s_j(Y) - s_k(Y)) \langle \cdot, e_j \rangle f_j \\
= \sum_{n=1}^{k} s_n(Y) \langle \cdot, e_n \rangle f_n + s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n = Y - V.
\]

We can also note that \( s_1(V) = \cdots = s_k(V) = s_k(Y) \), due to orthogonality of the systems \( \{e_n\} \) and \( \{f_n\} \). That will allows us to conclude that for all Ky Fan \( k \)-norms we have

\[
(2.4) \quad \left\| \sum_{n=1}^{N} C_n Y D_n \right\|_k \leq \sum_{n=1}^{N} C_n Z D_n \|_k + \sum_{n=1}^{N} C_n V D_n \|_k \\
\leq \|Z\|_1 + k\|\sum_{n=1}^{N} C_n Z D_n\|_\infty
\]

\[
(2.5) \quad \leq \sum_{n=1}^{k-1} (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \|\langle \cdot, e_j \rangle f_j\|_\infty + k\|V\|_\infty
\]

\[
(2.6) \quad \leq \sum_{n=1}^{k-1} n(s_n(Y) - s_{n+1}(Y)) + ks_k(Y) = \sum_{n=1}^{k} s_n(Y) = \|Y\|_k,
\]

with (2.4) following from (2.3) and (2.5) from (2.2).

Moreover, if \( Y \) is in \( C_\infty \) then also \( \sum_{n=1}^{\infty} C_n Y D_n \in C_\infty \). Indeed, elementary operators \( R_N(Y) = \sum_{n=1}^{N} C_n Y D_n \) acting on \( C_n^{(k)} \) represent a bounded family, because \( \|R_N(Y)\|_k \leq \|Y\|_k \) for all \( Y \in C_\infty \) by (2.6). Also, for one dimensional operators
$f \otimes g$ and $M > N$ we have

$$
\| R_M(f \otimes g) - R_N(f \otimes g) \|_k \leq \left( \sum_{n=N+1}^{M} D_n^* f \otimes C_n g \right)_k
$$

$$
\leq \sum_{n=N+1}^{M} \| D_n^* f \| \| C_n g \| \leq \left( \sum_{n=N+1}^{M} C_n C_n^* \right)^{1/2} \| g \| \left( \sum_{n=N+1}^{M} D_n^* D_n \right)^{1/2} \| f \|,
$$

which $\to 0$ as $M, N \to \infty$. Therefore $R_N(Y)$ converge in $C_{\infty}^{(k)}$ for all finite dimensional $Y$ to a compact operator. By the uniform boundedness principle the same is true for all $Y \in C_{\infty}^{(k)}$, due to its separability. So (2.1) holds for all Ky Fan $k$-norms, and we therefore invoke the Ky Fan dominance property to conclude that (2.1) holds for all unitarily invariant norms, as required.

In the sequel we will refer to a family $\{A_n\}_{n=1}^{\infty}$ in $B(H)$ as square summable if $\sum_{n=1}^{\infty} \| A_n f \|^2 < \infty$ for all $f \in H$. Though this means just the weak convergence of $\sum_{n=1}^{\infty} A_n^* A_n$, an appeal to the resonance principle shows that $\sum_{n=1}^{\infty} A_n^* A_n$ actually defines a bounded Hilbert space operator, and due to the monotonicity of its partial sums, the convergence is moreover strong. For such families the following Cauchy-Schwarz inequality holds:

**Theorem 2.2.** For a square summable families $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ of commuting normal operators

$$
\left( \sum_{n=1}^{\infty} A_n X B_n \right) \leq \left( \sum_{n=1}^{\infty} A_n^* A_n \right)^{1/2} X \left( \sum_{n=1}^{\infty} B_n^* B_n \right)^{1/2},
$$

for all $X \in B(H)$ and for all u.i. norms $\| \|$. If $C_1 \| \|_1$ is separable and $X \in C_{1 \| \|}$, then the left-hand side sum converges in the norm of this ideal.

**Proof.** First, we need a suitable factorization for Hilbert space operators $A_n$ and $B_n$. Let $A = (\sum_{n=1}^{\infty} A_n^* A_n)^{1/2}$ and $B = (\sum_{n=1}^{\infty} B_n^* B_n)^{1/2}$, and let $P$ and $Q$ denote respectively the orthogonal projections on $R(A)$ and $R(B)$. If for a given $f \in H$ we have that $Pf = \lim_{k \to \infty} A_n g_k$ for some sequence $\{g_k\}$ in $H$, then $\lim_{n \to \infty} A_n g_k$ exists for all $n \geq 1$ and does not depend on the chosen sequence. Indeed,

$$
\| A_n g_k - A_n g_l \| \leq \| A(g_k - g_l) \| \to 0
$$
as $k, l \to \infty$, and also $\| A_n g_k - A_n h_k \| \leq \| A(g_k - h_k) \| \to 0$ as $k \to \infty$ whenever $\lim_{k \to \infty} A_n h_k = Pf$ for some other sequence $\{h_k\}$. Thus we can correctly introduce operators $C_n, n = 1, 2, \cdots$, by $C_n f = \lim_{k \to \infty} A_n g_k$, where $\{g_k\}$ is any sequence in $H$ such that $\lim_{k \to \infty} A_n g_k = Pf$. Let us note that due to our definition every $C_n$ vanishes on $N(A)$, i.e., $C_n = C_n P$, and also $C_n A = A C_n = A_n$. Moreover, $\sum_{n=1}^{\infty} C_n^* C_n = P$. Indeed, $\sum_{n=1}^{\infty} C_n^* C_n A^2 = \sum_{n=1}^{\infty} A_n^* A_n = A^2$ implies $\sum_{n=1}^{\infty} C_n^* C_n P = P$, which together with the fact that $C_n (I - P) = 0$ gives the desired conclusion. For all $m, n = 1, 2, \cdots$, $C_m$ and $C_n$ commute on $R(A^2)$ and $R(N(A^2))$, and so also on all of $H$. Thus $\{C_n\}_{n=1}^{\infty}$ is a commuting family of normal contractions which realize the factorizations $C_n A = A C_n = A_n$, with $\sum_{n=1}^{\infty} C_n^* C_n = P$, and which commute with the family $\{A_n\}_{n=1}^{\infty}$. Similarly we get a commuting family $\{D_n\}_{n=1}^{\infty}$ of normal contractions which also commute with $\{B_n\}_{n=1}^{\infty}$ and satisfy $D_n B = BD_n = B_n$ and $\sum_{n=1}^{\infty} D_n^* D_n = Q$. One could easily derive the next explicit
formula: \( C_n = A_n^* A_n = A_n^* A_n \), where \( A_n^* \) denotes a (densely defined) Moore-Penrose (generalized) inverse for \( A \).

For \( Y = A X B \in C_{\|\cdot\|} \) (there is nothing to prove in the opposite case), an application of Theorem 2.1 gives
\[
\left\| \sum_{n=1}^{\infty} A_n X B_n \right\| = \left\| \sum_{n=1}^{\infty} C_n Y D_n \right\|
\]
(2.8)
\[
\leq \|Y\| = \left( \sum_{n=1}^{\infty} A_n^* A_n \right)^{1/2} X \left( \sum_{n=1}^{\infty} B_n^* B_n \right)^{1/2},
\]
which proves the first part of theorem.

Finally, if \( C_{\|\cdot\|} \) is separable, then for all \( N = 1, 2, \cdots \), an application of the just proven part of theorem combined with the arithmetic-geometric means inequality in [BhD] gives
\[
\left\| \sum_{n=N}^{\infty} A_n X B_n \right\| \leq \left\| \left( \sum_{n=N}^{\infty} A_n^* A_n \right)^{1/2} \right\| X \left( \sum_{n=N}^{\infty} B_n^* B_n \right)^{1/2}
\]
(2.9)
\[
= \left\| \left( \sum_{n=N}^{\infty} C_n^* C_n \right)^{1/2} AX \right\| \left( \sum_{n=N}^{\infty} D_n^* D_n \right)^{1/2}
\]
\[
\leq \frac{1}{2} \left\lVert \left( \sum_{n=N}^{\infty} C_n^* C_n \right) AXB + AXB \left( \sum_{n=N}^{\infty} D_n^* D_n \right) \right\|.
\]

We see by (2.8) that \( \left\{ \sum_{n=1}^{\infty} C_n^* C_n \right\}_{N=1}^{\infty} \) and \( \left\{ \sum_{n=1}^{\infty} D_n^* D_n \right\}_{N=1}^{\infty} \) represent bounded sequences of selfadjoint operators which strongly converge to 0 as \( N \to \infty \). As \( AXB \in C_{\|\cdot\|} \), which is separable, then the right-hand side of (2.9) tends to 0 as \( N \to \infty \) by Theorem 3.6.3. of [GK]. The conclusion follows.

**Corollary 2.1.** For normal \( A \) and \( B \) in \( B(H) \) and for all real \( r \geq 2 \),
\[
\left\| \frac{AX + XB}{2} \right\| \leq \left( \frac{1 + |A|^r}{2} \right)^{1/2} \left( \frac{1 + |B|^r}{2} \right)^{1/2}
\]
(2.10)
\[
\text{as well as}
\left\| \frac{X + AXB}{2} \right\| \leq \left( \frac{1 + |A|^r}{2} \right)^{1/2} \left( \frac{1 + |B|^r}{2} \right)^{1/2},
\]
(2.11)
for all \( X \in B(H) \) and for all u.i. norms \( \|\cdot\| \).

**Proof.** \( \{A, I\} \) and \( \{I, B\} \) are families of normal commuting operators, and so for \( r = 2 \) the desired conclusion follows by Theorem 2.2. For \( r > 2 \) the mapping \( t \to t^{\frac{r}{2}} \) is operator monotone by a well known Heinz therem, and therefore this is an operator concave mapping (see [BSh]). Specifically, \( \frac{1+|A|^2}{2} \leq \left( \frac{1+|A|^r}{2} \right)^{\frac{r}{2}} \), from which we obtain
\[
\left\| \frac{1+|A|^2}{2} \right\| ^{\frac{1}{2}} \left( \frac{1+|A|^r}{2} \right)^{-\frac{1}{2}} \leq 1\quad \text{and similarly} \quad \left\| \frac{1+|B|^2}{2} \right\| ^{\frac{1}{2}} \left( \frac{1+|B|^r}{2} \right)^{-\frac{1}{2}} \leq 1.
\]
Therefore
\[
\left\| \left( \frac{1+|A|^2}{2} \right)^{\frac{1}{2}} \frac{X}{2} \right\| ^{\frac{1}{2}} \left( \frac{1+|B|^2}{2} \right)^{\frac{1}{2}} \leq \left\| \left( \frac{1+|A|^r}{2} \right)^{\frac{1}{2}} \frac{X}{2} \right\| ^{\frac{1}{2}} \left( \frac{1+|B|^r}{2} \right)^{\frac{1}{2}},
\]
which completes the proof. \( \square \)
Corollary 2.2. For normal \( A \) and \( B \) in \( B(H) \) the inequality

\[
(2.12) \quad \left\| \frac{AX + XB}{2} \right\| \leq \left\| X \right\|^{1 - \frac{1}{2}} \left\| \frac{|A|^r X + |B|^r}{2} \right\|^{\frac{1}{2}}
\]

holds for all real \( r \geq 2 \), for all u.i. norms \( \| \cdot \| \) and for all \( X \in \mathcal{C}_1 \).

Proof. By Corollary 2.1, for all \( t > 0 \),

\[
\left\| \frac{AX + XB}{2} \right\| = t^{-1} \left\| \frac{tAX + XTB}{2} \right\|
\leq t^{-1} \left\| \left( \frac{1 + |A|^r}{2} \right)^{\frac{1}{2}} X \left( \frac{1 + |B|^r}{2} \right)^{\frac{1}{2}} \right\|,
\]

and therefore

\[
\left\| \frac{AX + XB}{2} \right\| \leq t^{-1} \left\| X \right\|^{1 - \frac{1}{2}} \left\| \left( \frac{1 + |A|^r}{2} \right)^{\frac{1}{2}} X \left( \frac{1 + |B|^r}{2} \right)^{\frac{1}{2}} \right\|^{\frac{2}{1 - \frac{1}{2}}},
\]

by [Ki], because \( \frac{2}{1} < 1 \). Therefore, the arithmetic-geometric mean inequality implies

\[
\left\| \frac{AX + XB}{2} \right\| \leq \frac{1}{2t} \left\| X \right\|^{1 - \frac{1}{2}} \left\| \frac{1 + |A|^r}{2} X + \frac{1 + |B|^r}{2} \right\|^{\frac{1}{2}}
\leq \frac{1}{2} \left\| X \right\|^{1 - \frac{1}{2}} \left( t^{-\frac{1}{2}} \left\| X \right\| + t^{\frac{1}{2}} \left\| \frac{|A|^r X + |B|^r}{2} \right\|^{\frac{1}{2}} \right).
\]

As the right-hand side equals \( \left\| X \right\|^{1 - \frac{1}{2}} \left\| \frac{|A|^r X + |B|^r}{2} \right\|^{\frac{1}{2}} \), which attains its minimum for \( t = \left\| X \right\|^{\frac{1}{2}} \left\| \frac{|A|^r X + |B|^r}{2} \right\|^{-\frac{1}{2}} \), the conclusion follows. \( \square \)

Theorem 2.3. For normal contractions \( A \) and \( B \) the inequality

\[
(2.14) \quad \left\| \left( I - A^* A \right)^{\frac{1}{2}} X (I - B^* B)^{\frac{1}{2}} \right\| \leq \left\| X - AXB \right\|,
\]

holds for all \( X \in B(H) \) and for all unitarily invariant norms \( \| \cdot \| \).

Proof. First, we note that \( s\text{-lim}_{n \to \infty} A^n (I - A^* A)^{\frac{1}{2}} = 0 \). Indeed, by a spectral theorem, for every \( f \in H \) there is a positive, finite Borel measure \( \mu \) concentrated on \( D = \{ z \in \mathbb{C} : |z| \leq 1 \} \) such that \( \| A^n (I - A^* A)^{\frac{1}{2}} f \|_2^2 = \int_D |z|^{2n} (1 - |z|^2) d\mu_f(z) \), whence the desired conclusion follows by Lebesgue’s dominating convergence theorem. Therefore

\[
\lim_{n \to \infty} \left( I - A^* A \right)^{\frac{1}{2}} (X - A^n X B^n) (I - B^* B)^{\frac{1}{2}} = (I - A^* A)^{\frac{1}{2}} X (I - B^* B)^{\frac{1}{2}}.
\]
So by Theorem 2.2 we get
\[
\| (I - A^*A)^{\frac{1}{2}} X (I - B^*B) \| \geq \lim_{n \to \infty} \| (I - A^*A)^{\frac{1}{2}} (X - A^nXB^n)(I - B^*B)^{\frac{1}{2}} \|
\]
\[
= \sum_{k=0}^{\infty} (I - A^*A)^{\frac{1}{2}} A^k (X - AXB) B^k (I - B^*B)^{\frac{1}{2}}
\]
\[
\leq \left( \sum_{k=0}^{\infty} (I - |A|^2)|A|^{2k} \right)^{\frac{1}{2}} (X - AXB) \left( \sum_{k=0}^{\infty} |B|^{2k} (I - |B|^2) \right)^{\frac{1}{2}}
\]
\[
= \left\| (I - P)(X - AXB)(I - Q) \right\| \leq \| X - AXB \|
\]
(2.15) where \( P \) and \( Q \) are the orthogonal projections on \( \text{Ker}(I - A^*A) \) and \( \text{Ker}(I - B^*B) \) respectively. This concludes the proof. 

\[\square\]

**References**


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