ON THE RELATIVE COMMUTANTS OF SUBFACTORS

M. KHOSHKAM AND B. MASHOOD

(Communicated by Palle E. T. Jorgensen)

Abstract. Let $A \subset B$ be factors generated by a periodic tower $A_n \subset B_n$ of finite dimensional $C^*$-algebras. We prove that for sufficiently large $n$, $A' \cap B$ is $*$-isomorphic to a subalgebra of $A'_n \cap B_n$.

INTRODUCTION

This paper is mainly concerned with the structure of the relative commutants $A' \cap B$ of a pair $A \subset B$ of type II$_1$ factors, generated by an infinite ladder $A_n \subset B_n$ of finite dimensional $C^*$-algebras. If $A_n \subset B_n$ is obtained by iteration of the basic construction on $B_0 \subset B_1$, then it is a well known result of A. Ocneanu that $A' \cap B \subset A'_n \cap B_n$; however, this is not true in general. For example, the tower of higher relative commutants associated with a pair of non-amenable subfactors does not satisfy this property (cf. [8]). We prove a weaker version of this property for general periodic towers. The periodicity is needed to ensure that the Jones index $[A : B]$ is finite. But we do not assume $A_n \subset B_n$ to be a tower of commuting squares. Specifically, we prove that $A' \cap B$ is $*$-isomorphic to a subalgebra of $A'_n \cap B_n$ for sufficiently large $n$. This will have several interesting consequences, one being a generalization of (Theorem 1.7, [9]). Our main tool is the perturbation technique developed by E. Christensen in [1]. Given $\epsilon > 0$, we prove that for sufficiently large $n$, $A' \cap B \subset A'_n \cap B_n$; then we are in a situation to use perturbation theory and get the desired isomorphism. The last section of the paper deals with towers of commuting squares. In particular, we obtain a sufficient condition for a tower $A_n \subset B_n$ to be a tower of commuting squares.

§1. Notations and preliminaries

Let $B$ be a type II$_1$ factor and let $tr$ be the faithful, normal and normalized trace on $B$. Denote by $L^2(B, tr)$ the Hilbert space closure of $B$ under the norm given by the inner product $\langle x, y \rangle = tr(y^*x)$. Then, $B$ acts on $L^2(B, tr)$ by left multiplication, and the identity of $B$ is a cyclic and separating vector for $B$ denoted by $\xi_0$. The involution $x \rightarrow x^*$ extends to a conjugate linear isometry on $L^2(B, tr)$ denoted by $J_B$. If $A$ is a von Neumann subalgebra of $B$, let $E_A$ be the conditional expectation

Received by the editors February 11, 1997.
1991 Mathematics Subject Classification. Primary 46L37.
Key words and phrases. Subfactors, von Neumann algebras, Jones index, commutant, commuting squares.

©1998 American Mathematical Society

2725
from \( B \) onto \( A \) associated with the trace, so that \( \text{tr}(E_A(x)) = \text{tr}(x) \) for every \( x \in B \). The extension of \( E_A \) to \( L^2(B, \text{tr}) \), denoted by \( e_A \), is the orthogonal projection of \( L^2(B, \text{tr}) \) onto the closure of \( A \) regarded as a subspace of \( L^2(B, \text{tr}) \). Then, \( \langle B, e_A \rangle \) the von Neumann algebra generated by \( B \) and \( e_A \) is called the basic construction. The following facts from [3] are frequently used in this the paper and are listed below for convenience.

(a) If \( x \in B \), then \( x \in A \Leftrightarrow xe_B = e_Bx \),
(b) if \( x \in B \), then \( e_Axe_A = E_N(x)e_A \),
(c) \( J_B(B, e_A)J_B = A' \),
(d) the map \( x \rightarrow xe_A \) is an isomorphism of \( A \) onto \( Ae_A \),
(e) \( e_A(B, e_A)e_A = Ae_A \).

The index of \( A \) in \( B \), denoted by \( [B : A] \), is defined to be \( (\text{tr}_{A'}(e_A))^{-1} \) if \( A' \) is finite and to be infinite otherwise (cf. [3]). It is a remarkable result of Jones that \([B : A] \in \{ 4 \cos^2 \frac{\pi}{n} : n = 3, 4, \ldots \} \cup \{ 4, \infty \} \), \( A' \cap B = C \) if \( [B : A] < 4 \), and \( A' \cap B \) is finite dimensional if \( [B : A] < \infty \).

If \( A \subset B \) are finite dimensional \( C^* \)-algebras with \( \{ e_i : i = 1, 2, \ldots, k \} \) and \( \{ f_j : j = 1, \ldots, l \} \) the sets of minimal central projections of \( A \) and \( B \) respectively, then the inclusion matrix \( T_B^A = (a_{ij}) \) is defined by

\[
a_{ij} = \begin{cases} 0, & \text{if } e_i f_j = 0; \\ \frac{\dim B_{e_i f_j}}{\dim A_{e_i f_j}}, & \text{if } e_i f_j \neq 0. 
\end{cases}
\]

The matrix \( T_B^A \) is unique up to permutations of the minimal central projections. It is an important fact that if \( \langle B, e_A \rangle \) is the basic construction, then \( \langle B, e_A \rangle \) is also finite dimensional and \( T_B^{(B, e_A)} = (T_A^B)^t \). The inclusion \( A \subset B \) may also be described by a bipartite graph called the Bratteli diagram of the pair \( A \subset B \), with the blocks of \( A \) and \( B \) forming the vertices of the graph and the edges being the multiplicities of subfactors of \( A \) in the subfactors of \( B \) (cf. [2], [3]). Increasing sequences of finite dimensional \( C^* \)-algebras \( \langle A_n \rangle, \langle B_n \rangle \), with \( A_n \subset B_n \), are said to constitute a periodic tower of algebras if for some \( h \) and all \( n \) we have:

\[
T_{B_{n+h}}^{B_{n+h+1}} = T_{B_{n+1}}^{B_n}, \quad T_{A_{n+h}}^{A_{n+h+1}} = T_{A_{n+1}}^{A_n}, \quad T_{A_n}^{B_n} = T_{A_{n+h}}^{B_{n+h}}.
\]

If there exists a faithful trace on \( \bigcup B_n \), then \( B = \overline{\bigcup B_n} \) and \( A = \overline{\bigcup A_n} \) are the closures with respect to the weak topology induced by the trace. We say that \( A \subset B \) is generated by the tower \( A_n \subset B_n \). If the inclusion matrices are indecomposable and the tower is periodic, then \( A \) and \( B \) are factors and \( [A : B] < \infty \). A diagram

\[
A \subset B \quad \cup \quad C \subset D
\]

of finite dimensional \( C^* \)-algebras is called a commuting square with respect to a trace, \( \text{tr} \), on \( B \) if

\[
A \xrightarrow{E_A} B \\
\cup \\
C \xleftarrow{E_C} D
\]

is a commutative diagram, i.e., for each \( x \in D \), \( E_A(x) \in C \). The expectations \( E_A \) and \( E_C \) correspond to the trace, and we assume \( \text{tr} \) is a Markov trace.
§2. Main results

Near inclusions $A \lessdot B$ of pairs of von Neumann subalgebras of a type II$_1$ von Neumann algebra $C$ were studied by E. Christensen in [1]. The relation $A \lessdot B$ means that for every $x \in A_1$, the unit ball of $A$, there exists an element $y \in B$ such that \(||x - y|| < \delta\), where $tr$ is the faithful, normal, and normalized trace on $C$. If $E_B$ denotes the canonical trace preserving conditional expectation from $C$ onto $B$, then

\[ A \lessdot B \iff ||x - E_B(x)|| < \delta \forall x \in A_1. \]

As we are considering pairs of factors generated by finite dimensional algebras, we only need to consider inclusions $A \lessdot B$ with $B$ finite dimensional. We need to modify several perturbations lemmas from [1]. However, the main ideas remain the same. Given $\delta > 0$, the following function is frequently needed:

\[ \gamma(\delta) = 2^{\frac{1}{2}}\delta^{\frac{1}{2}}(1 - 2^{\frac{1}{2}}\delta^{\frac{1}{2}})^{-1}. \]

**Lemma 2.1.** Let $B$ be a finite dimensional subalgebra of a type II$_1$ von Neumann algebra $C$. If $tr$ is the faithful, normal, and normalized trace on $B$, then there exists a semifinite trace, $tr'$, on $(C, e_B)$ such that

\[ tr'(xe_B) = tr(x) \forall x \in C, \]

\[ tr'(e_B) = 1. \]

**Proof.** Let $e_1, e_2, \ldots, e_m$ be the minimal projections of $B$. By [3, Proposition 3.1.5], $f_k = J_CE_kJ_C, 1 \leq k \leq m$, are the minimal central projections of the semifinite von Neumann algebra $(C, e_B)$. Hence, there exists a unique faithful normal semifinite trace, $tr'$, on $(C, e_B)$ such that $tr'(f_ke_B) = tr(e_k)$. Using the $*$-isomorphism $xf_ke_B \to E_B(x)e_k$ from $(C, e_B)f_ke_B$ onto $B e_k$ (cf. [3]) and the uniqueness of the trace on the factor $B e_k$, we have that $tr'(xf_ke_B) = tr(E_B(x)e_k)$ for all $x \in C$. Given $x \in C$, we have

\[ tr'(xe_B) = tr'\left(\sum_{k=1}^{m} xf_ke_B\right) \]
\[ = \sum_{k=1}^{m} tr'(xf_ke_B) \]
\[ = \sum_{k=1}^{m} tr(E_B(x)e_k) \]
\[ = tr(E_B(x)\sum_{k=1}^{m} e_k) = tr(x). \]

Let $x = 1$ in the above to get $tr'(e_B) = 1$. \qed

**Lemma 2.2.** Let $B$, $C$, and $e_1, e_2, \ldots, e_m$ be as above. Suppose that $A$ is a von Neumann subalgebra of $C$ such that $A \lessdot C$ with $\delta < \frac{1}{\sqrt{2}}$ and such that for every minimal projection $q \in B$, $\gamma^2(\delta) < tr(q)$. Then, there exists a $*$-homomorphism $\varphi$ from $A$ into $B$. Moreover, for each $x$ in the unit ball of $A$ we have:

\[ ||\varphi(x) - x|| < 26\gamma(\delta) + \delta, \quad \varphi(I) = I. \]
Proof. If \( u \in A \) is unitary, then
\[
\text{tr}(I - E_B(u^*)E_B(u)) = \|u - E_B(u)\|_2^{tr} < \delta^2.
\]
If \( tr' \) is as in Lemma 2.1, then we have
\[
tr'((e_B - u^*e_Bu)^2) = tr'(e_B + u^*e_Bu - u^*e_Bue_B - e_Bu^*e_Bu) = 2tr(e_B(1 - E_B(u^*)E_B(u)) = 2tr(I - E_B(u^*)E_B(u)) < 2\delta^2.
\]
As in [1, Theorem 3.1], we can find an element \( k \) of minimal trace norm \( \| \|_2^{tr'} \) in the ultrastrong closed convex hull of the set \( \{u^*e_Bu : u \text{ unitary in } B\} \) such that
\[
0 \leq k \leq I, \quad \|k - e_B\|_2^{tr'} < 2\frac{\delta}{2}.
\]
Since \( tr'(e_B) = 1 \), by the discussion preceding [1, Lemma 4.1] there exists a projection \( q \in A' \cap (A, e_B) \) such that
\[
\|q - e_B\|_2^{tr'} \leq \gamma(\delta) \quad \text{and} \quad |1 - tr'(q)| < \gamma(\delta)^2.
\]
Let \( \{f_1, ..., f_m\} \) be the set of minimal central projection of \( \langle C, e_B \rangle \) (see Lemma 2.1). Then,
\[
|tr'(qf_n) - tr'(e_Bf_n)| = |tr'((q - e_B)f_n)| \leq |tr'(q - e_B)| < |1 - tr'(q)| < \gamma(\delta)^2.
\]
Since the factor \( \langle C, e_B \rangle f_k \) is isomorphic to \( Be_k \) and \( tr(p) > \gamma(\delta)^2 \) for every projection \( p \in B \), the above inequality implies that \( tr'(qf_k) = tr'(e_Bf_k) \). Hence, \( qf_k \sim e_Bf_k \). Let \( v_k \in \langle C, e_B \rangle f_k \) be a partial isometry such that \( v_kv_k^* = qf_k \) and \( v_k^*v_k = e_Bf_k \). Then, \( q \sim e_B \) via the partial isometry \( v = v_1 + v_2 + \cdots + v_m \). It is easy to check that \( m \to v^*mv \) is a homomorphism from \( A \) into \( \langle C, e_B \rangle e_B = Be_B \).

Let \( \varphi \) be the composition of this with the canonical identification of \( Be_B \) with \( B \). If \( x \) is in the unit ball of \( A \), then by [2, Lemma 4.1], \( \|\varphi(x) - E_A(x)\| < 2\gamma(\delta) \).

Since \( \|x - E_A(x)\| < \delta \), the inequality (4) follows.

Proposition 2.3. Let \( A \subset B \) be factors generated by the tower \( A_n \subset B_n \) such that \( [B : A] < \infty \). Then, given \( \epsilon > 0 \), there exists \( N > 0 \) such that:
\[
A' \cap B \hat{\otimes} A'_n \cap B_n \forall n > N.
\]

Proof. Since \( B = \bigcup B_n \), given \( x \) in the unit ball of \( A' \cap B \) there exist an integer \( n \) and \( y \in B_n \) such that \( \|x - y\|_2^{tr'} < \frac{\delta}{2} \). Hence, \( \|x - E_B(x)\|_2^{tr} < \frac{\delta}{3} \). Since \( [B : A] < \infty \) by [3, Corollary 2.2.3] \( A' \cap B \) is finite dimensional, and we may use a standard compactness argument and find a positive \( N \) such that
\[
\|x - E_B(x)\|_2^{tr} < \epsilon \forall x \in (A' \cap B)_1 \quad \text{and} \quad \forall n > N.
\]
It is easy to see that if \( x \in A' \cap B \), then \( E_B(x) \in A'_n \cap B_n \) and hence \( A' \cap B \hat{\otimes} A'_n \cap B_n \) by (6) and (1).
Lemma 2.4. Let $A \subset B$ be factors generated by $A_n \subset B_n$. If $\lim \sup \|T_{A_n}^{B_n}\| \leq \infty$, then $[A : B] < \infty$.

Proof. It follows from [5, Propositions 2.6 and 3.4] that the entropy $H(B, A) \leq \lim \sup \|T_{A_n}^{B_n}\|$. If $A' \cap B$ had a completely non-atomic part, then as in the proof of [5, Theorem 4.4] we conclude that $H(B, A) = \infty$, which contradicts the earlier statement. Hence, $A' \cap B$ must be atomic and by [5, Theorem 4.4], we conclude that $[B : A] < \infty$. □

Corollary 2.5. Let $A \subset B$ be generated by a periodic tower $A_n \subset B_n$. Then, given $\varepsilon > 0$, there exists $N$ such that

$$A' \cap B \subset A_n' \cap B_n \forall n > N.$$ 

Proof. If $A_n \subset B_n$ is periodic, then the hypothesis of Lemma 2.4 is satisfied and hence $[A : B] < \infty$. Now Proposition 2.3 can be used. □

Theorem 2.6. Let $A \subset B$ be factors generated by $A_n \subset B_n$ and let $\{e_n^k\}_k$ be the set of minimal central projections of $A_n$. Suppose that

(a) $[B : A] < \infty$,
(b) $\lim \inf_{k,n} \text{tr}(e_n^k) > 0$,
(c) there exists $r > 0$ such that for every $n$ and every minimal projection $p \in A_n' \cap B_n$ we have $\text{tr}(p) > r$.

Then $A' \cap B$ is $*$-isomorphic to a subalgebra of $(A_n' \cap B_n)e_n^k$ for sufficiently large $n$.

Proof. Choose $d > 0$ such that $\lim \inf_{k,n} \text{tr}(e_n^k) > d > 0$ and choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{\sqrt{2}}, \gamma^2(\varepsilon) < r$, and $\gamma(\varepsilon) + \varepsilon < \text{tr}(p)$ for every minimal projection $p$ in the finite dimensional algebra $A' \cap B$. Next choose $\delta > 0$ such that $\frac{\delta}{\varepsilon} < \varepsilon$. By Proposition 2.3 there exists $N$ such that $A' \cap B \subset A_n' \cap B_n$ when $n > N$. Let $\tilde{\text{tr}}(x) = \frac{\text{tr}(x)}{\text{tr}(e_n^k)}$ for $x \in (B_n)e_n^k$. Then $\tilde{\text{tr}}_r$ is the canonical normalized trace on $(B_n)e_n^k$, and we obtain

$$(A' \cap B)e_n^k \subset (A_n' \cap B_n)e_n^k \forall k$$

where the above near inclusion is with respect to the $\tilde{\text{tr}}$ norm. Since, $\frac{\delta}{\varepsilon} < \varepsilon$ we have that

$$(A' \cap B)e_n^k \subset (A_n' \cap B_n)e_n^k \forall k, n > N.$$ 

By the choice of $\varepsilon$ the hypothesis of Lemma 2.2 is satisfied, and hence there exists a $*$-homomorphism $\varphi$ from $(A' \cap B)e_n^k$ into $(A_n' \cap B_n)e_n^k$. If $\varphi(x) = 0$ for some $x \in (A' \cap B)e_n^k$, then $\varphi(p) = 0$ for some projection. By (4), $\text{tr}(p) < \gamma(\varepsilon) + \varepsilon$. This contradicts the choice of $\varepsilon$. Hence, $\varphi$ must be one to one. Finally, as $e_n^k \in A$ and $A$ is a factor, $(A' \cap B)e_n^k$ is $*$-isomorphic to $A' \cap B$, and the proof is complete. □

Theorem 2.7. Suppose that $A \subset B$ is generated by a periodic tower $A_n \subset B_n$ of finite dimensional $C^*$-algebras, then $A' \cap B$ is $*$-isomorphic to a subalgebra of $A_n' \cap B_n$.

Proof. Since $A_n \subset B_n$ is periodic by Corollary 2.5, the index $[B : A]$ is finite. Conditions (b) and (c) of Theorem 2.6 are satisfied here because the inclusion matrices are periodic. We refer to [9] for a proof of this fact. Hence the result follows from Theorem 2.6. □
The following generalizes Theorem 1.7 of [9] in the sense that $A_n \subset B_n$ is not assumed to be a tower of commuting squares.

**Corollary 2.8.** If $A \subset B$ and $A_n \subset B_n$ are as above and there exists $m > 0$ such that $(A'_n \cap B_n)_{e_{m,n}} = C$ for some $k$ and every $n > m$. Then $A' \cap B = C$.

**Proof.** Follows directly from Theorems 2.5 and 2.7. □

**Corollary 2.9.** If $A \subset B$ and the tower $A_n \subset B_n$ are as before, and for a sequence $(n_k)$ the inclusion matrix $T_{A_{n_k}}$ has a column (or a row) of exactly one nonzero entry equal to one, then $A' \cap B = C$.

**Proof.** Our hypothesis implies that for some $l, (A'_n \cap B_{n_k})_{e_{l,n_k}} = C$ for all $n_k$. Now the result follows from Corollary 2.8. □

**Proposition 2.10.** Let $A \subset B$, with $[B : A] < \infty$, be factors generated by a tower $A_n \subset B_n$ of commuting squares. Let $\{e_{n,k}\}$ be the set of minimal central projections of $A_n$ and $B_n$ respectively and suppose that $\liminf_{k,n} (e_{n,k}) > 0$. Then, for sufficiently large $n$, $A' \cap B$ is $*$-isomorphic to a subalgebra of $A'_{n} \cap B_{n}$.

**Proof.** By [5, Proposition 2.6] $[B : A] = \lim \lambda(B_n : A_n)^{-1}$, and by [6, Theorem 2.3] $\lambda(B_n, A_n)^{-1} = \max_{k} (\sum_{l} a_{k,l}^{2} tr(e_{k,n}^{l}) / tr(e_{k,n}^{l} f_{n}^{l}))$ where $a_{k,l}^{2}$ is just the dimension of $(A'_{n} \cap B_{n}) e_{n,k} f_{n}^{l}$. Hence, there exists $N$ such that for every $n > N$ we have $a_{k,l}^{2} \cdot tr(e_{k,n}^{l}) / tr(e_{k,n}^{l} f_{n}^{l}) \leq [B : A]$.

Equivalently, for $n > N$

$$\frac{tr(e_{k,n}^{l} f_{n}^{l})}{a_{k,l}^{2}} \geq \frac{tr(e_{k,n}^{l}), f_{n}^{l})}{[B : A]}.$$

Now the left hand side of the above inequality is the trace of minimal projections in $(A'_{n} \cap B_{n}) e_{n,k} f_{n}^{l}$. Given that $\liminf_{k} tr(e_{n,k}^{l}) > 0$, it follows that condition (c) of Theorem 2.6 is satisfied here, and the result follows from that theorem. □

§3. Remarks on commuting squares

An important tool in construction and the study of subfactors is the concept of commuting squares (see Section 1 for the definition), developed by M. Pisner and S. Popa (cf. [5]), and used by many authors in relation with Jones index theory (cf. [4], [7], [9]). The construction and classification of subfactors are closely related to those of commuting squares. The results of this section are along these lines. If $B_2 = \langle B_1, e_1 \rangle$ is the basic construction on $A_1 \subset B_1$ and if $A_2 = \langle A_1, e_1 \rangle$, then the following is a commuting square:

$$A_2 \subset B_2$$
$$\cup \quad \cup$$
$$A_1 \subset B_1$$

By iterating the above procedure we obtain the tower $A_n \subset B_n$ of commuting squares. Note that the algebra $A_n$ contains the set $\{e_1, e_2, ..., e_{n-2}\}$ of projections corresponding to the basic construction. We want to show that the converse is also true in the following sense. Namely, let the tower $B_n \subset B_{n+1}$ be obtained by iterating the basic construction on the pair of finite dimensional $C^*$-algebras $B_1 \subset B_2$, and let $\{e_n\}$ be the corresponding Jones’ projection. Suppose that $\{A_n\}$ is an increasing sequence of $C^*$-algebras with $A_n \subset B_n$ such that for $n \geq 3$, $A_n$
contains the set \( \{e_1, e_2, ..., e_{n-2}\} \). Set \( A = \bigcup A_n \), \( B = \bigcup B_n \) and \( \tilde{A}_n = A \cap B_n \), where the closures are in the sense of the weak operator topology. With these notations, we state the following.

**Proposition 3.1.** \( \tilde{A}_n \subset B_n \) is a tower of commuting squares. Moreover, if \( [B : A] < \infty \), then the tower \( \tilde{A}_n \subset B_n \) is periodic.

**Proof.** We must show that if \( y \in \tilde{A}_n \), then \( E_{B_{n-1}}(y) \in \tilde{A}_{n-1} \). Since \( e_n e_{n-1} e_n = \lambda e_n \) with \( \lambda^{-1} = \|T_{B^n_{2}}\| \), we have that
\[
e_n e_{n-1} e_n e_n e_{n-1} e_n = e_n E_{B_{n-1}}(y)e_{n-1} e_n = E_{B_{n-1}}(y)e_{n-1} e_n = \lambda E_{B_{n-1}}(y)e_{n-1} e_n.
\]
Hence, \( e_n E_{B_{n-1}}(y) \in \tilde{A}_{n+2} \subset A \). It follows that \( e_m E_{B_{n-1}}(y) \in A \) for all \( m \geq n \). Hence, \( e_n \lor e_{n+1} \lor \cdots \lor e_m E_{B_{n-1}}(y) \in A \) for \( m > n \). Since \( tr(1 - e_{n+1} \lor \cdots \lor e_m) \) converges to zero as \( m \) tends to infinity (cf. §4, [2]), the projections \( e_n \lor \cdots \lor e_m \) converge to the identity as \( m \) tends to infinity. Hence, \( E_{B_{n-1}}(y) \in A \cap B_{n-1} = \tilde{A}_{n-1} \), proving the first statement. When \( [B : A] < \infty \) the periodicity is a consequence of the fact that \( [B : A] = \lim [B_n : A_n] \) (cf. [5]). \( \square \)

We obtain the following corollary regarding intermediate subfactors.

**Corollary 3.2.** Let \( A_n \subset B_n \) be obtained by the basic construction and let \( C \) be a middle subfactor, \( A \subset C \subset B \). Then there exists a periodic tower of commuting squares \( A_n \subset C_n \subset B_n \) such that \( C = \lim C_n \).

**Proof.** Let \( C_n = B_n \cap C \) and apply the previous lemma. \( \square \)

The following theorem shows that for factors \( A \subset B \) generated by a tower \( A_n \subset B_n \) the existence of proper intermediate subfactors may be reduced to a finite dimensional problem.

**Theorem 3.3.** Let \( A \subset B \) be generated by the tower \( A_n \subset B_n \) given by the basic construction. Then there exists a proper middle subfactor \( C \) if and only if, for some \( n \), there exist \( C^* \)-algebras \( C_n \subset C_{n+2} \) such that
\[
A_{n+2} \subset C_{n+2} \subset B_{n+2},
\]
\[
A_n \subset C_n \subset B_n,
\]
with \( T_{B_n}^{C_n} = T_{B_{n+2}}^{C_{n+2}} \).

**Proof.** The only if part follows from the previous corollary. Now suppose that \( C_n \subset C_{n+2} \) are as in the statement of the theorem. Since the identity is in \( A_{n+2} \cap C_{n+2} \), for every minimal central projection \( q \) in \( C_{n+2} \), \( q e_n \) is a nonzero projection. Then \( e_n E_{B_n}(C_{n+1}) = e_n C_{n+1} e_n \) is a subset of \( e_n(C_{n+2})e_n = (C_n)e_n \). The last equality holds because
\[
(C_n)e_n \subset (C_{n+2})e_n \subset (B_{n+2})e_n = (B_n)e_n.
\]
and $T_{C_n} = T_{C_n+2}$. Therefore, $E_{B_n}(C_{n+1}) \subseteq C_n$. But $C_n \subseteq E_{B_n}(C_{n+2})$, whence $E_{B_n}(C_{n+1}) = C_n$. This shows that

$$
C_{n+1} \subseteq B_{n+1} \cup U \cup C_n \subseteq B_n
$$

is a commuting square, and it is easy to see that $C_{n+2} = \langle C_{n+1}, e_n \rangle$. Then $C$ can be constructed inductively.

References


Department of Mathematics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6

E-mail address: khoshkam@math.usask.ca