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CENTRAL EXTENSIONS OF SOME LIE ALGEBRAS

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Abstract. We consider three Lie algebras: $Der\ \mathbb{C}((t))$, the Lie algebra of all derivations on the algebra $\mathbb{C}((t))$ of formal Laurent series; the Lie algebra of all differential operators on $\mathbb{C}((t))$; and the Lie algebra of all differential operators on $\mathbb{C}((t)) \otimes \mathbb{C}^n$. We prove that each of these Lie algebras has an essentially unique nontrivial central extension.

The Lie algebra of all derivations on the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$ can also be characterized as the Lie algebra of vector fields on the circle. The analogous object over a field F of characteristic $p > 0$, $Der\ F[t]/(t^p)$, is called the Witt algebra [C], and this name is sometimes applied to $Der\ \mathbb{C}[t, t^{-1}]$ as well. It is known [Bl] that $Der\ F[t]/(t^p)$ has an essentially unique nontrivial one-dimensional central extension, and also [GF] that $Der\ \mathbb{C}[t, t^{-1}]$ has an essentially unique nontrivial one-dimensional central extension. The proofs of these facts are similar. The nontrivial one-dimensional central extension of $Der\ \mathbb{C}[t, t^{-1}]$ is called the Virasoro algebra. It is one of the fundamental objects in representation theory as well as in theoretical physics.

For a positive integer n, the Lie algebra of all differential operators on $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}^n$ has a nontrivial one-dimensional central extension, and the extended Lie algebra is related to the representation theory of affine Lie algebras [KP]. It is proved in [L] that this extension is essentially unique (also see [F]). When $n = 1$, the Lie algebra of all differential operators on the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ can also be characterized as the Lie algebra of differential operators on the circle; the corresponding extension is referred to, particularly in the physics literature, as $\mathcal{W}_{1+\infty}$. Some representations of $\mathcal{W}_{1+\infty}$ have been studied recently (see, e.g., [KR], [FKRW]). In [FKRW], it is shown that some representations of $\mathcal{W}_{1+\infty}$ have natural structures of vertex operator algebras (see, e.g., [Bo] and [FLM] for definitions).

Each of these constructions involves the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$. This algebra is, of course, contained in $\mathbb{C}((t))$, the algebra of formal Laurent series. In this paper, we consider the Lie algebras obtained by replacing $\mathbb{C}[t, t^{-1}]$ by $\mathbb{C}((t))$ in each of these constructions. We show that each of the resulting Lie algebras has an essentially unique nontrivial one-dimensional central extension.

We work over the field of complex numbers, though all results hold over any field of characteristic zero.

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1. Some basic definitions

Let L and $\hat{L}$ be two Lie algebras over $\mathbb{C}$. The Lie algebra $\hat{L}$ is said to be a one-dimensional central extension of L if there is a Lie algebra exact sequence $0 \to \mathbb{C}c \to \hat{L} \to L \to 0$, where $\mathbb{C}c$ is the one-dimensional trivial Lie algebra and the image of $\mathbb{C}c$ is contained in the center of $\hat{L}$. It is well-known that $\hat{L}$ is a one-dimensional central extension of L if and only if $\hat{L}$ is the direct sum of L and $\mathbb{C}c$ as vector spaces and the Lie bracket $[\cdot,\cdot]_1$ in $\hat{L}$ is given by

$$[x,y]_1 = [x,y] + \varphi(x,y)c,$$
$$[x,c]_1 = 0$$

for all $x, y \in L$, where $[\cdot,\cdot]$ is the Lie bracket in L and $\varphi : L \times L \to \mathbb{C}$ is a bilinear form on L satisfying the following conditions:

$$\text{(i)} \quad \varphi(x,y) = -\varphi(y,x), \tag{1}$$

$$\text{(ii)} \quad \varphi([x,y],z) + \varphi([y,z],x) + \varphi([z,x],y) = 0 \tag{2}$$

for all $x, y, z \in L$. The bilinear form φ is called a 2-cocycle on L. A central extension is trivial if $\hat{L}$ is the direct sum of a subalgebra M and $\mathbb{C}c$ as Lie algebras, where the subalgebra M is isomorphic to L. A 2-cocycle φ corresponding to a trivial central extension is called a 2-coboundary, or a trivial 2-cocycle, and is given by a linear function f from L to $\mathbb{C}$:

$$\varphi(x,y) = f([x,y])$$

for all $x, y \in L$. The 2-coboundary defined by f is denoted by α_f. The set of all 2-cocycles on L is a vector space, denoted by $Z^2(L,\mathbb{C})$. The set of all 2-coboundaries is a subspace of $Z^2(L,\mathbb{C})$, denoted by $B^2(L,\mathbb{C})$. The quotient space $Z^2(L,\mathbb{C})/B^2(L,\mathbb{C})$ is called the 2nd cohomology group of L with coefficients in $\mathbb{C}$, and denoted by $H^2(L,\mathbb{C})$. If $\dim H^2(L,\mathbb{C})=1$, we say that L has an essentially unique nontrivial one-dimensional central extension. We say that 2-cocycles φ, ψ are equivalent if $\varphi - \psi$ is a 2-coboundary.

The following two lemmas will be used in the proofs of our main results.

Lemma 1. *Let L be a Lie algebra and S a subset of L such that S spans L and for each $x \in S$, $x = [y_x, z_x]$ for some $y_x, z_x \in L$. If a 2-cocycle φ satisfies $\varphi(y_x, z_x) = 0$ for all $x \in S$, then either $\varphi = 0$ or φ is nontrivial.*

Proof. Suppose that φ is trivial, so that $\varphi = \alpha_f$ for some linear function f. Then for each $x \in S$,

$$f(x) = f([y_x, z_x]) = \varphi(y_x, z_x) = 0.$$

Thus $f = 0$ since S spans L. This implies that $\varphi = \alpha_f = 0$. □

Lemma 2. *Let L be a Lie algebra and φ a 2-cocycle on L. Suppose there are linear endomorphisms E and F of L such that*

$$\varphi(Ex, y) = \varphi(x, Fy)$$

for all $x, y \in L$, E is surjective and F is locally nilpotent (i.e., for any $y \in L$, there is a positive integer n such that $F^n y = 0$). Then the 2-cocycle φ is 0.

Proof. For $x, y \in L$, let n be a positive integer such that $F^n y = 0$. Since E is surjective, we have $x' \in L$ such that $x = E^n x'$. Thus,

$$\varphi(x,y) = \varphi(E^n x', y) = \varphi(x', F^n y) = 0.$$ □

Let A be a (not necessarily associative) algebra. A linear map δ: $A \to A$ is called a derivation, if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$.

Now consider the algebra of all formal Laurent series

$$\mathbb{C}((t)) = \left\{ \sum_{i\in\mathbb{Z}, i\geq n} a_i t^i \middle| a_i \in \mathbb{C}, n \in \mathbb{Z} \right\}.$$

It is known that the set of all derivations on $\mathbb{C}((t))$ is

$$\mathcal{A} = \left\{ f(t)\frac{d}{dt} \middle| f(t) \in \mathbb{C}((t)) \right\},$$

where $\frac{d}{dt}$ is the formal derivation defined by $\frac{d}{dt}$: $\mathbb{C}((t)) \to \mathbb{C}((t))$, $\sum a_i t^i \longmapsto \sum i a_i t^{i-1}$. For convenience, we denote $\frac{d}{dt}$ by D and denote $\frac{d}{dt}f(t)$ by $f'(t)$. For $f(t) = \sum a_i t^i \in \mathbb{C}((t))$, define $Res\ f(t) = a_{-1}$. If $Res\ f(t) = 0$, we can define the formal integral of $f(t)$ as

$$\sum_{i\neq -1} \frac{a_i}{i+1} t^{i+1},$$

denoted by $\int f(t)$. Then $\mathcal{A}$ is a Lie algebra under the bracket operation:

$$\begin{aligned} &[f(t)D, g(t)D] \\ &= (f(t)D) \circ (g(t)D) - (g(t)D) \circ (f(t)D) \\ &= f(t)g'(t)D - g(t)f'(t)D \end{aligned}$$

for $f(t), g(t) \in \mathbb{C}((t))$, where the $\circ$ is the composition of operators.

More generally, consider the space of all differential operators on the algebra of formal Laurent series $\mathbb{C}((t))$:

$$\mathcal{B} = \operatorname{span}\left\{ f(t)D^l \middle| l \in \mathbb{N}, f(t) \in \mathbb{C}((t)) \right\}.$$

$\mathcal{B}$ is a Lie algebra under the bracket operation:

$$\begin{aligned} &[f(t)D^l, g(t)D^k] = (f(t)D^l) \circ (g(t)D^k) - (g(t)D^k) \circ (f(t)D^l) \\ &= \sum_{i=0}^{l} \binom{l}{i} f(t) \left(D^{l-i} g(t)\right) D^{k+i} - \sum_{j=0}^{k} \binom{k}{j} g(t) \left(D^{k-j} f(t)\right) D^{l+j}. \end{aligned}$$

Furthermore, we may consider $\mathcal{C} = \mathcal{B} \otimes gl_n(\mathbb{C})$, the space of differential operators on $\mathbb{C}((t)) \otimes \mathbb{C}^n$. Note that $\mathcal{C} \subset End\ (\mathbb{C}((t)) \otimes \mathbb{C}^n)$. Define the bracket operation on $\mathcal{C}$ by linearity and the commutator

$$\begin{aligned} &[f(t)D^l \otimes A, g(t)D^k \otimes B] \\ &= (f(t)D^l \otimes A) \circ (g(t)D^k \otimes B) - (g(t)D^k \otimes B) \circ (f(t)D^l \otimes A) \\ &= \sum_{i=0}^{l} \binom{l}{i} f(t) \left(D^{l-i} g(t)\right) D^{k+i} \otimes AB - \sum_{j=0}^{k} \binom{k}{j} g(t) \left(D^{k-j} f(t)\right) D^{l+j} \otimes BA. \end{aligned}$$

Hence $\mathcal{C}$ is a Lie algebra.

2. Main results and proofs

In this section, we will give our main results and their proofs. In each case we exhibit (using Lemma 1) a nontrivial 2-cocycle on the Lie algebra under consideration. The 2-cocycle is analogous to the standard nontrivial 2-cocycle on the Lie algebra obtained from Laurent polynomial algebra. Then for any given 2-cocycle on the Lie algebra, we reduce the 2-cocycle to a 2-cocycle which is equivalent to the original one and takes value 0 whenever the standard 2-cocycle takes value 0. We use Lemma 2 to show that the reduced 2-cocycle is a multiple of the standard one.

Theorem 1. *$dimH^2(\mathcal{A},\mathbb{C}) = 1$.*

Proof. Let β be a 2-cocycle on $\mathcal{A}$. Define a linear function $f_\beta : \mathcal{A} \to \mathbb{C}$ by

$$f_\beta(g(t)D) = \beta\left(D, \int g(t)D\right) \quad \text{for} \quad g(t) \in \mathbb{C}((t)),\ Res\ g(t) = 0$$

and

$$f_\beta\left(t^{-1}D\right) = \frac{1}{2}\beta\left(t^{-1}D, tD\right).$$

Then $\beta_1 = \beta - \alpha_{f_\beta}$ is a 2-cocycle on $\mathcal{A}$ which is equivalent to β.

For $f(t) = \sum_{i\neq 0} a_i t^i \in \mathbb{C}((t))$,

$$\begin{aligned} &\beta_1(D, f(t)D) \\ &= \beta(D, f(t)D) - f_\beta([D, f(t)D]) \\ &= \beta(D, f(t)D) - f_\beta(f'(t)D) = 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\beta_1\left(t^{-1}D, tD\right) \\ &= \beta\left(t^{-1}D, tD\right) - f_\beta\left([t^{-1}D, tD]\right) \\ &= \beta\left(t^{-1}D, tD\right) - f_\beta\left(2t^{-1}D\right) = 0. \end{aligned} \tag{4}$$

Lemma 3. *$\beta_1(D,\mathcal{A}) = 0$ and $\beta_1(tD,\mathcal{A}) = 0$.*

Proof of Lemma 3. From (3) and $\beta_1(D,D) = -\beta_1(D,D)$, we have $\beta_1(D,\mathcal{A}) = 0$. For $f(t) \in \mathbb{C}((t))$ and $Res\ f(t) = 0$,

$$\begin{aligned} &\beta_1(tD, f(t)D) \\ &= \beta_1\left(tD, \left[D, \int f(t)D\right]\right) \\ &= \beta_1\left([tD, D], \int f(t)D\right) + \beta_1\left(D, \left[tD, \int f(t)D\right]\right) = 0. \end{aligned}$$

$\beta_1(tD,\mathcal{A}) = 0$ follows from this and (4). □

Lemma 4. *$\beta_1(t^2D,\mathcal{A}) = 0$.*

Proof of Lemma 4. For $f(t) \in \mathbb{C}((t))$ and $Res\ f(t) = 0$, we have

$$\begin{aligned}
&\beta_1(t^2D, f(t)D)\\
&= \beta_1\left(t^2D, \left[D, \int f(t)D\right]\right)\\
&= \beta_1\left([t^2D, D], \int f(t)D\right) + \beta_1\left(D, \left[t^2D, \int f(t)D\right]\right) = 0.
\end{aligned}$$

Also

$$\begin{aligned}
&\beta_1\left(t^2D, t^{-1}D\right)\\
&= \beta_1\left(t^2D, -\frac{1}{2}\left[tD, t^{-1}D\right]\right)\\
&= -\frac{1}{2}\beta_1\left(\left[t^2D, tD\right], t^{-1}D\right) - \frac{1}{2}\beta_1\left(tD, \left[t^2D, t^{-1}D\right]\right)\\
&= -\frac{1}{2}\beta_1\left(-t^2D, t^{-1}D\right) = \frac{1}{2}\beta_1\left(t^2D, t^{-1}D\right).
\end{aligned}$$

This implies that $\beta_1\left(t^2D, t^{-1}D\right) = 0$. □

Lemma 5. *If $f(t) \in \mathbb{C}((t))$ and $Res\ f(t) = 0$, then $\beta_1(t^3D, f(t)D) = 0$.*

Proof of Lemma 5. We have

$$\begin{aligned}
&\beta_1(t^3D, f(t)D)\\
&= \beta_1\left(t^3D, \left[D, \int f(t)D\right]\right)\\
&= \beta_1\left(\left[t^3D, D\right], \int f(t)D\right) + \beta_1\left(D, \left[t^3D, \int f(t)D\right]\right)\\
&= \beta_1\left(-3t^2D, \int f(t)D\right) = 0.
\end{aligned}$$

□

Define $\alpha : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ by

$$\alpha\left(\sum_i a_i t^{i+1}D, \sum_j b_j t^{j+1}D\right) = \sum_i a_i b_{-i}(i^3 - i)$$

for $\sum_i a_i t^{i+1}D, \sum_j b_j t^{j+1}D \in \mathcal{A}$. Note that the sum on the right-hand side is finite and α is a 2-cocycle on $\mathcal{A}$. Let S be the subset of $\mathcal{A}$ given by

$$S = \left\{t^{-1}D\right\} \cup \left\{f(t)D \in \mathcal{A} \middle| Res\ f(t) = 0\right\}.$$

Now for $f(t) \in \mathbb{C}((t))$, $Res\ f(t) = 0$, we have

$$t^{-1}D = \left[\frac{1}{2}t^{-1}D, tD\right],$$

$$f(t)D = \left[D, \int f(t)D\right],$$

$$\alpha\left(t^{-1}D, tD\right) = 0,$$

and

$$\alpha\left(D, \int f(t)D\right) = 0.$$

Since α is nonzero (in fact, $\alpha(t^3D, t^{-1}D) = 6$), Lemma 1 shows that α is nontrivial. Also $\alpha = \alpha_1$. Applying Lemma 3, Lemma 4 and Lemma 5 to α, we have

$$\alpha(D, \mathcal{A}) = \alpha(tD, \mathcal{A}) = \alpha(t^2D, \mathcal{A}) = 0,$$

and

$$\alpha\left(t^3D, f(t)D\right) = 0 \quad \text{for} \quad f(t) \in \mathbb{C}((t)), \quad Res\ f(t) = 0.$$

Suppose that $\beta_1(t^3D, t^{-1}D) = 6r$ for some $r \in \mathbb{C}$. Define $\beta_2 = \beta_1 - r\alpha$; then we have

$$\beta_2(D, \mathcal{A}) = \beta_2(tD, \mathcal{A}) = \beta_2(t^2D, \mathcal{A}) = 0 \tag{5}$$

and

$$\beta_2(t^3D, \mathcal{A}) = 0. \tag{6}$$

We now show that $\beta_2 = 0$, completing the proof of Theorem 1.

Let ad: $\mathcal{A} \to \mathcal{A}$, $\mathrm{ad}(a)b = [a, b]$, be the adjoint operator; then

$$\begin{aligned} &\beta_2\left(\mathrm{ad}D(f(t)D), g(t)D\right) \\ &= \beta_2\left([D, f(t)D], g(t)D\right) \\ &= \beta_2\left([D, g(t)D], f(t)D\right) + \beta_2\left(D, [f(t)D, g(t)D]\right) \\ &= -\beta_2\left(f(t)D, \mathrm{ad}D(g(t)D)\right). \end{aligned} \tag{7}$$

Similarly,

$$\beta_2\left(\mathrm{ad}(tD)(f(t)D), g(t)D\right) = -\beta_2\left(f(t)D, \mathrm{ad}(tD)(g(t)D)\right), \tag{8}$$

$$\beta_2\left(\mathrm{ad}(t^3D)(f(t)D), g(t)D\right) = -\beta_2\left(f(t)D, \mathrm{ad}(t^3D)(g(t)D)\right). \tag{9}$$

Now we want to use formulas (7), (8) and (9) to construct two linear endomorphisms E and F on $\mathcal{A}$ so that we can use Lemma 2. Set

$$\begin{aligned} E &= (\mathrm{ad}D)^2\mathrm{ad}(t^3D) - (\mathrm{ad}tD)^3 - 3(\mathrm{ad}tD)^2 + 4\mathrm{ad}tD, \\ F &= -\mathrm{ad}(t^3D)(\mathrm{ad}D)^2 + (\mathrm{ad}tD)^3 - 3(\mathrm{ad}tD)^2 - 4\mathrm{ad}tD. \end{aligned}$$

Then, using (7), (8) and (9), we have

$$\beta_2(E(f(t)D), g(t)D) = \beta_2(f(t)D, F(g(t)D)). \tag{10}$$

For $f(t)=\sum_i a_i t^{i+1}, g(t)=\sum_j b_j t^{j+1}\in\mathbb{C}((t))$, we have

$$\begin{aligned}
&(\mathrm{ad}D)^2\mathrm{ad}(t^3D)(f(t)D)\\
&=\left[D,\left[D,\left[t^3D,f(t)D\right]\right]\right]\\
&=\left[D,\left[D,\sum_i(i-2)a_it^{i+3}D\right]\right]\\
&=\left[D,\sum_i(i-2)(i+3)a_it^{i+2}D\right]\\
&=\sum_i(i-2)(i+3)(i+2)a_it^{i+1}D\\
&=\sum_i(i^3+3i^2-4i-12)a_it^{i+1}D.
\end{aligned}$$

Also

$$(\mathrm{ad}tD)^k(f(t)D)=\sum_i i^k a_i t^{i+1}D$$

for all $k\in\mathbb{N}$. This implies that $E(f(t)D)=-12f(t)D$ for all $f(t)\in\mathbb{C}((t))$. Thus E is invertible. Similarly, for $g(t)=\sum_j b_j t^{j+1}\in\mathbb{C}((t))$, we have

$$-\mathrm{ad}(t^3D)(\mathrm{ad}\,D)^2(g(t)D)=\sum_j(-j^3+3j^2+4j)b_jt^{j+1}D.$$

Thus $F(g(t)D)=0$ for all $g(t)\in\mathbb{C}((t))$. From Lemma 2, we have $\beta_2=0$ or $\beta_1=r\alpha$. □

Theorem 2. *$dimH^2(\mathcal{B},\mathbb{C})=1$.*

Proof. For any 2-cocycle ψ on $\mathcal{B}$, define

$$f_\psi\left(g(t)D^l\right)=-\frac{1}{l+1}\psi\left(t,g(t)D^{l+1}\right)$$

for $g(t)\in\mathbb{C}((t))$. Then $\psi_1=\psi-\alpha_{f_\psi}$ is a 2-cocycle and equivalent to ψ. We have

$$\psi_1\left(t,g(t)D^{l+1}\right)=0$$

for all $g(t)\in\mathbb{C}((t))$ and $l\in\mathbb{N}$.

Lemma 6. *If $g(t)\in\mathbb{C}((t))$ and Res $g(t)=0$, then $\psi_1\left(t,g(t)\right)=0$.*

Proof of Lemma 6. For $f(t)\in\mathbb{C}((t))$ and *Res* $f(t)=0$, we have

$$\begin{aligned}
&\psi_1\left(t,f(t)\right)\\
&=\psi_1\left([tD,t],f(t)\right)\\
&=\psi_1\left(tD,[t,f(t)]\right)+\psi_1\left([tD,f(t)],t\right)\\
&=-\psi_1\left(t,tf'(t)\right).
\end{aligned}$$

Therefore, $\psi_1(t,f(t)+tf'(t))=0$. Note that every element $g(t)$ in $\mathbb{C}((t))$ with *Res* $g(t)=0$ can be written in the form $f(t)+tf'(t)$ for some $f(t)\in\mathbb{C}((t))$ with *Res* $f(t)=0$. □

Define $\varphi : \mathcal{B} \times \mathcal{B} \to \mathbb{C}$,

$$\varphi\left(\sum_m a_m t^{l+m} D^l, \sum_n b_n t^{k+n} D^k\right) = \sum_m a_m b_{-m} (-1)^l l! k! \binom{m+l}{l+k+1}.$$

Then φ is a 2-cocycle on $\mathcal{B}$. Let S be the subset of $\mathcal{B}$ given by

$$S = \left\{ f(t)D^l \middle| l \in \mathbb{N}, f(t) \in \mathbb{C}((t)) \right\}.$$

For any $l \in \mathbb{N}$ and $f(t) \in \mathbb{C}((t))$,

$$f(t)D^l = -\frac{1}{l+1}\left[t, f(t)D^{l+1}\right]$$

and

$$\varphi\left(t, f(t)D^{l+1}\right) = 0.$$

From Lemma 1 and the fact that $\varphi(t, t^{-1}) = 1$, we have that φ is nontrivial. If $\psi_1(t, t^{-1}) = s$, we define $\psi_2 = \psi_1 - s\varphi$. Then using Lemma 6, we have $\psi_2(t, \mathcal{B}) = 0$. Note that

$$\begin{aligned} &\psi_2\left(\mathrm{ad}t(f(t)D^l), g(t)D^k\right) \\ &= \psi_2\left(\left[t, f(t)D^l\right], g(t)D^k\right) \\ &= \psi_2\left(\left[t, g(t)D^k\right], f(t)D^l\right) + \psi_2\left(t, \left[f(t)D^l, g(t)D^k\right]\right) \\ &= -\psi_2\left(f(t)D^l, \mathrm{ad}t(g(t)D^k)\right). \end{aligned} \tag{11}$$

For $f(t) = \sum_m a_m t^{l+m} \in \mathbb{C}((t))$,

$$[t, f(t)D^l] = -l f(t) D^{l-1}.$$

Therefore the operator ad$t$ is surjective and locally nilpotent. Let $E = \mathrm{ad}\, t$ and $F = -\mathrm{ad}\, t$. From equation (11) and Lemma 2, we have $\psi_2 = 0$. This gives $\psi_1 = s\varphi$. □

Remark. This method gives a simplified proof of Theorem 2.1 of [L].

Consider the Lie algebra $\mathcal{C} = \mathcal{B} \otimes gl_n(\mathbb{C})$. Define a bilinear map $\phi : \mathcal{C} \times \mathcal{C} \to \mathbb{C}$ by

$$\begin{aligned} &\phi\left(\sum_m a_m t^{l+m} D^l \otimes A, \sum_n b_n t^{k+n} D^k \otimes B\right) \\ = \quad &\sum_m a_m b_{-m} (-1)^l l! k! \binom{m+l}{l+k+1} tr(AB) \end{aligned}$$

for $\sum_m a_m t^{l+m}, \sum_n b_n t^{k+n} \in \mathbb{C}((t))$, $l, k \in \mathbb{N}$ and $A, B \in gl_n(\mathbb{C})$. Then ϕ is a 2-cocycle on $\mathcal{C}$. Using a method similar to the proof of Theorem 2.2 of [L], we have

Corollary. $dim H^2(\mathcal{C}, \mathbb{C}) = 1$.

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