CENTRAL EXTENSIONS OF SOME LIE ALGEBRAS

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Abstract. We consider three Lie algebras: $\text{Der} \mathbb{C}((t))$, the Lie algebra of all derivations on the algebra $\mathbb{C}((t))$ of formal Laurent series; the Lie algebra of all differential operators on $\mathbb{C}((t))$; and the Lie algebra of all differential operators on $\mathbb{C}((t)) \otimes \mathbb{C}^n$. We prove that each of these Lie algebras has an essentially unique nontrivial central extension.

The Lie algebra of all derivations on the Laurent polynomial algebra $\mathbb{C}[t,t^{-1}]$ can also be characterized as the Lie algebra of vector fields on the circle. The analogous object over a field $F$ of characteristic $p > 0$, $\text{Der} F[t]/(t^p)$, is called the Witt algebra $\mathfrak{w}$, and this name is sometimes applied to $\text{Der} \mathbb{C}[t,t^{-1}]$ as well. It is known [Bl] that $\text{Der} F[t]/(t^p)$ has an essentially unique nontrivial one-dimensional central extension, and also [GF] that $\text{Der} \mathbb{C}[t,t^{-1}]$ has an essentially unique nontrivial one-dimensional central extension. The proofs of these facts are similar. The nontrivial one-dimensional central extension of $\text{Der} \mathbb{C}[t,t^{-1}]$ is called the Virasoro algebra. It is one of the fundamental objects in representation theory as well as in theoretical physics.

For a positive integer $n$, the Lie algebra of all differential operators on $\mathbb{C}[t,t^{-1}] \otimes \mathbb{C}^n$ has a nontrivial one-dimensional central extension, and the extended Lie algebra is related to the representation theory of affine Lie algebras [KP]. It is proved in [L] that this extension is essentially unique (also see [F]). When $n = 1$, the Lie algebra of all differential operators on the Laurent polynomial ring $\mathbb{C}[t,t^{-1}]$ can also be characterized as the Lie algebra of differential operators on the circle; the corresponding extension is referred to, particularly in the physics literature, as $\mathcal{W}_{1+\infty}$. Some representations of $\mathcal{W}_{1+\infty}$ have been studied recently (see, e.g., [KR], [FKRW]). In [FKRW], it is shown that some representations of $\mathcal{W}_{1+\infty}$ have natural structures of vertex operator algebras (see, e.g., [Bo] and [FLM] for definitions).

Each of these constructions involves the Laurent polynomial algebra $\mathbb{C}[t,t^{-1}]$. This algebra is, of course, contained in $\mathbb{C}((t))$, the algebra of formal Laurent series. In this paper, we consider the Lie algebras obtained by replacing $\mathbb{C}[t,t^{-1}]$ by $\mathbb{C}((t))$ in each of these constructions. We show that each of the resulting Lie algebras has an essentially unique nontrivial one-dimensional central extension.

We work over the field of complex numbers, though all results hold over any field of characteristic zero.

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1. Some Basic Definitions

Let \( L \) and \( \hat{L} \) be two Lie algebras over \( \mathbb{C} \). The Lie algebra \( \hat{L} \) is said to be a one-dimensional central extension of \( L \) if there is a Lie algebra exact sequence
\[
0 \rightarrow \mathbb{C}c \rightarrow \hat{L} \rightarrow L \rightarrow 0,
\]
where \( \mathbb{C}c \) is the one-dimensional trivial Lie algebra and the image of \( \mathbb{C}c \) is contained in the center of \( \hat{L} \). It is well-known that \( \hat{L} \) is a one-dimensional central extension of \( L \) if and only if \( \hat{L} \) is the direct sum of \( L \) and \( \mathbb{C}c \) as vector spaces and the Lie bracket \([ \cdot , \cdot ]_1\) in \( \hat{L} \) is given by
\[
[x, y]_1 = [x, y] + \varphi(x, y)c,
\]
\[
[x, c]_1 = 0
\]
for all \( x, y \in L \), where \([ \cdot , \cdot ]\) is the Lie bracket in \( L \) and \( \varphi : L \times L \rightarrow \mathbb{C} \) is a bilinear form on \( L \) satisfying the following conditions:
\[
(1) \quad \varphi(x, y) = -\varphi(y, x),
\]
\[
(2) \quad \varphi([x, y], z) + \varphi([y, z], x) + \varphi([z, x], y) = 0
\]
for all \( x, y, z \in L \). The bilinear form \( \varphi \) is called a 2-cocycle on \( L \). A central extension is trivial if \( \hat{L} \) is the direct sum of a subalgebra \( M \) and \( \mathbb{C}c \) as Lie algebras, where the subalgebra \( M \) is isomorphic to \( L \). A 2-cocycle \( \varphi \) corresponding to a trivial central extension is called a 2-coboundary, or a trivial 2-cocycle, and is given by a linear function \( f \) from \( L \) to \( \mathbb{C} \):
\[
\varphi(x, y) = f([x, y])
\]
for all \( x, y \in L \). The 2-coboundary defined by \( f \) is denoted by \( \alpha_f \). The set of all 2-cocycles on \( L \) is a vector space, denoted by \( Z^2(L, \mathbb{C}) \). The set of all 2-coboundaries is a subspace of \( Z^2(L, \mathbb{C}) \), denoted by \( B^2(L, \mathbb{C}) \). The quotient space \( Z^2(L, \mathbb{C})/B^2(L, \mathbb{C}) \) is called the 2nd cohomology group of \( L \) with coefficients in \( \mathbb{C} \), and denoted by \( H^2(L, \mathbb{C}) \). If \( \dim H^2(L, \mathbb{C}) = 1 \), we say that \( L \) has an essentially unique nontrivial one-dimensional central extension. We say that 2-cocycles \( \varphi, \psi \) are equivalent if \( \varphi - \psi \) is a 2-coboundary.

The following two lemmas will be used in the proofs of our main results.

**Lemma 1.** Let \( L \) be a Lie algebra and \( S \) a subset of \( L \) such that \( S \) spans \( L \) and for each \( x \in S \), \( x = [y_e, z_e] \) for some \( y_e, z_e \in L \). If a 2-cocycle \( \varphi \) satisfies \( \varphi(y_e, z_e) = 0 \) for all \( x \in S \), then either \( \varphi = 0 \) or \( \varphi \) is nontrivial.

**Proof.** Suppose that \( \varphi \) is trivial, so that \( \varphi = \alpha_f \) for some linear function \( f \). Then for each \( x \in S \),
\[
f(x) = f([y_e, z_e]) = \varphi(y_e, z_e) = 0.
\]
Thus \( f = 0 \) since \( S \) spans \( L \). This implies that \( \varphi = \alpha_f = 0 \). \( \square \)

**Lemma 2.** Let \( L \) be a Lie algebra and \( \varphi \) a 2-cocycle on \( L \). Suppose there are linear endomorphisms \( E \) and \( F \) of \( L \) such that
\[
\varphi(Ex, y) = \varphi(x, Fy)
\]
for all \( x, y \in L \), \( E \) is surjective and \( F \) is locally nilpotent (i.e., for any \( y \in L \), there is a positive integer \( n \) such that \( F^n y = 0 \)). Then the 2-cocycle \( \varphi \) is 0.

**Proof.** For \( x, y \in L \), let \( n \) be a positive integer such that \( F^n y = 0 \). Since \( E \) is surjective, we have \( x' \in L \) such that \( x = E^n x' \). Thus,
\[
\varphi(x, y) = \varphi(E^n x', y) = \varphi(x', F^n y) = 0.
\]
\( \square \)
Let $A$ be a (not necessarily associative) algebra. A linear map $\delta : A \to A$ is called a derivation, if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$.

Now consider the algebra of all formal Laurent series

$$\mathbb{C}((t)) = \left\{ \sum_{i \in \mathbb{Z}, i \geq n} a_i t^i \mid a_i \in \mathbb{C}, n \in \mathbb{Z} \right\}.$$  

It is known that the set of all derivations on $\mathbb{C}((t))$ is

$$\mathcal{A} = \left\{ f(t) \frac{d}{dt} \mid f(t) \in \mathbb{C}((t)) \right\},$$

where $\frac{d}{dt}$ is the formal derivation defined by $\frac{d}{dt} : \mathbb{C}((t)) \to \mathbb{C}((t))$, $\sum a_i t^i \mapsto \sum ia_it^{i-1}$. For convenience, we denote $\frac{d}{dt}$ by $D$ and denote $\frac{d}{dt}f(t)$ by $f'(t)$. For $f(t) = \sum a_i t^i \in \mathbb{C}((t))$, define $\text{Res } f(t) = a_{-1}$. If $\text{Res } f(t) = 0$, we can define the formal integral of $f(t)$ as

$$\sum_{i \neq -1} \frac{a_i}{i+1} t^{i+1},$$

denoted by $\int f(t)$. Then $\mathcal{A}$ is a Lie algebra under the bracket operation:

$$[f(t)D, g(t)D] = (f(t)D) \circ (g(t)D) - (g(t)D) \circ (f(t)D) = f(t)g'(t)D - g(t)f'(t)D$$

for $f(t), g(t) \in \mathbb{C}((t))$, where the $\circ$ is the composition of operators.

More generally, consider the space of all differential operators on the algebra of formal Laurent series $\mathbb{C}((t))$:

$$\mathcal{B} = \text{span} \left\{ f(t)D^l \mid l \in \mathbb{N}, f(t) \in \mathbb{C}((t)) \right\},$$

$\mathcal{B}$ is a Lie algebra under the bracket operation:

$$[f(t)D^l, g(t)D^k] = (f(t)D^l) \circ (g(t)D^k) - (g(t)D^k) \circ (f(t)D^l)$$

$$= \sum_{i=0}^l \binom{l}{i} f(t) (D^{l-i}g(t)) D^{k+i} - \sum_{j=0}^k \binom{k}{j} g(t) (D^{k-j}f(t)) D^{l+j}.$$   

Furthermore, we may consider $\mathcal{C} = \mathcal{B} \otimes \mathfrak{gl}_n(\mathbb{C})$, the space of differential operators on $\mathbb{C}((t)) \otimes \mathbb{C}^n$. Note that $\mathcal{C} \subset \text{End } (\mathbb{C}((t)) \otimes \mathbb{C}^n)$. Define the bracket operation on $\mathcal{C}$ by linearity and the commutator

$$[f(t)D^l \otimes A, g(t)D^k \otimes B] = (f(t)D^l \otimes A) \circ (g(t)D^k \otimes B) - (g(t)D^k \otimes B) \circ (f(t)D^l \otimes A)$$

$$= \sum_{i=0}^l \binom{l}{i} f(t) (D^{l-i}g(t)) D^{k+i} \otimes AB - \sum_{j=0}^k \binom{k}{j} g(t) (D^{k-j}f(t)) D^{l+j} \otimes BA.$$ 

Hence $\mathcal{C}$ is a Lie algebra.
2. Main results and proofs

In this section, we will give our main results and their proofs. In each case we exhibit (using Lemma 1) a nontrivial 2-cocycle on the Lie algebra under consideration. The 2-cocycle is analogous to the standard nontrivial 2-cocycle on the Lie algebra obtained from Laurent polynomial algebra. Then for any given 2-cocycle on the Lie algebra, we reduce the 2-cocycle to a 2-cocycle which is equivalent to the original one and takes value 0 whenever the standard 2-cocycle takes value 0. We use Lemma 2 to show that the reduced 2-cocycle is a multiple of the standard one.

**Theorem 1.** \( \dim H^2(A, \mathbb{C}) = 1. \)

**Proof.** Let \( \beta \) be a 2-cocycle on \( A \). Define a linear function \( f_\beta : A \to \mathbb{C} \) by

\[
f_\beta(g(t)D) = \beta(D, \int g(t)D) \quad \text{for} \quad g(t) \in \mathbb{C}(\!(t)\!), \; \text{Res} \; g(t) = 0
\]

and

\[
f_\beta(t^{-1}D) = \frac{1}{2} \beta(t^{-1}D, tD).
\]

Then \( \beta_1 = \beta - \alpha f_\beta \) is a 2-cocycle on \( A \) which is equivalent to \( \beta \).

For \( f(t) = \sum_{i \neq 0} a_i t^i \in \mathbb{C}(\!(t)\!) \),

\[
\beta_1(D, f(t)D) = \beta(D, f(t)D) - f_\beta([D, f(t)D]) \quad (3)
\]

and

\[
\beta_1(t^{-1}D, tD) = \beta(t^{-1}D, tD) - f_\beta([t^{-1}D, tD]) \quad (4)
\]

**Lemma 3.** \( \beta_1(D, A) = 0 \) and \( \beta_1(tD, A) = 0 \).

**Proof of Lemma 3.** From (3) and \( \beta_1(D, D) = -\beta_1(D, D) \), we have \( \beta_1(D, A) = 0 \).

For \( f(t) \in \mathbb{C}(\!(t)\!) \) and \( \text{Res} \; f(t) = 0 \),

\[
\beta_1(tD, f(t)D) = \beta_1(tD, \left[ D, \int f(t)D \right])
\]

and

\[
\beta_1(t^{-1}D, tD) = \beta(t^{-1}D, tD) - f_\beta(2t^{-1}D) = 0.
\]

\( \beta_1(tD, A) = 0 \) follows from this and (4).

**Lemma 4.** \( \beta_1(t^2D, A) = 0 \).
Proof of Lemma 4. For \( f(t) \in \mathbb{C}(t) \) and \( \text{Res } f(t) = 0 \), we have
\[
\beta_1(t^2 D, f(t)D) = \beta_1 \left( t^2 D, \left[ D, \int f(t)D \right] \right) = \beta_1 \left( [t^2 D, D], \int f(t)D \right) + \beta_1 \left( D, \left[ t^2 D, \int f(t)D \right] \right) = 0.
\]
Also
\[
\beta_1 (t^2 D, t^{-1}D) = \beta_1 \left( t^2 D, -\frac{1}{2} [tD, t^{-1}D] \right) = -\frac{1}{2} \beta_1 ([t^2 D, tD], t^{-1}D) - \frac{1}{2} \beta_1 (tD, [t^2 D, t^{-1}D]) = -\frac{1}{2} \beta_1 (-t^2 D, t^{-1}D) = \frac{1}{2} \beta_1 (t^2 D, t^{-1}D).
\]
This implies that \( \beta_1 (t^2 D, t^{-1}D) = 0 \). \( \square \)

Lemma 5. If \( f(t) \in \mathbb{C}(t) \) and \( \text{Res } f(t) = 0 \), then \( \beta_1(t^3 D, f(t)D) = 0 \).

Proof of Lemma 5. We have
\[
\beta_1 (t^3 D, f(t)D) = \beta_1 \left( t^3 D, \left[ D, \int f(t)D \right] \right) = \beta_1 \left( [t^3 D, D], \int f(t)D \right) + \beta_1 \left( D, \left[ t^3 D, \int f(t)D \right] \right) = \beta_1 \left( -3t^2 D, \int f(t)D \right) = 0. \quad \square
\]

Define \( \alpha : \mathcal{A} \times \mathcal{A} \to \mathbb{C} \) by
\[
\alpha \left( \sum_i a_i t^{i+1} D, \sum_j b_j t^{j+1} D \right) = \sum_i a_i b_{-i} (i^3 - i)
\]
for \( \sum_i a_i t^{i+1} D, \sum_j b_j t^{j+1} D \in \mathcal{A} \). Note that the sum on the right-hand side is finite and \( \alpha \) is a 2-cocycle on \( \mathcal{A} \). Let \( S \) be the subset of \( \mathcal{A} \) given by
\[
S = \{ t^{-1} D \} \cup \left\{ f(t)D \in \mathcal{A} \left| \text{Res } f(t) = 0 \right. \right\}.
\]
Now for \( f(t) \in \mathbb{C}(t) \), \( \text{Res } f(t) = 0 \), we have
\[
t^{-1} D = \left[ \frac{1}{2} t^{-1} D, tD \right], \quad f(t)D = \left[ D, \int f(t)D \right], \quad \alpha (t^{-1} D, tD) = 0,
\]
and

\[ \alpha \left( D, \int f(t)D \right) = 0. \]

Since \( \alpha \) is nonzero (in fact, \( \alpha(t^3D, t^{-1}D) = 6 \)), Lemma 1 shows that \( \alpha \) is nontrivial. Also \( \alpha = \alpha_1 \). Applying Lemma 3, Lemma 4 and Lemma 5 to \( \alpha \), we have

\[ \alpha(D, A) = \alpha(tD, A) = \alpha(t^2D, A) = 0, \]

and

\[ \alpha \left( t^3D, f(t)D \right) = 0 \text{ for } f(t) \in \mathbb{C}((t)), \text{ Res } f(t) = 0. \]

Suppose that \( \beta_1(t^3D, t^{-1}D) = 6r \) for some \( r \in \mathbb{C} \). Define \( \beta_2 = \beta_1 - r\alpha \); then we have

(5) \[ \beta_2(D, A) = \beta_2(tD, A) = \beta_2(t^2D, A) = 0 \]

and

(6) \[ \beta_2(t^3D, A) = 0. \]

We now show that \( \beta_2 = 0 \), completing the proof of Theorem 1.

Let \( \text{ad} : A \to A \), \( \text{ad}(a)b = [a, b] \), be the adjoint operator; then

(7) \[ \beta_2 \left( \text{ad}(D)(f(t)D), g(t)D \right) = \beta_2 \left( [D, f(t)D], g(t)D \right) = \beta_2 \left( [D, g(t)D], f(t)D \right) + \beta_2 \left( D, [f(t)D, g(t)D] \right) = -\beta_2 \left( f(t)D, \text{ad}(D)(g(t)D) \right). \]

Similarly,

(8) \[ \beta_2 \left( \text{ad}(tD)(f(t)D), g(t)D \right) = -\beta_2 \left( f(t)D, \text{ad}(tD)(g(t)D) \right), \]

(9) \[ \beta_2 \left( \text{ad}(t^3D)(f(t)D), g(t)D \right) = -\beta_2 \left( f(t)D, \text{ad}(t^3D)(g(t)D) \right). \]

Now we want to use formulas (7), (8) and (9) to construct two linear endomorphisms \( E \) and \( F \) on \( A \) so that we can use Lemma 2. Set

\[ E = (\text{ad}D)^2 \text{ad}(t^3D) - (\text{ad}tD)^3 - 3(\text{ad}D)^2 + 4\text{ad}tD, \]
\[ F = -\text{ad}(t^3D)(\text{ad}D)^2 + (\text{ad}tD)^3 - 3(\text{ad}D)^2 - 4\text{ad}tD. \]

Then, using (7), (8) and (9), we have

(10) \[ \beta_2(E(f(t)D), g(t)D) = \beta_2(f(t)D, F(g(t)D)). \]
For $f(t) = \sum_i a_it^{i+1}$, $g(t) = \sum_j b_jt^{j+1} \in \mathbb{C}(t)$, we have

$$(\text{ad}D)^2(\text{ad}(t^3D))(f(t)D)$$

$$= [D, [D, [t^3D, f(t)D]]]$$

$$= [D, \sum_i (i-2)a_it^{i+3}D]$$

$$= \sum_i (i-2)(i+3)a_it^{i+2}D$$

$$= \sum_i (i-2)(i+3)(i+2)a_it^{i+1}D$$

$$= \sum_i (i^3 + 3i^2 - 4i - 12)a_it^{i+1}D.$$

Also

$$(\text{ad}tD)^k(f(t)D) = \sum_i i^k a_it^{i+1}D$$

for all $k \in \mathbb{N}$. This implies that $E(f(t)D) = -12f(t)D$ for all $f(t) \in \mathbb{C}(t))$. Thus $E$ is invertible. Similarly, for $g(t) = \sum_j b_jt^{j+1} \in \mathbb{C}(t)$, we have

$$-\text{ad}(t^3D)(\text{ad}D)^2(g(t)D) = \sum_j (-j^3 + 3j^2 + 4j)b_jt^{j+1}D.$$

Thus $F(g(t)D) = 0$ for all $g(t) \in \mathbb{C}(t)$. From Lemma 2, we have $\beta_2 = 0$ or $\beta_1 = r\alpha$.

**Theorem 2.** $\dim H^2(B, \mathbb{C}) = 1$.

**Proof.** For any 2-cocycle $\psi$ on $B$, define

$$f_\psi (g(t)D^l) = -\frac{1}{l+1} \psi \left( t, g(t)D^{l+1} \right)$$

for $g(t) \in \mathbb{C}(t))$. Then $\psi_1 = \psi - \alpha f_\psi$ is a 2-cocycle and equivalent to $\psi$. We have

$$\psi_1 \left( t, g(t)D^{l+1} \right) = 0$$

for all $g(t) \in \mathbb{C}(t)$ and $l \in \mathbb{N}$.

**Lemma 6.** If $g(t) \in \mathbb{C}(t)$ and $\text{Res} \ g(t) = 0$, then $\psi_1 \left( t, g(t) \right) = 0$.

**Proof of Lemma 6.** For $f(t) \in \mathbb{C}(t))$ and $\text{Res} \ f(t) = 0$, we have

$$\psi_1 \left( t, f(t) \right)$$

$$= \psi_1 \left( [tD, t], f(t) \right)$$

$$= \psi_1 \left( [tD, [t, f(t)]], t \right)$$

$$= -\psi_1 \left( t, tf'(t) \right).$$

Therefore, $\psi_1(t, f(t) + tf'(t)) = 0$. Note that every element $g(t) \in \mathbb{C}(t))$ with $\text{Res} \ g(t) = 0$ can be written in the form $f(t) + tf'(t)$ for some $f(t) \in \mathbb{C}(t))$ with $\text{Res} \ f(t) = 0$. \qed
Define \( \varphi : \mathcal{B} \times \mathcal{B} \to \mathbb{C} \),
\[
\varphi \left( \sum_m a_m t^{l+m} D^l, \sum_n b_n t^{k+n} D^k \right) = \sum_m a_m b_{-m} (-1)^l l! k! \left( \frac{m+l}{l+k+1} \right).
\]
Then \( \varphi \) is a 2-cocycle on \( \mathcal{B} \). Let \( S \) be the subset of \( \mathcal{B} \) given by
\[
S = \left\{ f(t) D^l \mid l \in \mathbb{N}, f(t) \in \mathbb{C}((t)) \right\}.
\]
For any \( l \in \mathbb{N} \) and \( f(t) \in \mathbb{C}((t)) \),
\[
f(t) D^l = -\frac{1}{l+1} \left[ t, f(t) D^{l+1} \right]
\]
and
\[
\varphi \left( t, f(t) D^{l+1} \right) = 0.
\]
From Lemma 1 and the fact that \( \varphi(t, t^{-1}) = 1 \), we have that \( \varphi \) is nontrivial. If \( \psi_1(t, t^{-1}) = s \), we define \( \psi_2 = \psi_1 - s \varphi \). Then using Lemma 6, we have \( \psi_2(t, B) = 0 \).

Note that
\[
\psi_2 \left( \text{ad}t(f(t) D^l), g(t) D^k \right) = \psi_2 \left( [t, f(t) D^l], g(t) D^k \right) = \psi_2 \left( [t, g(t) D^k], f(t) D^l \right) + \psi_2 \left( f(t) D^l, [g(t) D^k] \right)
\]
(11)

For \( f(t) = \sum_m a_m t^{l+m} \in \mathbb{C}((t)) \),
\[
[t, f(t) D^l] = -t f(t) D^{l-1}.
\]
Therefore the operator \( \text{ad}t \) is surjective and locally nilpotent. Let \( E = \text{ad}t \) and \( F = -\text{ad}t \). From equation (11) and Lemma 2, we have \( \psi_2 = 0 \). This gives \( \psi_1 = s \varphi \).

**Remark.** This method gives a simplified proof of Theorem 2.1 of [L].

Consider the Lie algebra \( \mathcal{C} = \mathcal{B} \otimes \text{gl}_n(\mathbb{C}) \). Define a bilinear map \( \phi : \mathcal{C} \times \mathcal{C} \to \mathbb{C} \) by
\[
\phi \left( \sum_m a_m t^{l+m} D^l \otimes A, \sum_n b_n t^{k+n} D^k \otimes B \right) = \sum_m a_m b_{-m} (-1)^l l! k! \left( \frac{m+l}{l+k+1} \right) \text{tr}(AB)
\]
for \( \sum_m a_m t^{l+m}, \sum_n b_n t^{k+n} \in \mathbb{C}((t)), l, k \in \mathbb{N} \) and \( A, B \in \text{gl}_n(\mathbb{C}) \). Then \( \phi \) is a 2-cocycle on \( \mathcal{C} \). Using a method similar to the proof of Theorem 2.2 of [L], we have

**Corollary.** \( \text{dim} H^2(\mathcal{C}, \mathcal{C}) = 1 \).

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