A STRUCTURAL RESULT OF IRREDUCIBLE INCLUSIONS OF TYPE III$_{\lambda}$ FACTORS, $\lambda \in (0, 1)$

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Abstract. Given an irreducible inclusion of factors with finite index $N \subset M$, where $M$ is of type III$_{\lambda}$$_{1/m}$, $N$ of type III$_{\lambda}$$_{1/n}$, $0 < \lambda < 1$, and $m, n$ are relatively prime positive integers, we will prove that if $N \subset M$ satisfies a commuting square condition, then its structure can be characterized by using fixed point algebras and crossed products of automorphisms acting on the middle inclusion of factors associated with $N \subset M$. Relations between $N \subset M$ and a certain $G$-kernel on subfactors are also discussed.

1. Introduction

Let $N \overset{E_M}{\subset} M$ be an irreducible inclusion of type III factors such that $M$ is of type III$_{\lambda}$$_{1/m}$ and $N$ of type III$_{\lambda}$$_{1/n}$, where $\lambda \in (0, 1), m, n$ are positive relatively prime integers, and $E_M : M \to N$ is a normal faithful conditional expectation with finite index. We are interested in studying the structure of such an inclusion and its relevance to the classification problem. By the results in [17], such an inclusion can be decomposed into separate sub-inclusions, each of which admits a simple description using automorphisms; more specifically, there exist type III$_{\lambda}$ subfactors $P$ and $Q$ with $N \overset{E_Q}{\subset} Q \overset{E_P}{\subset} P \overset{E_M}{\subset} M$ and such that $\text{Ind}(E_P^M) = m$, $\text{Ind}(E_Q^N) = n$. Moreover, $P$ is the fixed point algebra of the restriction of a modular automorphism of order $m$ on $M$, whereas $Q$ is the crossed product of $N$ with a modular automorphism of order $n$ on $N$. As for $Q \subset P$, there exists a joint discrete decomposition in the sense that there exist type II$_{\infty}$ factors $Q^\infty \subset P^\infty$ and an automorphism $\theta$ which acts simultaneously on $Q^\infty \subset P^\infty$ with $\text{mod}(\theta) = \lambda$ and $Q \subset P$ is isomorphic to $Q^\infty \times_\theta \mathbb{Z} \subset P^\infty \times_\theta \mathbb{Z}$. The classification of these sub-inclusions is now well understood. For instance, the top and bottom inclusions $P \subset M$ and $N \subset Q$ are each uniquely determined (cf. [17]) and the middle inclusion $Q \subset P$ is classified by the type II core $Q^\infty \subset P^\infty$, the module of $\theta$ and the standard invariant of $\theta$ on the tower of higher relative commutants of $Q^\infty \subset P^\infty$ by [23].

Despite the success in classifying these separate sub-inclusions, it remains an open problem to classify the original inclusion $N \subset M$. In this paper, we continue to investigate the feasibility of characterizing the structure of a general inclusion of factors such as $N \subset M$ in terms of automorphisms. First we observe that using Takesaki duality, we can find properly outer, periodic automorphisms $\alpha$ and $\beta$
with non-trivial modules, acting on $P$ and $Q$ respectively, so that $M$ is the crossed product $P \times_{\alpha} \mathbb{Z}_m$ and $N$ is the fixed point algebra $Q^\mathbb{Z}_n$ via $\beta$. However, it is not possible, in general, to have $\alpha$ and $\beta$ act on the inclusion $Q \subset P$. In fact we will prove that this is the case if and only if the inclusions $N \subset Q \subset P \subset M$ satisfy certain commuting square conditions which are equivalent to some restriction and extension conditions of the Longo canonical endomorphism of $Q \subset P$ and this property is also equivalent to the existence of two trace-scaling automorphisms $\theta_1$ and $\theta_2$ on $Q^\infty \subset P^\infty$ such that: $\theta = \theta_1^m = \theta_2^n$, $M = P^\infty \times_{\theta_1} \mathbb{Z}$ and $N_1 = Q^\infty \times_{\theta_2} \mathbb{Z}$, where $N_1$ is the basic construction of $N \subset Q$. We can then associate to $N \subset M$ a $G$-kernel on $Q \subset P$, i.e., a homomorphism of the group $G$ into $\text{Aut}(P, Q) / \text{Int}(Q)$, arising from the subgroup generated by $\alpha$ and $\beta$ modulo $\text{Int}(Q)$. We can then show that the isomorphism class of $N \subset M$ is determined by the conjugacy class of the corresponding $G$-kernel.

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2. Main results

We begin by recalling the definition of commuting squares and some basic results about them that we will need.

An inclusion of factors

$$
P \subset M \cup Q \subset N
$$

is said to be a commuting square with respect to the expectations of minimal indices $E^M_N, E^M_P, E^N_Q$ and $E^P_Q$ if $E^M_N|P = E^P_Q$ or equivalently, if $E^M_P|N = E^N_Q$ (cf. [8]).

The commuting square

$$
P \subset M \cup Q \subset N
$$

is called co-commuting (cf. [24]) or non-degenerate (cf. [22]) if

$$
N' \subset Q' \cup M' \subset P'
$$

is also a commuting square with respect to the expectations with minimal indices. We refer the reader to [13, 22, 24] for additional properties of commuting and co-commuting squares. We only mention the following result in [24, 13]: the commuting square

$$
P \subset M \cup Q \subset N
$$

is co-commuting if and only if $\text{Ind} E^M_N = \text{Ind} E^P_Q$ or $\text{Ind} E^M_P = \text{Ind} E^N_Q$. 

For convenience, unless otherwise stated, $E^A_B$ will denote the conditional expectation of minimal index from $A$ onto $B$ and $[A: B]_0$ the minimal index value.

**Lemma 1.** Let

$$
\begin{align*}
P & \subset M \\
U & \subset U \\
Q & \subset N
\end{align*}
$$

be a commuting square (with respect to the expectations with minimal indices). Let $S \subset R$ be intermediate subfactors such that $P \subset R \subset M, Q \subset S \subset N, E^M_R = E^R_P E^M_R$ and $E^N_Q = E^N_Q E^N_S$. Then

$$
\begin{align*}
P & \subset R \\
U & \subset U \\
Q & \subset S
\end{align*}
$$

is also a commuting square.

**Proof.** For any $s \in S, E^R_P(s) = E^M_P(s)$, which belongs to $Q$ because

$$
\begin{align*}
P & \subset M \\
U & \subset U \\
Q & \subset N
\end{align*}
$$

is a commuting square by assumption. And so by [8],

$$
\begin{align*}
P & \subset R \\
U & \subset U \\
Q & \subset S
\end{align*}
$$

is a commuting square. Q.E.D.

**Lemma 2.** Let

$$
\begin{align*}
P & \subset M \\
U & \subset U \\
Q & \subset N
\end{align*}
$$

be a commuting square of factors with respect to the expectations with minimal indices. Then $[M: N]_0 \geq [P: Q]_0$ and $[M: P]_0 \geq [N: Q]_0$.

**Proof.** By [21], for any $x \in P_+, E^M_N(x) \geq [M: N]^{-1}_0 x$. Since $E^P_P = E^M_N | P, E^P_Q(x) \geq [M: N]^{-1}_0 x$. Using [21] again, we have $[P: Q]^{-1}_0 \geq [M: N]^{-1}_0$ and so $[M: N]_0 \geq [P: Q]_0$. Similarly, $[M: P]_0 \geq [N: Q]_0$. Q.E.D.

In the following, all factors under consideration are of type III and thus without loss of generality, we may assume that they have a common cyclic and separating vector and are acting standardly on the same Hilbert space by the results of [5]. Let us also recall that for an inclusion of factors $N \subset M$ which act standardly on the same Hilbert space, the canonical endomorphism of $N \subset M$ is defined by $\gamma_{M, N} = \text{Ad} J_N J_M | M$. For additional properties on the canonical endomorphism in relation to the index theory of $N \subset M$, we refer the reader to [16].

**Proposition 1.** Let $N \subset Q \subset P$ be an irreducible inclusion of type III factors with finite index. The following are equivalent.
(i) There is a subfactor $N_0$ such that
\[ Q \subset P \cup N \subset N_0 \]
is a commuting and co-commuting square.

(ii) There is a choice of the canonical endomorphisms $\gamma_{Q,N}$ and $\gamma_{P,Q}$ such that $\gamma_{Q,N}(P) \subset P$, $\gamma_{P,Q}(N) \subset N$ and
\[ \gamma_{P,Q}(Q) \subset Q \]
\[ \cup \subset \cup \]
\[ \gamma_{P,Q}(N) \subset N \]
is a commuting and co-commuting square.

Proof. (i) ⇒ (ii) By [9], there is a choice of $\gamma_{Q,N}$ and $\gamma_{P,Q}$ of $Q \subset P$ such that $\gamma_{Q,N}(P) \subset P$ and $\gamma_{P,Q}(N) \subset N$. It follows that $\gamma_{P,Q}(Q)$ and $\gamma_{P,Q}(N)$ are the second factors in the downward basic constructions of $Q \subset P$ and $N \subset N_0$, respectively. Since
\[ Q \subset P \]
\[ N \subset N_0 \]
is a co-commuting and commuting square, it is easy to check that
\[ \gamma_{P,Q}(Q) \subset Q \]
\[ \cup \subset \cup \]
\[ \gamma_{P,Q}(N) \subset N \]
is a commuting square which is also co-commuting because $[Q:N]_0 = [\gamma_{P,Q}(Q):\gamma_{P,Q}(N)]_0$.

(ii) ⇒ (i) Let $J_P$ and $J_Q$ be the modular conjugate operators on their respective factor such that $\gamma_{P,Q} = \text{Ad}(J_QJ_P)$ and $\gamma_{Q,N} = \text{Ad}(J_NJ_Q)$. Since $\gamma_{P,N} = \gamma_{Q,N} \cdot \gamma_{P,Q}$, we have the following inclusions of factors:
\[ \gamma_{P,Q}(Q) \subset \gamma_{P,Q}(P) \subset Q \]
\[ \cup \subset \cup \]
\[ \gamma_{P,Q}(N) \subset N \]
\[ \gamma_{P,N}(Q) \subset \gamma_{P,N}(P) \subset \gamma_{Q,N}(Q) \]
\[ \cup \subset \cup \]
\[ \gamma_{P,N}(N) \]
Since
\[ \gamma_{P,Q}(Q) \subset Q \]
\[ \cup \subset \cup \]
\[ \gamma_{P,Q}(N) \subset N \]
is a commuting and co-commuting square by assumption, Proposition 2.3 in [9] implies that $\gamma_{Q,N}$ restricts to a canonical endomorphism of $\gamma_{P,Q}(N) \subset \gamma_{P,Q}(Q)$ and thus $\gamma_{P,Q}(N)$ is the basic construction of $\gamma_{P,N}(N) \subset \gamma_{P,N}(Q)$. Let $\tilde{N}$ be the von Neumann algebra generated by $\gamma_{P,Q}(N)$ and $\gamma_{P,N}(P)$. As $N' \cap P = C$, $\gamma_{P,N}(P)$ is irreducible in $N$ and hence $\tilde{N}$ is a factor. Also since $\gamma_{Q,N}(P) \subset P$, $\gamma_{P,N}(P) \subset \gamma_{P,Q}(P)$ and so $\tilde{N} \subset \gamma_{P,Q}(P)$ as well. By repeated applications of Takesaki’s
criterion in [26], we see that there exist conditional expectations on each of the following inclusions: $\gamma_{P,Q}(N) \subset \tilde{N}$, $\gamma_{P,N}(P) \subset \tilde{N}$, $N_0 \subset \gamma_{P,Q}(P)$ and $\tilde{N} \subset N$. Moreover, as these expectations all have finite indices, we may just assume that they have minimal indices.

From the assumption that $\gamma_{P,Q}(Q) \subset Q$ 
\[
\gamma_{P,Q}(N) \subset N
\]
is a commuting and co-commuting square, it follows that $\gamma_{P,Q}(N) \subset N$ 
\[
\gamma_{P,N}(Q) \subset \gamma_{Q,N}(Q)
\]
is also a commuting and co-commuting square. Hence by Lemma 1, 
\[
\gamma_{P,Q}(Q) \subset \gamma_{P,Q}(P) \quad \gamma_{P,Q}(N) \subset \tilde{N}
\]
\[
\gamma_{P,Q}(N) \subset \tilde{N} \quad \gamma_{P,N}(Q) \subset \gamma_{P,N}(P)
\]
are commuting squares. Thus by Lemma 2, 
$[P: Q]_0 = [\gamma_{P,Q}(P): \gamma_{P,Q}(Q)]_0 \geq [\tilde{N}: \gamma_{P,Q}(N)]_0 \geq [\gamma_{P,N}(P): \gamma_{P,N}(Q)]_0 = [P: Q]_0$ 
and so $[\tilde{N}: \gamma_{P,Q}(N)]_0 = [P: Q]_0$. Therefore 
\[
\gamma_{P,Q}(Q) \subset \gamma_{P,Q}(P) \\
\gamma_{P,Q}(N) \subset \tilde{N}
\]
is a commuting and co-commuting square by [24]. Now let $N_0 = \gamma_{P,Q}^{-1}(\tilde{N})$; then 
\[
Q \subset P \quad \gamma_{P,Q}(N) \subset N_0 
\]
is a commuting and co-commuting square. Q.E.D.

Let us also state and prove the following dual version of Proposition 1.

**Proposition 2.** Let $Q \subset P \subset M$ be an irreducible inclusion of type III factors with finite indices. The following are equivalent.

(i) There is a factor $M_0$ such that 
\[
\begin{array}{c}
P \subset M \\
\cup \\
Q \subset M_0
\end{array}
\]
is a commuting and co-commuting square.

(ii) There is a choice of the canonical endomorphisms $\gamma_{M,P}$ and $\gamma_{P,Q}$ such that $\gamma_{M,P}(Q) \subset Q$, $\gamma_{P,Q}(M) \subset M$ and 
\[
\begin{array}{c}
\gamma_{P,Q}(M) \subset M \\
\cup \\
\gamma_{P,Q}(P) \subset P
\end{array}
\]
is a commuting and co-commuting square.
Proof. (i) ⇒ (ii) By Proposition 2.3 of [9], there exist canonical endomorphisms $\gamma_{M,P}$ and $\gamma_{P,Q}$ that satisfy the stated extension and restriction conditions. Moreover, $\gamma_{P,Q}(P)$ and $\gamma_{P,Q}(M)$ are the basic constructions of $Q \subset P$ and $M_0 \subset M$, respectively, and hence

$$\gamma_{P,Q}(M) \subset M$$

or

$$\gamma_{P,Q}(P) \subset P$$

is a commuting and co-commuting square because

$$P \subset M_0$$

is commuting and co-commuting by assumption.

(ii) ⇒ (i) By taking the commutants of the factors, the assumptions in (ii) imply that the inclusion $M' \subset P' \subset Q'$ satisfies the hypotheses of (ii) of Proposition 1. Therefore there is a factor $M_0$ such that

$$P' \subset Q'$$

is commuting and co-commuting. Passing to the commutants of these factors will yield the desired commuting and co-commuting square. Q.E.D.

In order to describe the structure of an inclusion of type III factors that satisfies both the commuting and co-commuting square conditions of Propositions 1 and 2, we need the information about the Connes-Takesaki modules (cf. [4]) of the associated automorphisms provided by the next lemma.

Lemma 3. (i) Let $P$ be a factor of type $\text{III}_\lambda$, $\lambda \in (0,1)$, and $m \in \mathbb{N}$. Let $\varphi$ be a generalized trace on $P$ and $T = [2\pi/\ln \lambda]$. Then $P^{\sigma_{T/m}}_\varphi$ and $P \times_{\sigma_{T/m}_\varphi} Z_m$ are both of type $\text{III}_{\lambda m}$ and if $\alpha$ is either the dual or the pre-dual automorphism of $\sigma_{T/m}_\varphi$, then $\text{mod}(\alpha) \equiv \lambda$.

(ii) Let $\lambda \in (0,1)$ and $Q \overset{E}{\subset} P$ be an inclusion of type $\text{III}_\lambda$ factors with a common discrete decomposition. Then for any $\alpha \in \text{Aut}(P,Q)$ which commutes with $E$, $\text{mod}(\alpha|P) = \text{mod}(\alpha|Q)$.

Proof. (i) We will only prove the statement about the module of $\alpha$ as it is well known that $P^{\sigma_{T/m}_\varphi}$ and $P \times_{\sigma_{T/m}_\varphi} Z_m$ are both of type $\text{III}_{\lambda m}$. Suppose that $\alpha$ is the automorphism that is pre-dual to $\sigma_{T/m}_\varphi$ on $P^{\sigma_{T/m}_\varphi}$. Then $P = P^{\sigma_{T/m}_\varphi} \times_{\alpha} Z_m$ and $\sigma_{T/m}_\varphi(U) = e^{-2\pi i/m}U$, where $U$ is the unitary in $P$ that implements $\alpha$. On the other hand, it follows from [17] that the pair $P^{\sigma_{T/m}_\varphi} \subset P$ is isomorphic to $P^\infty \times_{\vartheta} Z \subset P^\infty \times_{\vartheta} Z$, where $\{P^\infty, \vartheta\}$ is a discrete decomposition of $P$. Let $V$ be the implementing unitary of $\vartheta$; then as $\varphi$ is a generalized trace on $P$ (cf. [3]), $\sigma_{T/m}(V) = \lambda^t V$, for all $t \in \mathbb{R}$. In particular, $\sigma_{T/m}(V) = \lambda^{T/m} V = e^{-2\pi i/m} V$. Hence $W = UV^* \in P^{\sigma_{T/m}_\varphi} \subset P$ and $\theta(x) = \text{Ad} W^* \cdot \alpha(x)$ for every $x \in P^\infty$. Since $\text{mod}(\theta) = \lambda$, we deduce that $\text{mod}(\alpha) \equiv \lambda$ (cf. [4]).

If $\alpha$ is the dual automorphism to $\sigma_{T/m}_\varphi$, then using the just established result for the predual of $\sigma_{T/m}_\varphi$ and the Takesaki Duality Theorem, we infer that $\text{mod}(\alpha) \equiv \lambda$. 

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(ii) Let \(\varphi\) be a normal faithful semi-finite weight on \(Q\). Then \(\varphi \cdot \alpha \sim \mu^{-1} \varphi\). Since \(\alpha\) and \(E^\varphi_Q\) commute, \(\varphi \cdot E^\varphi_Q \cdot \alpha \sim \mu^{-1} \varphi \cdot E^\varphi_Q\) as well, i.e., \(\text{mod}(\alpha) = \text{mod}(\alpha|Q)\).

Q.E.D.

We can now prove the following characterization based on automorphisms of an inclusion of the form \(N \subset Q \subset P \subset M\) that satisfies the commuting and co-commuting square condition.

**Theorem 1.** Let \(\lambda \in (0, 1)\), and \(m, n \in \mathbb{N}\) be relatively prime. Suppose that \(N \subset Q \subset P \subset M\) is an irreducible inclusion of factors such that \(Q\) and \(P\) are both of type III\textsubscript{\lambda}, \(M\) of type III\textsubscript{\lambda}1\textsubscript{1}/m, and \(N\) of type III\textsubscript{\lambda}1\textsubscript{1}/n. The following are equivalent.

(i) There exist canonical endomorphisms \(\gamma_{Q, N}\), \(\gamma_{P, Q}\) and \(\gamma_{M, P}\) which satisfy:

\[
\begin{align*}
\gamma_{P, Q}(Q) & \subset Q \\
\gamma_{M, P}(Q) & \subset Q \\
\gamma_{M, P}(N) & \subset N
\end{align*}
\]

are commuting and co-commuting squares.

(ii) There exist properly outer and periodic automorphisms \(\alpha, \beta\) acting on \(Q \subset P\) such that \(\alpha\) has order \(m\), \(\beta\) has order \(n\), \(\text{mod}(\alpha) \equiv \lambda^1/m\), \(\text{mod}(\beta) \equiv \lambda^1/n\) and such that

\[
N = Q^\beta \text{ and } P = M \times \theta_1 Z_m,
\]

where \(N_1\) is the basic construction of \(N \subset Q\).

Proof: The equivalence between (ii) and (iii) was established in [19] and so we only need to prove that (i) and (ii) are equivalent.

(ii) \Rightarrow (i) Assuming that \(\alpha, \beta\) exist, then it is easy to check that

\[
\begin{align*}
P \subset P \times_\alpha Z_m = M \\
Q \subset Q \times_\alpha Z_m
\end{align*}
\]

are commuting co-commuting squares and (ii) follows from Propositions 1 and 2.

(iii) Let \(\{Q^\infty \subset P^\infty, \theta\} \) be a common discrete decomposition of \(Q \subset P\). Then \(\theta\) is both an \(m\) and an \(n\) power, i.e., there exist trace-scaling automorphisms \(\theta_1\) and \(\theta_2\) on \(Q^\infty \subset P^\infty\) such that \(\theta = \theta_1^m = \theta_2^n\), \(M = P^\infty \times \theta_1 Z\) and \(N_1 = Q^\infty \times \theta_2 Z\), where \(N_1\) is the basic construction of \(N \subset Q\).
\[ \sigma^\psi_{E^Q_T} = \text{Id} \text{ on } P \text{ and, as proved in [17], } P = M^\psi_{E^M_Q} \psi_{E^M_Q}. \text{ It follows that } Q \subset M^\psi_{E^M_Q} Q \subset M_0. \text{ But because}

\[
P = M^\psi_{E^M_Q} \subset M \cup \cup M^\psi_{E^M_0} \subset M_0
\]

is a commuting and co-commuting square, the subfactor \( M^\psi_{E^M_0} \) has index value \( m \) in \( M_0 \), hence \( Q = M^\psi_{E^M_0} \). Now if \( \alpha \) is the predual automorphism of \( \sigma^\psi_{E^Q_T} \) on \( Q \), then \( M_0 = Q \times_\alpha \mathbb{Z}_m \) and similarly \( M = P \times_\alpha \mathbb{Z}_m \). By Lemma 3, we have \( \text{mod}(\alpha) = \text{mod}(\alpha(Q) \equiv \lambda^{1/m} \). As for the commuting and co-commuting square

\[
N_0 \subset P \cup N \subset Q,
\]

since \( N \) is of type \( \text{III}_{\lambda^{1/n}} \), there exists a generalized trace \( \varphi \) on \( N \) such that \( \sigma^\varphi_{E^N_T} = \text{Id} \) on \( N \), where \( T \) is as before. Now \( \sigma^\varphi_{E^N_T} \) is inner on \( Q \), say \( \sigma^\varphi_{E^N_T} = \text{Ad} u \) for some unitary \( u \) in \( Q \); then we may assume that \( u^n = 1 \) as \( N' \cap Q = \mathbb{C} \) and hence \( \sigma^\varphi_{E^N_T} = \text{Id} \) on \( N \). As \( Q' \cap P = \mathbb{C} \) and \( P \) is of type \( \text{III}_\lambda \), we also have \( \sigma^\varphi_{E^N_T} E^Q_P = \text{Ad} u \) on \( P \) and because \( E^Q_N \cdot E^Q_{P} E^N_{N_0} \cdot E^P_{N_0} \), \( \sigma^\varphi_{E^N_T} E^N_{N_0} E^P_{N_0} = \text{Ad} u \) on \( P \) as well. Hence for \( x \in N_0, \sigma^\varphi_{E^N_{N_0}}(x) = \sigma^\varphi_{E^N_{N_0}} E^P_{N_0}(x) = u x u^* \) so that \( u N_0 u^* = N_0 \) and \( \text{Ad} u \) defines a properly outer action of \( \mathbb{Z}_n \) on \( N_0 \). Moreover, \( E^Q_N(u^j) = E^Q_N(u^j) = 0 \) for \( 0 \leq j \leq n - 1 \). Thus \( \{N_0, u\}^n \) is the crossed product of \( N_0 \) by \( \sigma^\varphi_{E^N_{N_0}} \) and so it contains \( P \) as a subfactor with index value \( n \). Thus \( \{N_0, u\}^n = P \). Similarly, \( Q \) is the crossed product \( N \) by \( \sigma^\varphi_{E^N} \). Now if we let \( \beta \) be the dual action of \( \sigma^\varphi_{E^N_{N_0}} \), then \( N = Q^\beta \) and \( N_0 = P^\beta \). By Lemma 3, \( \text{mod}(\beta) = \text{mod}(\beta|Q) \equiv \lambda^{1/n} \). Q.E.D.

**Remarks.** 1) In [2] inclusions of the form \( R^H \subset R \times K \), where \( H \) and \( K \) are finite groups of outer automorphisms on a type \( \text{II}_1 \) factor \( R \), are studied and thus inclusions satisfying any one of the equivalent conditions of Theorem 1 may be viewed as a subfactor analogue of group-like inclusions.

2) In view of condition (ii) in Theorem 1, we can define a \( G \)-kernel on \( Q \subset P \), i.e., a homomorphism of \( G \) into \( \text{Aut}(P, Q)/\text{Int}(Q) \), where \( G \) is the group generated by \( \alpha \) and \( \beta \) in \( \text{Aut}(P, Q)/\text{Int}(Q) \) and the homomorphism is given by the quotient map. Such a \( G \)-kernel may be viewed as a subfactor analogue to those for single factors that were studied in [11, 20, 25].

3) Let \( N \subset Q \subset P \subset M \) be an irreducible inclusion of factors that satisfy the hypotheses of the equivalent conditions in Theorem 1. It is easy to see that the isomorphism class of \( N \subset M \) determines the isomorphism class of the associated
commuting and co-commuting square

\[
N_0 \subset P \subset M \\
U \quad \quad \quad \quad \quad U \\
N \subset Q \subset M_0
\]

and vice versa. Indeed, let \( \varphi \) be a generalized trace on \( N \), put \( \psi = \varphi \cdot E_{N}^{\lambda} \) and let \( T = |2\pi/\ln \lambda| \); then by [17] \( Q = N \times \sigma_{\varphi} \mathbb{Z} \) and \( P = M^{\sigma_{\varphi}} \). By the uniqueness of the generalized trace \( \varphi \), we see that \( Q \subset P \) is invariant under isomorphisms of \( N \subset M \). Using the spatial uniqueness of the standard form as proved in [10], the extension and restriction properties of the canonical endomorphisms: \( \gamma_{Q,N} \), \( \gamma_{P,Q} \) and \( \gamma_{M,P} \) and the commuting and co-commuting square conditions of

\[
\gamma_{P,Q}(Q) \subset Q \\
\cup \quad \quad \quad \quad \cup \\
\gamma_{P,Q}(M) \subset M \\
\cup \quad \quad \quad \quad \cup \\
\gamma_{P,Q}(N) \subset N \\
\cup \quad \quad \quad \quad \cup \\
\gamma_{P,Q}(P) \subset P
\]

are also preserved under isomorphisms of \( N \subset M \). Finally, from the proof of Propositions 1 and 2, the constructions of the subfactors \( M_0 \) and \( N_0 \) are also preserved under isomorphisms of \( N \subset M \).

We now turn to the study of inclusions that satisfy the commuting square conditions explained above by means of the discrete decomposition. We recall the following result proved in [19].

**Proposition 3.** Let \( \lambda \in (0, 1) \) and \( m \in \mathbb{N} \). Suppose that

\[
P \subset M \\
U \quad \quad \quad \quad \quad U \\
Q \subset M_0
\]

is a commuting square of factors with finite indices such that: \( Q \subset P \) are both of type III\( \lambda \) and \( M_0 \subset M \) are of type III\( \lambda/\lambda \), \( [M_0 : Q] = [M : P] = m \). Then there exist type II\( \infty \) factors \( Q^\infty \subset P^\infty \) and a trace-scaling automorphism \( \theta \in \text{Aut}(P^\infty, Q^\infty) \) such that \( \text{mod}(\theta) = \lambda \) and

\[
P \subset M \\
U \quad \quad \quad \quad \quad U \\
Q \subset M_0
\]

is isomorphic to

\[
P^\infty \times_{\theta} \mathbb{Z} \subset P^\infty \times_{\theta} \mathbb{Z} \\
U \quad \quad \quad \quad \quad U \\
Q^\infty \times_{\theta} \mathbb{Z} \subset Q^\infty \times_{\theta} \mathbb{Z}.
\]

Using the classification result in [23] for trace-scaling automorphisms on strongly amenable type II\( \infty \) inclusions, we obtain the following classification result of inclusions of type III factors satisfying the hypotheses of Proposition 1.

**Corollary 1.** Let

\[
P \subset M \\
U \quad \quad \quad \quad \quad U \\
Q \subset M_0
\]

be as in Proposition 1 and assume further that \( Q \subset P \) is strongly amenable, i.e., its type II core is strongly amenable in the sense defined in [22]. Then the commuting
square is classified by its type II core $Q^\infty \subset P^\infty$ and the standard invariant of $\theta$ on $Q^\infty \subset P^\infty$.

Similarly, let

$$N_0 \subset P$$

$$\cup \quad \cup$$

$$N \subset Q$$

be a commuting square of factors with finite indices such that: $Q \subset P$ are both of type III, $N \subset N_0$ are of type III, $[N_0 : N]_0 = [P : Q]_0 = n$, and let $M_0$ be the basic construction of $N \subset Q$ and $M$ the basic construction of $N_0 \subset P$. We then obtain a commuting square satisfying the same assumptions as in Proposition 1 and thus

$$N_0 \subset P$$

$$\cup \quad \cup$$

$$N \subset Q$$

is also classified by the type II invariants associated with $Q^\infty \subset P^\infty$ as in Corollary 1 when $Q \subset P$ is strongly amenable.

As an application of Theorem 1, we are going to show that the middle inclusion $Q \subset P$ and the associated $G$-kernel can be used to study the structure of $N \subset M$. The following proposition is an easy extension to the subfactor case of the results proved in [2] for group-like inclusions.

**Proposition 4.** Let $E_N \subset Q \subset P \subset M$ be an irreducible inclusion of type III factors satisfying the hypotheses of the equivalent conditions in Theorem 1. Then $N \subset M$ has finite depth if and only if $Q \subset P$ has finite depth and $G$ is finite.

**Proof.** Suppose that $N \subset M$ has finite depth. Then by [1], $Q \subset P$ and $P^\beta \subset P \times_\alpha \mathbb{Z}_m$ both have finite depth, and so $\langle \alpha, \beta \rangle / \langle \alpha, \beta \rangle \cap \text{Int}(P)$ is finite by [2]. On the other hand, as $Q \subset P$ has finite index, $\langle \alpha, \beta \rangle \cap \text{Int}(P) / \langle \alpha, \beta \rangle \cap \text{Int}(Q)$ is finite by [21] and hence $G = \langle \alpha, \beta \rangle / \langle \alpha, \beta \rangle \cap \text{Int}(Q)$ is also finite.

Conversely, suppose that $Q \subset P$ has finite depth and $G$ is finite. Then $N = Q^\beta \subset Q \times_\alpha \mathbb{Z}_m$ and $P^\beta \subset M = P \times_\alpha \mathbb{Z}_m$ both have finite depth by [2]. Since

$$P^\beta \subset P$$

$$\cup \quad \cup$$

$$N = Q^\beta \subset Q$$

are commuting squares,

$$P^\beta \subset M$$

$$\cup \quad \cup$$

$$N = Q^\beta \subset Q \times_\alpha \mathbb{Z}_m$$

is also a commuting square, and hence $N \subset M$ has finite depth by [27]. Q.E.D.

Let $G$ be a countable discrete group and let $\Theta_1$ and $\Theta_2$ be two $G$-kernels on an inclusion of factors $Q \subset P$. As in the single factor case that was studied in [11, 25] we say that $\Theta_1$ and $\Theta_2$ are conjugate if there exists $\Phi \in \text{Aut}(P, Q) / \text{Int}(Q)$ such that $\Phi \cdot \Theta_1 \cdot \Phi^{-1} = \Theta_2$. According to [22] an inclusion of factors of the form $R^\beta \subset R \times_\alpha \mathbb{Z}_m$, where $\alpha$ and $\beta$ are outer automorphisms with order $m$ and $n$, respectively, on a type II factor $R$, is classified by the conjugacy class of the $G$-kernel coming from $\langle \alpha, \beta \rangle / \text{Int}(R)$. It is thus not surprising that a similar
result holds for an inclusion of AFD factors that satisfies any one of the equivalent conditions in Theorem 1.

**Theorem 2.** Let $N^O \subset Q \subset P \subset M$ be an irreducible inclusion of type III factors that satisfies any one of the equivalent conditions in Theorem 1. Let $\alpha$ and $\beta$ be the associated automorphisms and $\{G, \Theta\}$ be the kernel on $Q \subset P$ arising from the subgroup generated by $\alpha$ and $\beta$ modulo $\text{Int}(Q)$. Then the isomorphism class of $N \subset M$ is determined by the conjugacy class of $G$.

**Proof.** Let $N_1 \subset M_1$ be another irreducible inclusion satisfying the same properties as $N \subset M$. Let $Q_1 \subset P_1, \alpha_1, \beta_1$ and $\Theta_1$ be defined accordingly from $N_1 \subset M_1$.

Suppose first that $\Phi$ is an isomorphism of $N \subset M$ onto $N_1 \subset M_1$. Then as noted in the Remarks after Theorem 1 above, $\Phi$ actually maps $N \subset Q \subset P \subset M$ onto $N_1 \subset Q_1 \subset P_1 \subset M_1$ and so we can identify all the respective factors: $N = N_1$, $Q = Q_1$, $P = P_1$, $M = M_1$. As a result we see that $\Phi$ can be extended to an isomorphism mapping

\[
P \subset P \times_{\alpha} Z_m \quad P \subset P \times_{\beta} Z_n \quad \text{onto} \quad P \subset P \times_{\alpha_1} Z_{m_1} \quad Q \subset Q \times_{\beta} Z_n
\]

onto

\[
P \subset P \times_{\alpha_1} Z_{m_1} \quad P \subset P \times_{\beta_1} Z_{n_1} \quad Q \subset Q \times_{\beta_1} Z_{n_1}.
\]

It is then straightforward to prove that there exist unitaries $u$ and $v$ in $Q$ such that $\Phi \cdot \alpha \cdot \Phi^{-1} = \text{Ad } u \cdot \alpha_1$ and $\Phi \cdot \beta \cdot \Phi^{-1} = \text{Ad } v \cdot \beta_1$. Hence $\Theta$ and $\Theta_1$ are conjugate.

Conversely, suppose that $\Theta$ and $\Theta_1$ are two $G$-kernels on $Q \subset P$ that are conjugate via an isomorphism $\Phi$ of $Q \subset P$ onto $Q_1 \subset P_1$. Then there exist unitaries $u$ and $v$ in $Q_1$ such that $\Phi \cdot \alpha \cdot \Phi^{-1} = \text{Ad } u \cdot \alpha_1$ and $\Phi \cdot \beta \cdot \Phi^{-1} = \text{Ad } v \cdot \beta_1$, where $l$ and $n$ are relatively prime. Then by standard arguments, $\Phi$ can be extended to isomorphisms between

\[
P \subset P \times_{\alpha} Z_m \quad P \subset P \times_{\beta} Z_n \quad \text{onto} \quad P \subset P \times_{\alpha_1} Z_{m_1} \quad Q \subset Q \times_{\beta} Z_n
\]

and

\[
P_1 \subset P_1 \times_{\alpha_1} Z_{m_1} \quad P_1 \subset P_1 \times_{\beta_1} Z_{n_1} \quad Q_1 \subset Q_1 \times_{\beta_1} Z_{n_1}.
\]

As $\Phi(N)$ is the downward construction of $Q_1 \subset Q_1 \times_{\beta_1} Z_{n_1}$, we can find a unitary $w$ in $Q_1$ such that $\text{Ad } w \cdot \Phi(N) = N_1$ and therefore $\text{Ad } w \cdot \Phi$ is an isomorphism between $N \subset M$ and $N_1 \subset M_1$. \quad \text{Q.E.D.}

Recall that if $R_0$ is the hyperfinite type $\text{II}_1$ factor, then by [11] in the finite case, and by [20] in the amenable case, $G$-kernels of $R_0$ are classified up to conjugacy by their obstructions as defined in [11, 25], which are elements of $H^3(G, T)$: the third cohomology group of $G$ with coefficients in the unit circle $T$. It would thus be an interesting problem to classify $G$-kernels on subfactors by their cohomological invariants, their standard invariants (cf. [18]) and, in the case of type III$_\lambda$ factors, their Connes-Takesaki modules.
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