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THE FUGLEDE-PUTNAM THEOREM AND A GENERALIZATION OF BARRÍA'S LEMMA

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Abstract. Let A and B be bounded linear operators, and let C be a partial isometry on a Hilbert space. Suppose that (1) $CA = BC$, (2) $\|A\| \geq \|B\|$, (3) $(C^*C)A = A(C^*C)$ and (4) $C(\|A\|^2 - AA^*)^{1/2} = 0$. Then we have $CA^* = B^*C$.

Let $\mathcal{H}$ be a complex Hilbert space. An operator means a bounded linear operator on $\mathcal{H}$. The familiar Fuglede-Putnam theorem is stated as follows:

Theorem A (Fuglede-Putnam [3, Theorem IX.6.7]). *If A and B are normal operators on $\mathcal{H}$ and C is an operator such that $CA = BC$, then $CA^* = B^*C$.*

Several authors have relaxed the normality hypothesis on A and B in Theorem A in various ways (for example, to hyponormality), still without restrictions on C, and have reached the same conclusion. However, it appears that few have attempted to place conditions on the operator C in order to remove the normality hypotheses on A and B. In this note we wish to generalize the following lemma of Barría from this point of view.

Lemma B (Barría [1, Lemma 2]). *Assume that $V_1^*V_2 = V_2V_1^*$, where V_1 and V_2 are isometries. Then $V_1V_2 = V_2V_1$.*

Now we state our result. The proof is elementary, depending on partially isometric extensions of contractions.

Theorem. *Let A and B be bounded linear operators, and let C be a partial isometry. Suppose that*
(1) $CA = BC$,
(2) $\|A\| \geq \|B\|$,
(3) $(C^*C)A = A(C^*C)$ *and*
(4) $C(\|A\|^2 - AA^*)^{1/2} = 0$.
Then we have $CA^* = B^*C$.

Proof. We assume first that A, B and C are partial isometries. Then condition (4) becomes $C^*C \leq AA^*$. In particular, C^*C and AA^* commute. It follows from [6, Lemma 2] that $CA = BC$ is a partial isometry. Therefore, B^*B and CC^* commute

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by [6, Lemma 2] again. Then we have

$$\begin{aligned} B^*C &= B^*CAA^* = B^*BCA^* = B^*BCC^*CA^* \\ &= CC^*B^*BCA^* = C(BC)^*(BC)A^* \\ &= C(CA)^*(CA)A^* \\ &= CA^*C^*CAA^* = CC^*CA^*AA^* = CA^*. \end{aligned}$$

Thus the theorem is true for partial isometries A, B and C.

Now, let A, B and C satisfy the hypotheses of the theorem. Dividing by $\|A\|$, we may assume that $\|A\| = 1$ and $\|B\| \leq 1$.

We define operator matrices $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$ by

$$\widetilde{A} = \begin{bmatrix} A & (1 - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B & (1 - BB^*)^{1/2} \\ 0 & 0 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$ are partial isometries on $\mathcal{H} \oplus \mathcal{H}$ and satisfy

$$\widetilde{C}^*\widetilde{C} = \begin{bmatrix} C^*C & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \widetilde{A}\widetilde{A}^*,$$

$$(\widetilde{C}^*\widetilde{C})\widetilde{A} = \begin{bmatrix} (C^*C)A & (C^*C)(1 - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix} = \widetilde{A}(\widetilde{C}^*\widetilde{C}),$$

and

$$\widetilde{C}\widetilde{A} = \begin{bmatrix} CA & C(1 - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} BC & 0 \\ 0 & 0 \end{bmatrix} = \widetilde{B}\widetilde{C}.$$

Therefore, by the first paragraph of the proof, we have $\widetilde{C}\widetilde{A}^* = \widetilde{B}^*\widetilde{C}$. This implies that $CA^* = B^*C$.

Now we present an example which is not covered by the Fuglede-Putnam theorem or an existing generalization of it.

Example. Let

$$A = \begin{bmatrix} 0 & 0 & a_1 & & & & \\ & 0 & 0 & a & & & 0 \\ & & 0 & 0 & a_2 & & \\ & & & 0 & 0 & a & \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & a & & & & 0 \\ & 0 & 0 & b_1 & & & \\ & & 0 & 0 & a & & \\ & & & 0 & 0 & b_2 & \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & & & & 0 \\ & 0 & 0 & 0 & & & \\ & & 0 & 1 & 0 & & \\ & & & 0 & 0 & 0 & \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $\{a_n\}$ and $\{b_n\}$ are bounded sequences of complex numbers such that $a = \sup_n |a_n| \geq \sup_n |b_n|$.

Then B is not M-hyponormal [7], indeed, in general it is not even a dominant [9] or $\mathcal{Y}$-class operator. (We point out that an operator T is said to be M-hyponormal if there exists a constant $M \geq 1$ such that $(T-\lambda)(T-\lambda)^* \leq M^2(T-\lambda)^*(T-\lambda)$ for any complex number λ. An operator T is said to be dominant if for any complex number λ there exists a number $M_\lambda \geq 1$ such that $(T-\lambda)(T-\lambda)^* \leq M_\lambda^2 (T-\lambda)^*(T-\lambda)$.

For $\alpha > 0$ an operator T is said to be in $\mathcal{Y}_\alpha$ if there exists a number $M_\alpha \geq 1$ such that $|T^*T - TT^*|^\alpha \leq M_\alpha^2 (T-\lambda)^*(T-\lambda)$ for any complex number λ. An operator T is said to be in $\mathcal{Y}$ (or to be of $\mathcal{Y}$-class) if T is in $\mathcal{Y}_\alpha$ for some $\alpha \geq 1$.)

Therefore, neither the Fuglede-Putnam theorem nor any existing generalizations apply. However, since A, B and C satisfy the hypotheses of our theorem, we can conclude that $CA^* = B^*C$.

Remark 1. We cannot merely drop condition (3) in the Theorem. For example, let U be the unilateral shift. Take $A = B = U^*$ and $C = (U^*)^2$. Then A, B and C satisfy (1), (2) and (4), but $CA^* \neq B^*C$ immediately.

Remark 2. We cannot merely drop condition (2) in the Theorem. For example, put

$$A = \begin{bmatrix} 0 & 0 & 1 & & & & \\ & 0 & 0 & 1 & & & \Large 0 \\ & & 0 & 0 & 1 & & \\ & & & 0 & 0 & 1 & \\ \Large 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 1 & & & & \\ & 0 & 0 & 1 & & & \Large 0 \\ & & 0 & 0 & 1 & & \\ & & & 0 & 0 & 1 & \\ \Large 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & & & & \\ & 0 & 0 & 0 & & & \Large 0 \\ & & 0 & 1 & 0 & & \\ & & & 0 & 0 & 0 & \\ \Large 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then A, B and C satisfy (1), (3) and (4), but $CA^* \neq B^*C$.

Remark 3. We cannot merely drop condition (4) in the Theorem. For example, put

$$A = \begin{bmatrix} 0 & 0 & 2 & & & & \\ & 0 & 0 & 1 & & & \Large 0 \\ & & 0 & 0 & 1 & & \\ & & & 0 & 0 & 1 & \\ \Large 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \Large 0 \\ & 1 & 0 & 0 & 1 & & & \\ & & 0 & 0 & 0 & 1 & & \\ & & & 0 & 0 & 0 & 1 & \\ \Large 0 & & & & \ddots & \ddots & \ddots & \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & & & \Large 0 \\ & 0 & 0 & 0 & & \\ & & 0 & 1 & 0 & \\ & & & 0 & 0 & 0 \\ \Large 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then A, B and C satisfy (1), (2) and (3), but $CA^* \neq B^*C$.

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