

## NOTES ON HOMOLOGY COBORDISMS OF PLUMBED HOMOLOGY 3-SPHERES

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**ABSTRACT.** The gauge-theoretical invariants of Donaldson and Seiberg-Witten are used to detect some infinite order elements in the homology cobordism group of integral homology 3-spheres.

This paper is concerned with the homology cobordism group  $\Theta_{\mathbb{Z}}^3$  of oriented integral homology 3-spheres. We use S. Donaldson's [D] and M. Furuta's [F] (see also [A]) theorems and the  $\bar{\mu}$ -invariant introduced by W. Neumann [N] and L. Siebenmann [S] to prove the following two theorems.

**Theorem 1.** *Let a homology sphere  $\Sigma$  be the link of an algebraic singularity. If  $\Sigma$  is homology cobordant to zero, then  $\bar{\mu}(\Sigma) \geq 0$ .*

All Seifert fibered homology spheres are the links of algebraic singularities. Therefore, one can apply this theorem to recover some of the results of R. Fintushel and R. Stern [FS] on homology cobordisms of Seifert fibered homology spheres. For example, it easily follows that for any relatively prime positive integers  $p$  and  $q$ , Seifert fibered homology spheres  $\Sigma(p, q, pqk - 1)$ ,  $k \geq 1$ , have infinite order in the group  $\Theta_{\mathbb{Z}}^3$ . Theorem 1 implies some new results as well; see Section 2.

**Theorem 2.** *Let  $\Sigma$  be a plumbed homology sphere and assume that  $\bar{\mu}(\Sigma) \neq 0$ . If  $\Sigma$  bounds a plumbed 4-manifold with even intersection form of rank not exceeding  $10|\bar{\mu}(\Sigma)|$ , then  $\Sigma$  has infinite order in the group  $\Theta_{\mathbb{Z}}^3$ .*

One can easily see [NR] that any Seifert sphere  $\Sigma(a_1, \dots, a_n)$  with one of the  $a_i$ 's even bounds an (essentially unique) plumbed manifold with even intersection form. This intersection form is very often, though not always, of the desired rank not exceeding  $10|\bar{\mu}|$ . Moreover, in many cases the rank can be reduced with the help of Kirby calculus without changing  $\bar{\mu}$  so that the conclusion of Theorem 2 still holds. Such an approach leads in particular to the following result [S1].

**Corollary 1.** *For any relatively prime integers  $p, q \geq 2$  and any odd integer  $k \geq 1$  the Seifert fibered homology sphere  $\Sigma(p, q, pqk + 1)$  has infinite order in the group  $\Theta_{\mathbb{Z}}^3$ .*

This result cannot be extended to  $\Sigma(p, q, pqk + 1)$  with even  $k$ , because, for instance,  $\Sigma(2, 3, 13)$  is known to bound a contractible manifold.

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It is worth mentioning that in fact  $\bar{\mu}(\Sigma)$  vanishes for all known examples of plumbed homology spheres  $\Sigma$  homology cobordant to zero. In [N], W. Neumann conjectured that  $\bar{\mu}$  is a homology cobordism invariant. The results of this paper give further evidence which weighs for a positive solution to this conjecture.

### 1. DEFINITION AND BASIC PROPERTIES OF THE $\bar{\mu}$ -INVARIANT

We recall the definition of the invariant  $\bar{\mu}$  by W. Neumann [N]. Note that our definition differs from the original one by a factor of  $1/8$ .

Let  $\Gamma$  be a connected plumbing graph, that is, a connected graph with no cycles, each of whose vertices carries an integer weight  $e_i$ ,  $i = 1, \dots, s$ . The matrix  $A(\Gamma) = (a_{ij})_{i,j=1,\dots,s}$  with the entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j, \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge,} \\ 0, & \text{otherwise,} \end{cases}$$

is the intersection matrix of the 4-dimensional manifold  $P(\Gamma)$  obtained by plumbing  $D^2$ -bundles over 2-spheres according to  $\Gamma$ . This manifold is simply connected.

Disconnected graphs are also allowed. Namely, if  $\Gamma = \Gamma_0 + \Gamma_1$  is a disjoint union of  $\Gamma_0$  and  $\Gamma_1$ , then  $P(\Gamma)$  is the boundary connected sum  $P(\Gamma_0) \natural P(\Gamma_1)$ .

If  $\Gamma$  is a plumbing graph as above, then  $M(\Gamma) = \partial P(\Gamma)$  is an integral homology sphere if and only if  $\det A(\Gamma) = \pm 1$ . For example, all Seifert fibered homology spheres  $\Sigma(a_1, \dots, a_n)$  are of the form  $\partial P(\Gamma)$  where  $\Gamma$  is a star-shaped graph; see [NR].

If  $M(\Gamma)$  is a homology sphere, there is a unique homology class  $w \in H_2(P(\Gamma); \mathbb{Z})$  satisfying the following two conditions. First,  $w$  is *characteristic*, that is (dot represents intersection number)

$$(1) \quad w \cdot x \equiv x \cdot x \pmod{2} \quad \text{for all } x \in H_2(P(\Gamma); \mathbb{Z}),$$

and second, all coordinates of  $w$  are either 0 or 1 in the natural basis of  $H_2(P(\Gamma); \mathbb{Z})$ . We call  $w$  *the integral Wu class* for  $P(\Gamma)$ . It follows from [N] that the integer  $\text{sign } P(\Gamma) - w \cdot w$  depends only on  $M(\Gamma)$  and not on  $\Gamma$ . This integer is divisible by 8, so one can define the Neumann-Siebenmann invariant by the formula

$$(2) \quad \bar{\mu}(M(\Gamma)) = \frac{1}{8}(\text{sign } P(\Gamma) - w \cdot w).$$

It is also easily seen that (1) implies that no two coordinates of  $w$  corresponding to adjacent vertices in  $\Gamma$  can both be 1, so it follows that  $w$  is spherical, and the modulo 2 reduction of the  $\bar{\mu}$ -invariant is the usual Rohlin invariant  $\mu$ ; see [NR].

### 2. THE $\bar{\mu}$ -INVARIANT OF ALGEBRAIC LINKS

The plumbed homology spheres have been classified in [EN]. In particular, it has been shown that a plumbed homology sphere  $\Sigma$  is an algebraic link if and only if there exists a plumbing graph  $\Gamma$  such that the intersection form of the manifold  $P(\Gamma)$  with  $\Sigma = \partial P(\Gamma)$  is negative definite. The simplest case is the Seifert fibered case: any Seifert fibered homology sphere  $\Sigma(a_1, \dots, a_n)$  is the link of the singularity of  $f^{-1}(0)$ , where  $f : \mathbb{C}^n \rightarrow \mathbb{C}^{n-2}$  is a map of the form

$$f(z_1, \dots, z_n) = \left( \sum_{k=1}^n b_{1,k} z_k^{a_k}, \dots, \sum_{k=1}^n b_{n-2,k} z_k^{a_k} \right)$$

with sufficiently general coefficient matrix  $(b_{i,j})$ ; see [NR]. For instance,

$$\Sigma(p, q, r) = \{z \in \mathbb{C}^3 \mid \|z\| = \varepsilon \text{ and } z_1^p + z_2^q + z_3^r = 0\}$$

for  $\varepsilon > 0$  small enough.

The next simplest case is the following: if  $p, q, r$  are pairwise relatively prime integers, as are  $p', q', r'$ , then the homology sphere  $\Sigma$  obtained by splicing  $\Sigma(p, q, r)$  and  $\Sigma(p', q', r')$  along the singular fibers of degrees  $r$  and  $r'$  is the link of singularity if and only if  $rr' > pp'qq'$ ; see [NW], § 4.

**Theorem 1.** *Let a homology sphere  $\Sigma$  be the link of an algebraic singularity. If  $\Sigma$  is homology cobordant to zero, then  $\bar{\mu}(\Sigma) \geq 0$ .*

*Proof.* Since  $\Sigma$  is an algebraic link, one may assume that  $\Sigma$  is the boundary of a plumbed 4-manifold  $P(\Gamma)$  whose intersection form is negative definite. Suppose that  $\Sigma$  bounds a smooth homology ball  $M$ . Let us consider the manifold  $W = P(\Gamma) \cup_{\Sigma} (-M)$ . This is a smooth closed oriented manifold whose intersection form is naturally isomorphic to the intersection form of  $P(\Gamma)$ , in particular, is negative definite. By S. Donaldson’s Theorem 1 from [D], this form is diagonalizable over the integers.

We use this fact to evaluate  $\bar{\mu}(\Sigma)$ . Obviously,  $\text{sign } P(\Gamma) = -s$ , where  $s$  is the number of vertices in the graph  $\Gamma$ , so we only need to find the Wu class  $w$ . In the standard basis associated with the plumbing, the matrix  $A$  of the intersection form of  $P(\Gamma)$  takes the form  $A = U^t(-E)U$ , where  $U \in \text{SL}_s(\mathbb{Z})$  and  $E$  is the identity matrix. The defining relation (1) translates to

$$w^t U^t(-E)Ux \equiv x^t U^t(-E)Ux \pmod{2} \quad \text{for all } x \in H_2(P(\Gamma); \mathbb{Z}),$$

or, equivalently,

$$(Uw)^t(-E)y \equiv y^t(-E)y \pmod{2} \quad \text{for all } y \in H_2(P(\Gamma); \mathbb{Z}).$$

Therefore,  $Uw$  is characteristic for  $-E$ , in particular, all the coordinates of  $Uw$  are odd. Now, we have

$$w.w = w^t U^t(-E)Uw = -(Uw)^t(Uw),$$

which is equal to minus the square of the Euclidean length of the vector  $Uw$ . Since all the coordinates of  $Uw$  are odd,  $w.w \leq -s$ , and therefore  $\bar{\mu}(\Sigma) \geq 0$ .  $\square$

**Corollary 2.** *If a plumbed homology sphere  $\Sigma$  is an algebraic link and  $\bar{\mu}(\Sigma) < 0$ , then  $\Sigma$  has infinite order in the group  $\Theta_{\mathbb{Z}}^3$ .*

*Proof.* Let  $m\Sigma = \Sigma \# \dots \# \Sigma$  ( $m$  times). Since  $\bar{\mu}$  is additive with respect to connected sums,  $\bar{\mu}(m\Sigma) = m\bar{\mu}(\Sigma) < 0$ , therefore,  $m\Sigma$  is not homology cobordant to zero for any  $m$ .  $\square$

**Example.** For any relatively prime integers  $p, q > 0$ , one can easily see that  $\bar{\mu}(\Sigma(p, q, pq - 1)) < 0$ . Therefore, all these homology spheres are of infinite order in  $\Theta_{\mathbb{Z}}^3$ . In particular, the Poincaré homology sphere  $\Sigma(2, 3, 5)$  is of infinite order.

**Example.** Homology spheres  $\Sigma(p, q, pqk - 1)$  have infinite order in  $\Theta_{\mathbb{Z}}^3$  for any  $k \geq 1$ . If  $k$  is odd, this holds since  $\bar{\mu}(\Sigma(p, q, pqk - 1))$  equals  $\bar{\mu}(\Sigma(p, q, pq - 1))$  and is therefore negative. If  $k$  is even, then  $\bar{\mu}(\Sigma(p, q, pqk - 1)) = 0$ , and the argument of Theorem 1 cannot be applied directly. Let  $A$  be the intersection form of a negative definite plumbing  $X$  bounding  $\Sigma(p, q, pqk - 1)$ ,  $k$  even. We perform a  $(-1)$ -surgery

on the singular fiber of degree  $pqk - 1$  to get  $\Sigma(p, q, pq(k + 1) - 1)$ . The union of the cobordism  $X$  with the trace of this surgery is a plumbed manifold with the intersection form  $A \oplus (-1)$ . Now, if  $\Sigma(p, q, pqk - 1)$  bounded an acyclic manifold,  $A$  would be diagonalizable, and so would be  $A \oplus (-1)$ . This contradicts the fact that  $\bar{\mu}(\Sigma(p, q, pq(k + 1) - 1)) < 0$ . A similar argument shows that none of the multiples of  $\Sigma(p, q, pqk - 1)$  is homology cobordant to zero.

Many Seifert homology spheres  $\Sigma(a_1, \dots, a_n)$  having infinite order in the group  $\Theta_{\mathbb{Z}}^3$  by Corollary 2 can also be detected by the  $R$ -invariant of R. Fintushel and R. Stern,

$$R(a_1, \dots, a_n) = \frac{2}{a} - 3 + n + \sum_{i=1}^n \frac{2}{a_i} \sum_{k=1}^{a_i-1} \cot\left(\frac{\pi ak}{a_i^2}\right) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi k}{a_i}\right),$$

where  $a = a_1 \cdot \dots \cdot a_n$ . A theorem in [FS] implies that if  $\Sigma(a_1, \dots, a_n)$  bounds a homology ball, then  $R(a_1, \dots, a_n) < 0$ . There are however Seifert fibered homology spheres which are not homology cobordant to zero and which can be detected by  $\bar{\mu}$  and not by  $R$ , and vice versa.

**Example.** Both Seifert spheres  $\Sigma(2, 11, 19)$  and  $\Sigma(3, 5, 7)$  are not homology cobordant to zero. As to the former one, this follows from the fact that  $\bar{\mu}(\Sigma(2, 11, 19)) = -1$  is negative (though  $R(2, 11, 19) = -1$ ); the latter one has  $\bar{\mu}(\Sigma(3, 5, 7)) = 0$  and the result follows from  $R(3, 5, 7) = 1$ .

The  $\bar{\mu}$ -invariant also detects some plumbed homology spheres which have infinite order in  $\Theta_{\mathbb{Z}}^3$  but are not Seifert fibered.

**Example.** Let  $\Sigma$  be the splice of  $\Sigma(4, 7, 9)$  and  $\Sigma(2, 3, 25)$  along the singular fibers of degrees 9 and 25. This manifold is an algebraic link. By using the additivity of the  $\bar{\mu}$ -invariant proven in [Du] and [S2], we find that

$$\bar{\mu}(\Sigma) = \bar{\mu}(\Sigma(4, 7, 9)) + \bar{\mu}(\Sigma(2, 3, 25)) = -2 + 0 < 0.$$

Therefore,  $\Sigma$  has infinite order in  $\Theta_{\mathbb{Z}}^3$  (although the Rohlin invariant  $\mu(\Sigma)$  equals 0 modulo 2).

### 3. MORE CONSTRAINTS FROM EVEN PLUMBING

In 1995, M. Furuta [F] used the Seiberg-Witten invariants to prove the so-called 10/8-conjecture, which says that if  $A$  is the intersection form of a smooth closed spin 4-manifold, then  $\text{rank } A / |\text{sign } A| > 10/8$ . In this section we use this result to prove the following theorem.

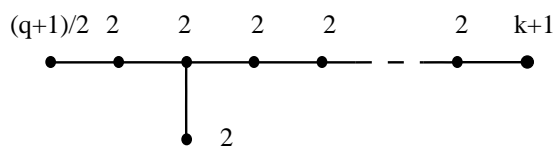
**Theorem 2.** *Let  $\Sigma$  be a plumbed homology sphere and assume that  $\bar{\mu}(\Sigma) \neq 0$ . If  $\Sigma$  bounds a plumbed 4-manifold with even intersection form of rank not exceeding  $10|\bar{\mu}(\Sigma)|$ , then  $\Sigma$  has infinite order in the group  $\Theta_{\mathbb{Z}}^3$ .*

*Proof.* Let  $P(\Gamma)$  be a plumbed manifold with boundary  $\partial P(\Gamma) = \Sigma$ , such that its intersection form  $A = A(\Gamma)$  is even and has rank not exceeding  $10|\bar{\mu}(\Sigma)|$ . Since  $A$  is even, its Wu class  $w$  vanishes. Therefore,  $8 \cdot \bar{\mu}(\Sigma) = \text{sign } A$ , and moreover  $A$  is isomorphic over the integers to  $|\bar{\mu}(\Sigma)| \cdot E_8 \oplus b \cdot H$ . Since  $\text{rank}(A) \leq 10|\bar{\mu}(\Sigma)|$ , we get that  $\text{rank } A / |\text{sign } A| \leq 10/8$ .

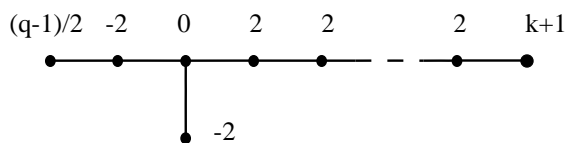
Now, suppose that  $\Sigma$  bounds a smooth homology ball  $M$ , and consider the manifold  $W = P(\Gamma) \cup_{\Sigma} (-M)$ . This is a smooth closed oriented spin 4-manifold whose intersection form  $Q$  is isomorphic to  $|\bar{\mu}(\Sigma)| \cdot E_8 \oplus b \cdot H$ . Since  $\text{rank } Q / |\text{sign } Q| \leq 10/8$ , we get a contradiction with the 10/8 conjecture.

The argument can be repeated with  $\Sigma$  replaced by any of its multiples, which proves the theorem.  $\square$

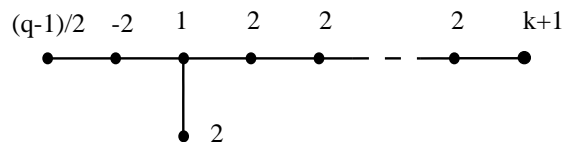
For a complete proof of Corollary 1 formulated in the introduction we refer the reader to [S1], and only prove it here in the simplest case  $p = 2$ . If  $q \equiv 3 \pmod 4$ , then  $\Sigma(2, q, 2qk + 1)$ ,  $k$  odd, is the boundary of the 4-manifold obtained by plumbing according to the following diagram:



The intersection form of this manifold is isomorphic to  $\frac{q+1}{4} \cdot E_8 \oplus H$ . If  $q \equiv 1 \pmod 4$ , the homology sphere  $\Sigma(2, q, 2qk + 1)$  bounds plumbing



with intersection form  $\frac{q-1}{4} \cdot E_8 \oplus 3 \cdot H$ . To deal with this case, we need to reduce the rank by surgering out two “hyperbolics”  $H$ . This can be done as follows. First, redraw the plumbing diagram:



After two obvious blow-downs we get the manifold in Figure 1 with the intersection form  $\frac{q-1}{4} \cdot E_8 \oplus 2 \cdot H$ . An equivalent link description is shown in Figure 2. After two more blow-downs we get rid of another “hyperbolic”  $H$ . The final link description is shown in Figure 3.

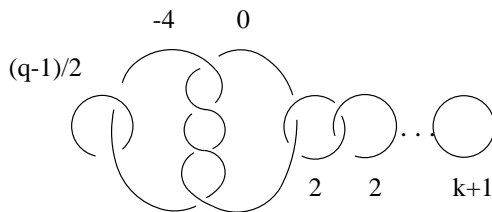


FIGURE 1

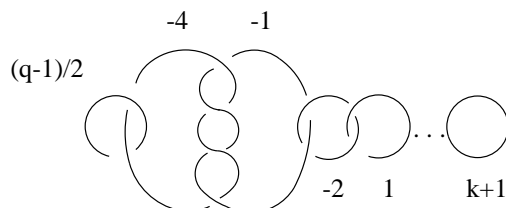


FIGURE 2

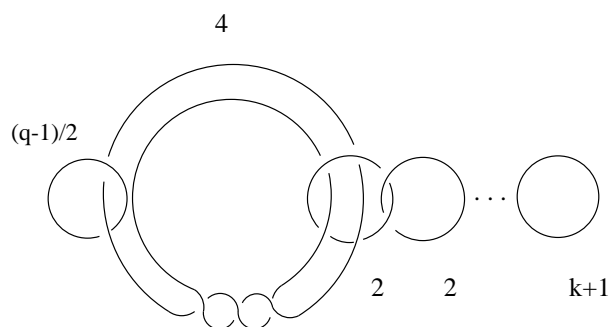


FIGURE 3

The corresponding intersection form is isomorphic to  $\frac{q-1}{4} \cdot E_8 \oplus H$ , and an argument similar to that in Theorem 2 can be used to show that  $\Sigma(2, q, 2qk + 1)$ ,  $k$  odd, has infinite order in  $\Theta_{\mathbb{Z}}^3$ .

**Example.** In addition to those listed in Corollary 1, there are of course many homology spheres having infinite order in  $\Theta_{\mathbb{Z}}^3$  due to Theorem 2. For example,  $\Sigma(8, 13, 21)$  is of this sort because it has  $\bar{\mu}$ -invariant equal 4 and can be obtained by plumbing on an even weighted graph of rank 34.

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