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NOTES ON HOMOLOGY COBORDISMS OF PLUMBED HOMOLOGY 3-SPHERES

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ABSTRACT. The gauge-theoretical invariants of Donaldson and Seiberg-Witten are used to detect some infinite order elements in the homology cobordism group of integral homology 3-spheres.

This paper is concerned with the homology cobordism group $\Theta^3_{\mathbb{Z}}$ of oriented integral homology 3-spheres. We use S. Donaldson's [D] and M. Furuta's [F] (see also [A]) theorems and the $\bar{\mu}$-invariant introduced by W. Neumann [N] and L. Siebenmann [S] to prove the following two theorems.

Theorem 1. *Let a homology sphere Σ be the link of an algebraic singularity. If Σ is homology cobordant to zero, then $\bar{\mu}(\Sigma) \geq 0$.*

All Seifert fibered homology spheres are the links of algebraic singularities. Therefore, one can apply this theorem to recover some of the results of R. Fintushel and R. Stern [FS] on homology cobordisms of Seifert fibered homology spheres. For example, it easily follows that for any relatively prime positive integers p and q, Seifert fibered homology spheres $\Sigma(p,q,pqk-1)$, $k \geq 1$, have infinite order in the group $\Theta^3_{\mathbb{Z}}$. Theorem 1 implies some new results as well; see Section 2.

Theorem 2. *Let Σ be a plumbed homology sphere and assume that $\bar{\mu}(\Sigma) \neq 0$. If Σ bounds a plumbed 4-manifold with even intersection form of rank not exceeding $10\,|\bar{\mu}(\Sigma)|$, then Σ has infinite order in the group $\Theta^3_{\mathbb{Z}}$.*

One can easily see [NR] that any Seifert sphere $\Sigma(a_1, \dots, a_n)$ with one of the a_i's even bounds an (essentially unique) plumbed manifold with even intersection form. This intersection form is very often, though not always, of the desired rank not exceeding $10\,|\bar{\mu}|$. Moreover, in many cases the rank can be reduced with the help of Kirby calculus without changing $\bar{\mu}$ so that the conclusion of Theorem 2 still holds. Such an approach leads in particular to the following result [S1].

Corollary 1. *For any relatively prime integers $p, q \geq 2$ and any odd integer $k \geq 1$ the Seifert fibered homology sphere $\Sigma(p,q,pqk+1)$ has infinite order in the group $\Theta^3_{\mathbb{Z}}$.*

This result cannot be extended to $\Sigma(p,q,pqk+1)$ with even k, because, for instance, $\Sigma(2,3,13)$ is known to bound a contractible manifold.

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It is worth mentioning that in fact $\bar{\mu}(\Sigma)$ vanishes for all known examples of plumbed homology spheres Σ homology cobordant to zero. In [N], W. Neumann conjectured that $\bar{\mu}$ is a homology cobordism invariant. The results of this paper give further evidence which weighs for a positive solution to this conjecture.

1. Definition and basic properties of the $\bar{\mu}$-invariant

We recall the definition of the invariant $\bar{\mu}$ by W. Neumann [N]. Note that our definition differs from the original one by a factor of 1/8.

Let Γ be a connected plumbing graph, that is, a connected graph with no cycles, each of whose vertices carries an integer weight e_i , $i = 1, \dots, s$. The matrix $A(\Gamma) = (a_{ij})_{i,j=1,\dots,s}$ with the entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j, \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge,} \\ 0, & \text{otherwise,} \end{cases}$$

is the intersection matrix of the 4-dimensional manifold $P(\Gamma)$ obtained by plumbing D^2-bundles over 2-spheres according to Γ. This manifold is simply connected.

Disconnected graphs are also allowed. Namely, if $\Gamma = \Gamma_0 + \Gamma_1$ is a disjoint union of Γ_0 and Γ_1, then $P(\Gamma)$ is the boundary connected sum $P(\Gamma_0)\natural P(\Gamma_1)$.

If Γ is a plumbing graph as above, then $M(\Gamma) = \partial P(\Gamma)$ is an integral homology sphere if and only if $\det A(\Gamma) = \pm 1$. For example, all Seifert fibered homology spheres $\Sigma(a_1, \dots, a_n)$ are of the form $\partial P(\Gamma)$ where Γ is a star-shaped graph; see [NR].

If $M(\Gamma)$ is a homology sphere, there is a unique homology class $w \in H_2(P(\Gamma); \mathbb{Z})$ satisfying the following two conditions. First, w is *characteristic*, that is (dot represents intersection number)

$$w.x \equiv x.x \mod 2 \quad \text{for all } x \in H_2(P(\Gamma); \mathbb{Z}), \tag{1}$$

and second, all coordinates of w are either 0 or 1 in the natural basis of $H_2(P(\Gamma); \mathbb{Z})$. We call *$w$ the integral Wu class* for $P(\Gamma)$. It follows from [N] that the integer $\operatorname{sign} P(\Gamma) - w.w$ depends only on $M(\Gamma)$ and not on Γ. This integer is divisible by 8, so one can define the Neumann-Siebenmann invariant by the formula

$$\bar{\mu}(M(\Gamma)) = \frac{1}{8}(\operatorname{sign} P(\Gamma) - w.w). \tag{2}$$

It is also easily seen that (1) implies that no two coordinates of w corresponding to adjacent vertices in Γ can both be 1, so it follows that w is spherical, and the modulo 2 reduction of the $\bar{\mu}$-invariant is the usual Rohlin invariant μ; see [NR].

2. The $\bar{\mu}$-invariant of algebraic links

The plumbed homology spheres have been classified in [EN]. In particular, it has been shown that a plumbed homology sphere Σ is an algebraic link if and only if there exists a plumbing graph Γ such that the intersection form of the manifold $P(\Gamma)$ with $\Sigma = \partial P(\Gamma)$ is negative definite. The simplest case is the Seifert fibered case: any Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ is the link of the singularity of $f^{-1}(0)$, where $f : \mathbb{C}^n \to \mathbb{C}^{n-2}$ is a map of the form

$$f(z_1, \dots, z_n) = \left(\sum_{k=1}^{n} b_{1,k} z_k^{a_k}, \dots, \sum_{k=1}^{n} b_{n-2,k} z_k^{a_k} \right)$$

with sufficiently general coefficient matrix $(b_{i,j})$; see [NR]. For instance,

$$\Sigma(p,q,r) = \{z \in \mathbb{C}^3 |\ \|z\| = \varepsilon \text{ and } z_1^p + z_2^q + z_3^r = 0\}$$

for $\varepsilon > 0$ small enough.

The next simplest case is the following: if p, q, r are pairwise relatively prime integers, as are p', q', r', then the homology sphere Σ obtained by splicing $\Sigma(p,q,r)$ and $\Sigma(p',q',r')$ along the singular fibers of degrees r and r' is the link of singularity if and only if $rr' > pp'qq'$; see [NW], § 4.

Theorem 1. *Let a homology sphere Σ be the link of an algebraic singularity. If Σ is homology cobordant to zero, then $\bar{\mu}(\Sigma) \geq 0$.*

Proof. Since Σ is an algebraic link, one may assume that Σ is the boundary of a plumbed 4-manifold $P(\Gamma)$ whose intersection form is negative definite. Suppose that Σ bounds a smooth homology ball M. Let us consider the manifold $W = P(\Gamma) \cup_\Sigma (-M)$. This is a smooth closed oriented manifold whose intersection form is naturally isomorphic to the intersection form of $P(\Gamma)$, in particular, is negative definite. By S. Donaldson's Theorem 1 from [D], this form is diagonalizable over the integers.

We use this fact to evaluate $\bar{\mu}(\Sigma)$. Obviously, $\operatorname{sign} P(\Gamma) = -s$, where s is the number of vertices in the graph Γ, so we only need to find the Wu class w. In the standard basis associated with the plumbing, the matrix A of the intersection form of $P(\Gamma)$ takes the form $A = U^t(-E)U$, where $U \in \mathrm{SL}_s(\mathbb{Z})$ and E is the identity matrix. The defining relation (1) translates to

$$w^t U^t (-E) U x \equiv x^t U^t (-E) U x \mod 2 \quad \text{for all } \ x \in H_2(P(\Gamma); \mathbb{Z}),$$

or, equivalently,

$$(Uw)^t (-E) y \equiv y^t (-E) y \mod 2 \quad \text{for all } \ y \in H_2(P(\Gamma); \mathbb{Z}).$$

Therefore, Uw is characteristic for $-E$, in particular, all the coordinates of Uw are odd. Now, we have

$$w.w = w^t U^t (-E) U w = -(Uw)^t (Uw),$$

which is equal to minus the square of the Euclidean length of the vector Uw. Since all the coordinates of Uw are odd, $w.w \leq -s$, and therefore $\bar{\mu}(\Sigma) \geq 0$. □

Corollary 2. *If a plumbed homology sphere Σ is an algebraic link and $\bar{\mu}(\Sigma) < 0$, then Σ has infinite order in the group $\Theta^3_{\mathbb{Z}}$.*

Proof. Let $m\Sigma = \Sigma \# \ldots \# \Sigma$ (m times). Since $\bar{\mu}$ is additive with respect to connected sums, $\bar{\mu}(m\Sigma) = m\bar{\mu}(\Sigma) < 0$, therefore, $m\Sigma$ is not homology cobordant to zero for any m. □

Example. For any relatively prime integers $p, q > 0$, one can easily see that $\bar{\mu}(\,\Sigma(p,q,pq-1)) < 0$. Therefore, all these homology spheres are of infinite order in $\Theta^3_{\mathbb{Z}}$. In particular, the Poincaré homology sphere $\Sigma(2,3,5)$ is of infinite order.

Example. Homology spheres $\Sigma(p,q,pqk-1)$ have infinite order in $\Theta^3_{\mathbb{Z}}$ for any $k \geq 1$. If k is odd, this holds since $\bar{\mu}(\Sigma(p,q,pqk-1))$ equals $\bar{\mu}(\Sigma(p,q,pq-1))$ and is therefore negative. If k is even, then $\bar{\mu}(\Sigma(p,q,pqk-1)) = 0$, and the argument of Theorem 1 cannot be applied directly. Let A be the intersection form of a negative definite plumbing X bounding $\Sigma(p,q,pqk-1)$, k even. We perform a (-1)–surgery on the singular fiber of degree $pqk-1$ to get $\Sigma(p,q,pq(k+1)-1)$. The union of the cobordism X with the trace of this surgery is a plumbed manifold with the intersection form $A\oplus(-1)$. Now, if $\Sigma(p,q,pqk-1)$ bounded an acyclic manifold, A would be diagonalizable, and so would be $A\oplus(-1)$. This contradicts the fact that $\bar\mu(\Sigma(p,q,pq(k+1)-1))<0$. A similar argument shows that none of the multiples of $\Sigma(p,q,pqk-1)$ is homology cobordant to zero.

Many Seifert homology spheres $\Sigma(a_1,\dots,a_n)$ having infinite order in the group $\Theta^3_{\mathbb{Z}}$ by Corollary 2 can also be detected by the R-invariant of R. Fintushel and R. Stern,

$$R(a_1,\dots,a_n)=\frac{2}{a}-3+n+\sum_{i=1}^{n}\frac{2}{a_i}\sum_{k=1}^{a_i-1}\cot\left(\frac{\pi a k}{a_i^2}\right)\cot\left(\frac{\pi k}{a_i}\right)\sin^2\left(\frac{\pi k}{a_i}\right),$$

where $a=a_1\cdot\ldots\cdot a_n$. A theorem in [FS] implies that if $\Sigma(a_1,\dots,a_n)$ bounds a homology ball, then $R(a_1,\dots,a_n)<0$. There are however Seifert fibered homology spheres which are not homology cobordant to zero and which can be detected by $\bar\mu$ and not by R, and vice versa.

Example. Both Seifert spheres $\Sigma(2,11,19)$ and $\Sigma(3,5,7)$ are not homology cobordant to zero. As to the former one, this follows from the fact that $\bar\mu(\Sigma(2,11,19))=-1$ is negative (though $R(2,11,19)=-1$); the latter one has $\bar\mu(\Sigma(3,5,7))=0$ and the result follows from $R(3,5,7)=1$.

The $\bar\mu$-invariant also detects some plumbed homology spheres which have infinite order in $\Theta^3_{\mathbb{Z}}$ but are not Seifert fibered.

Example. Let Σ be the splice of $\Sigma(4,7,9)$ and $\Sigma(2,3,25)$ along the singular fibers of degrees 9 and 25. This manifold is an algebraic link. By using the additivity of the $\bar\mu$–invariant proven in [Du] and [S2], we find that

$$\bar\mu(\Sigma)=\bar\mu(\Sigma(4,7,9))+\bar\mu(\Sigma(2,3,25))=-2+0<0.$$

Therefore, Σ has infinite order in $\Theta^3_{\mathbb{Z}}$ (although the Rohlin invariant $\mu(\Sigma)$ equals 0 modulo 2).

3. More constraints from even plumbing

In 1995, M. Furuta [F] used the Seiberg-Witten invariants to prove the so-called 10/8-conjecture, which says that if A is the intersection form of a smooth closed spin 4-manifold, then $\operatorname{rank} A/|\operatorname{sign} A|>10/8$. In this section we use this result to prove the following theorem.

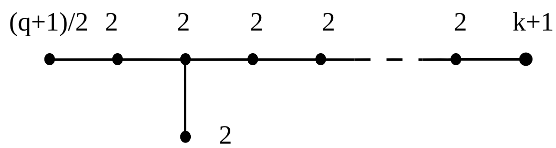
Theorem 2. *Let Σ be a plumbed homology sphere and assume that $\bar\mu(\Sigma)\neq 0$. If Σ bounds a plumbed 4-manifold with even intersection form of rank not exceeding $10\,|\bar\mu(\Sigma)|$, then Σ has infinite order in the group $\Theta^3_{\mathbb{Z}}$.*

Proof. Let $P(\Gamma)$ be a plumbed manifold with boundary $\partial P(\Gamma)=\Sigma$, such that its intersection form $A=A(\Gamma)$ is even and has rank not exceeding $10\,|\bar\mu(\Sigma)|$. Since A is even, its Wu class w vanishes. Therefore, $8\cdot\bar\mu(\Sigma)=\operatorname{sign} A$, and moreover A is isomorphic over the integers to $|\bar\mu(\Sigma)|\cdot E_8\oplus b\cdot H$. Since $\operatorname{rank}(A)\le 10\,|\bar\mu(\Sigma)|$, we get that $\operatorname{rank} A/|\operatorname{sign} A|\le 10/8$.

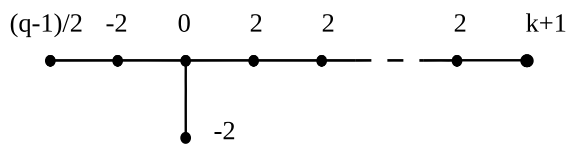
Now, suppose that Σ bounds a smooth homology ball M, and consider the manifold $W = P(\Gamma) \cup_\Sigma (-M)$. This is a smooth closed oriented spin 4-manifold whose intersection form Q is isomorphic to $|\bar{\mu}(\Sigma)| \cdot E_8 \oplus b \cdot H$. Since $\operatorname{rank} Q / |\operatorname{sign} Q| \leq 10/8$, we get a contradiction with the 10/8 conjecture.

The argument can be repeated with Σ replaced by any of its multiples, which proves the theorem. □

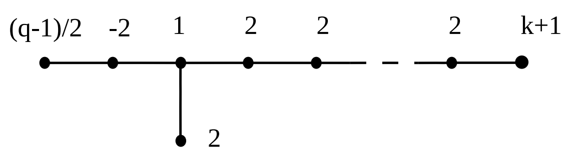
For a complete proof of Corollary 1 formulated in the introduction we refer the reader to [S1], and only prove it here in the simplest case $p = 2$. If $q \equiv 3 \mod 4$, then $\Sigma(2, q, 2qk+1)$, k odd, is the boundary of the 4-manifold obtained by plumbing according to the following diagram:



The intersection form of this manifold is isomorphic to $\frac{q+1}{4} \cdot E_8 \oplus H$. If $q \equiv 1 \mod 4$, the homology sphere $\Sigma(2, q, 2qk+1)$ bounds plumbing



with intersection form $\frac{q-1}{4} \cdot E_8 \oplus 3 \cdot H$. To deal with this case, we need to reduce the rank by surgering out two "hyperbolics" H. This can be done as follows. First, redraw the plumbing diagram:



After two obvious blow-downs we get the manifold in Figure 1 with the intersection form $\frac{q-1}{4} \cdot E_8 \oplus 2 \cdot H$. An equivalent link description is shown in Figure 2. After two more blow-downs we get rid of another "hyperbolic" H. The final link description is shown in Figure 3.

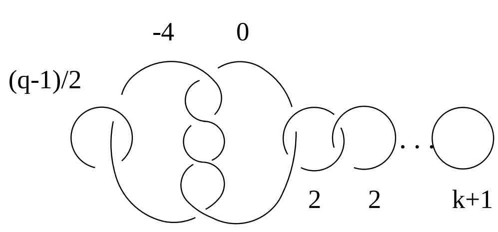


FIGURE 1

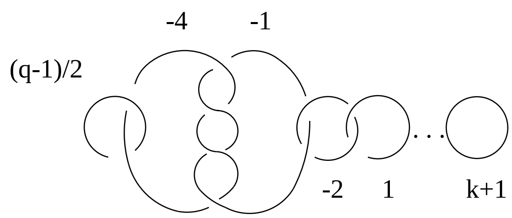


FIGURE 2

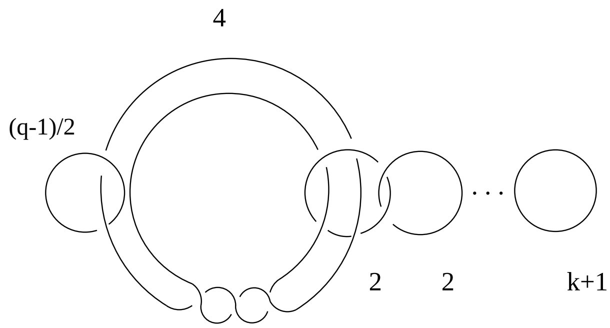


FIGURE 3

The corresponding intersection form is isomorphic to $\frac{q-1}{4} \cdot E_8 \oplus H$, and an argument similar to that in Theorem 2 can be used to show that $\Sigma(2, q, 2qk+1)$, k odd, has infinite order in $\Theta^3_{\mathbb{Z}}$.

Example. In addition to those listed in Corollary 1, there are of course many homology spheres having infinite order in $\Theta^3_{\mathbb{Z}}$ due to Theorem 2. For example, $\Sigma(8, 13, 21)$ is of this sort because it has $\bar{\mu}$-invariant equal 4 and can be obtained by plumbing on an even weighted graph of rank 34.

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