NOTES ON HOMOLOGY COBORDISMS
OF PLUMBED HOMOLOGY 3-SPHERES

NIKOLAI SAVELIEV

(Communicated by Ronald A. Fintushel)

Abstract. The gauge-theoretical invariants of Donaldson and Seiberg-Witten are used to detect some infinite order elements in the homology cobordism group of integral homology 3-spheres.

This paper is concerned with the homology cobordism group $\Theta_3^\mathbb{Z}$ of oriented integral homology 3-spheres. We use S. Donaldson’s [D] and M. Furuta’s [F] (see also [A]) theorems and the $\bar{\mu}$-invariant introduced by W. Neumann [N] and L. Siebenmann [S] to prove the following two theorems.

Theorem 1. Let a homology sphere $\Sigma$ be the link of an algebraic singularity. If $\Sigma$ is homology cobordant to zero, then $\bar{\mu}(\Sigma) \geq 0$.

All Seifert fibered homology spheres are the links of algebraic singularities. Therefore, one can apply this theorem to recover some of the results of R. Fintushel and R. Stern [FS] on homology cobordisms of Seifert fibered homology spheres. For example, it easily follows that for any relatively prime positive integers $p$ and $q$, Seifert fibered homology spheres $\Sigma(p, q, pqk - 1)$, $k \geq 1$, have infinite order in the group $\Theta_3^\mathbb{Z}$. Theorem 1 implies some new results as well; see Section 2.

Theorem 2. Let $\Sigma$ be a plumbed homology sphere and assume that $\bar{\mu}(\Sigma) \neq 0$. If $\Sigma$ bounds a plumbed 4-manifold with even intersection form of rank not exceeding $10|\bar{\mu}(\Sigma)|$, then $\Sigma$ has infinite order in the group $\Theta_3^\mathbb{Z}$.

One can easily see [NR] that any Seifert sphere $\Sigma(a_1, \ldots, a_n)$ with one of the $a_i$’s even bounds an (essentially unique) plumbed manifold with even intersection form. This intersection form is very often, though not always, of the desired rank not exceeding $10|\bar{\mu}|$. Moreover, in many cases the rank can be reduced with the help of Kirby calculus without changing $\bar{\mu}$ so that the conclusion of Theorem 2 still holds. Such an approach leads in particular to the following result [S1].

Corollary 1. For any relatively prime integers $p, q \geq 2$ and any odd integer $k \geq 1$ the Seifert fibered homology sphere $\Sigma(p, q, pqk + 1)$ has infinite order in the group $\Theta_3^\mathbb{Z}$.

This result cannot be extended to $\Sigma(p, q, pqk + 1)$ with even $k$, because, for instance, $\Sigma(2, 3, 13)$ is known to bound a contractible manifold.
It is worth mentioning that in fact $\bar{\mu}(\Sigma)$ vanishes for all known examples of plumbed homology spheres $\Sigma$ homology cobordant to zero. In [N], W. Neumann conjectured that $\bar{\mu}$ is a homology cobordism invariant. The results of this paper give further evidence which weighs for a positive solution to this conjecture.

1. Definition and Basic Properties of the $\bar{\mu}$-Invariant

We recall the definition of the invariant $\bar{\mu}$ by W. Neumann [N]. Note that our definition differs from the original one by a factor of $1/8$.

Let $\Gamma$ be a connected plumbing graph, that is, a connected graph with no cycles, each of whose vertices carries an integer weight $e_i$, $i = 1, \ldots, s$. The matrix $A(\Gamma) = (a_{ij})_{i,j=1,\ldots,s}$ with the entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j, \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge}, \\ 0, & \text{otherwise}, \end{cases}$$

is the intersection matrix of the 4-dimensional manifold $P(\Gamma)$ obtained by plumbing $D^2$-bundles over 2-spheres according to $\Gamma$. This manifold is simply connected.

Disconnected graphs are also allowed. Namely, if $\Gamma = \Gamma_0 + \Gamma_1$ is a disjoint union of $\Gamma_0$ and $\Gamma_1$, then $P(\Gamma)$ is the boundary connected sum $P(\Gamma_0) \# P(\Gamma_1)$.

If $\Gamma$ is a plumbing graph as above, then $M(\Gamma) = \partial P(\Gamma)$ is an integral homology sphere if and only if $\det A(\Gamma) = \pm 1$. For example, all Seifert fibered homology spheres $\Sigma(a_1, \ldots, a_n)$ are of the form $\partial P(\Gamma)$ where $\Gamma$ is a star-shaped graph; see [NR].

If $M(\Gamma)$ is a homology sphere, there is a unique homology class $w \in H_2(P(\Gamma); \mathbb{Z})$ satisfying the following two conditions. First, $w$ is characteristic, that is (dot represents intersection number)

$$w.x \equiv x.x \pmod{2} \quad \text{for all } x \in H_2(P(\Gamma); \mathbb{Z}),$$

and second, all coordinates of $w$ are either 0 or 1 in the natural basis of $H_2(P(\Gamma); \mathbb{Z})$. We call $w$ the integral Wu class for $P(\Gamma)$. It follows from [N] that the integer $\text{sign } P(\Gamma) - w.w$ depends only on $M(\Gamma)$ and not on $\Gamma$. This integer is divisible by 8, so one can define the Neumann-Siebenmann invariant by the formula

$$\bar{\mu}(M(\Gamma)) = \frac{1}{8}(\text{sign } P(\Gamma) - w.w).$$

It is also easily seen that (1) implies that no two coordinates of $w$ corresponding to adjacent vertices in $\Gamma$ can both be 1, so it follows that $w$ is spherical, and the modulo 2 reduction of the $\bar{\mu}$-invariant is the usual Rohlin invariant $\mu$; see [NR].

2. The $\bar{\mu}$-Invariant of Algebraic Links

The plumbed homology spheres have been classified in [EN]. In particular, it has been shown that a plumbed homology sphere $\Sigma$ is an algebraic link if and only if there exists a plumbing graph $\Gamma$ such that the intersection form of the manifold $P(\Gamma)$ with $\Sigma = \partial P(\Gamma)$ is negative definite. The simplest case is the Seifert fibered case: any Seifert fibered homology sphere $\Sigma(a_1, \ldots, a_n)$ is the link of the singularity of $f^{-1}(0)$, where $f : \mathbb{C}^n \to \mathbb{C}^{n-2}$ is a map of the form

$$f(z_1, \ldots, z_n) = \left( \sum_{k=1}^{n} b_{1,k} z_k^{a_1}, \ldots, \sum_{k=1}^{n} b_{n-2,k} z_k^{a_{n-2}} \right)$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
with sufficiently general coefficient matrix \((b_{i,j})\); see [NR]. For instance,
\[
\Sigma(p, q, r) = \{ z \in \mathbb{C}^3 \mid \| z \| = \varepsilon \text{ and } z_1^p + z_2^q + z_3^r = 0 \}
\]
for \(\varepsilon > 0\) small enough.

The next simplest case is the following: if \(p, q, r\) are pairwise relatively prime integers, as are \(p', q', r'\), then the homology sphere \(\Sigma\) obtained by splicing \(\Sigma(p, q, r)\) and \(\Sigma(p', q', r')\) along the singular fibers of degrees \(r\) and \(r'\) is the link of singularity if and only if \(rr' > pp'qq'\); see [NW], § 4.

**Theorem 1.** Let a homology sphere \(\Sigma\) be the link of an algebraic singularity. If \(\Sigma\) is homology cobordant to zero, then \(\bar{\mu}(\Sigma) \geq 0\).

**Proof.** Since \(\Sigma\) is an algebraic link, one may assume that \(\Sigma\) is the boundary of a plumbed 4-manifold \(P(\Gamma)\) whose intersection form is negative definite. Suppose that \(\Sigma\) bounds a smooth homology ball \(M\). Let us consider the manifold \(W = P(\Gamma) \cup_{\Sigma} (-M)\). This is a smooth closed oriented manifold whose intersection form is naturally isomorphic to the intersection form of \(P(\Gamma)\), in particular, is negative definite. By S. Donaldson’s Theorem 1 from [D], this form is diagonalizable over the integers.

We use this fact to evaluate \(\bar{\mu}(\Sigma)\). Obviously, sign \(P(\Gamma) = -s\), where \(s\) is the number of vertices in the graph \(\Gamma\), so we only need to find the Wu class \(w\). In the standard basis associated with the plumbing, the matrix \(A\) of the intersection form of \(P(\Gamma)\) takes the form \(A = U^t(-E)U\), where \(U \in SL_s(\mathbb{Z})\) and \(E\) is the identity matrix. The defining relation (1) translates to
\[
w^tU^t(-E)Ux \equiv x^tU^t(-E)Ux \mod 2 \quad \text{for all } x \in H_2(P(\Gamma); \mathbb{Z}),
\]
or, equivalently,
\[
(Uw)^t(-E)y \equiv y^t(-E)y \mod 2 \quad \text{for all } y \in H_2(P(\Gamma); \mathbb{Z}).
\]
Therefore, \(Uw\) is characteristic for \(-E\), in particular, all the coordinates of \(Uw\) are odd. Now, we have
\[
w.w = w^tU^t(-E)Uw = -(Uw)^t(Uw),
\]
which is equal to minus the square of the Euclidean length of the vector \(Uw\). Since all the coordinates of \(Uw\) are odd, \(w.w \leq -s\), and therefore \(\bar{\mu}(\Sigma) \geq 0\).

**Corollary 2.** If a plumbed homology sphere \(\Sigma\) is an algebraic link and \(\bar{\mu}(\Sigma) < 0\), then \(\Sigma\) has infinite order in the group \(\Theta^3_2\).

**Proof.** Let \(m\Sigma = \Sigma \# \ldots \# \Sigma\) \((m\text{ times})\). Since \(\bar{\mu}\) is additive with respect to connected sums, \(\bar{\mu}(m\Sigma) = m\bar{\mu}(\Sigma) < 0\), therefore, \(m\Sigma\) is not homology cobordant to zero for any \(m\).

**Example.** For any relatively prime integers \(p, q > 0\), one can easily see that \(\bar{\mu}(\Sigma(p, q, pq - 1)) < 0\). Therefore, all these homology spheres are of infinite order in \(\Theta^3_2\). In particular, the Poincaré homology sphere \(\Sigma(2, 3, 5)\) is of infinite order.

**Example.** Homology spheres \(\Sigma(p, q, pqk - 1)\) have infinite order in \(\Theta^3_2\) for any \(k \geq 1\). If \(k\) is odd, this holds since \(\bar{\mu}(\Sigma(p, q, pqk - 1))\) equals \(\bar{\mu}(\Sigma(p, q, pq - 1))\) and is therefore negative. If \(k\) is even, then \(\bar{\mu}(\Sigma(p, q, pqk - 1)) = 0\), and the argument of Theorem 1 cannot be applied directly. Let \(A\) be the intersection form of a negative definite plumbing \(X\) bounding \(\Sigma(p, q, pqk - 1), k\) even. We perform a \((-1)\)-surgery
on the singular fiber of degree \( pqk - 1 \) to get \( \Sigma(p, q, pq(k + 1) - 1) \). The union of the cobordism \( X \) with the trace of this surgery is a plumbed manifold with the intersection form \( A \oplus (-1) \). Now, if \( \Sigma(p, q, pqk - 1) \) bounded an acyclic manifold, \( A \) would be diagonalizable, and so would be \( A \oplus (-1) \). This contradicts the fact that \( \bar{\mu}(\Sigma(p, q, pq(k + 1) - 1)) < 0 \). A similar argument shows that none of the multiples of \( \Sigma(p, q, pqk - 1) \) is homology cobordant to zero.

Many Seifert homology spheres \( \Sigma(a_1, \ldots, a_n) \) having infinite order in the group \( \Theta^3_\mathbb{Z} \) by Corollary 2 can also be detected by the \( R \)-invariant of R. Fintushel and R. Stern,

\[
R(a_1, \ldots, a_n) = \frac{2}{a} - 3 + n + \sum_{i=1}^{n} \frac{a_i}{a} \sum_{k=1}^{a_i-1} \cot \left( \frac{\pi a k}{a_i} \right) \cot \left( \frac{\pi k}{a_i} \right) \sin^2 \left( \frac{\pi k}{a_i} \right),
\]

where \( a = a_1 \cdot \ldots \cdot a_n \). A theorem in [FS] implies that if \( \Sigma(a_1, \ldots, a_n) \) bounds a homology ball, then \( R(a_1, \ldots, a_n) < 0 \). There are however Seifert fibered homology spheres which are not homology cobordant to zero and which can be detected by \( \bar{\mu} \) and not by \( R \), and vice versa.

**Example.** Both Seifert spheres \( \Sigma(2, 11, 19) \) and \( \Sigma(3, 5, 7) \) are not homology cobordant to zero. As to the former one, this follows from the fact that \( \bar{\mu}(\Sigma(2, 11, 19)) = -1 \) is negative (though \( R(2, 11, 19) = -1 \)); the latter one has \( \bar{\mu}(\Sigma(3, 5, 7)) = 0 \) and the result follows from \( R(3, 5, 7) = 1 \).

The \( \bar{\mu} \)-invariant also detects some plumbed homology spheres which have infinite order in \( \Theta^3_\mathbb{Z} \) but are not Seifert fibered.

**Example.** Let \( \Sigma \) be the splice of \( \Sigma(4, 7, 9) \) and \( \Sigma(2, 3, 25) \) along the singular fibers of degrees 9 and 25. This manifold is an algebraic link. By using the additivity of the \( \bar{\mu} \)-invariant proven in [Du] and [S2], we find that

\[
\bar{\mu}(\Sigma) = \bar{\mu}(\Sigma(4, 7, 9)) + \bar{\mu}(\Sigma(2, 3, 25)) = -2 + 0 < 0.
\]

Therefore, \( \Sigma \) has infinite order in \( \Theta^3_\mathbb{Z} \) (although the Rohlin invariant \( \mu(\Sigma) \) equals 0 modulo 2).

### 3. More constraints from even plumbing

In 1995, M. Furuta [F] used the Seiberg-Witten invariants to prove the so-called 10/8-conjecture, which says that if \( A \) is the intersection form of a smooth closed spin 4-manifold, then \( \text{rank } A |\text{sign } A| > 10/8 \). In this section we use this result to prove the following theorem.

**Theorem 2.** Let \( \Sigma \) be a plumbed homology sphere and assume that \( \bar{\mu}(\Sigma) \neq 0 \). If \( \Sigma \) bounds a plumbed 4-manifold with even intersection form of rank not exceeding 10|\( \bar{\mu}(\Sigma) \)|, then \( \Sigma \) has infinite order in the group \( \Theta^3_\mathbb{Z} \).

**Proof.** Let \( P(\Gamma) \) be a plumbed manifold with boundary \( \partial P(\Gamma) = \Sigma \), such that its intersection form \( A = A(\Gamma) \) is even and has rank not exceeding 10|\( \bar{\mu}(\Sigma) \)|. Since \( A \) is even, its Wu class \( w \) vanishes. Therefore, \( 8 \cdot \bar{\mu}(\Sigma) = \text{sign } A \), and moreover \( A \) is isomorphic over the integers to \( |\bar{\mu}(\Sigma)| \cdot E_8 \oplus b \cdot H \). Since \( \text{rank}(A) \leq 10 |\bar{\mu}(\Sigma)| \), we get that \( \text{rank } A / |\text{sign } A| \leq 10/8 \).
Now, suppose that $\Sigma$ bounds a smooth homology ball $M$, and consider the manifold $W = P(\Gamma) \cup_\Sigma (-M)$. This is a smooth closed oriented spin 4-manifold whose intersection form $Q$ is isomorphic to $|\bar{\mu}(\Sigma)|E_8 \oplus bH$. Since rank $Q/|\text{sign } Q| \leq 10/8$, we get a contradiction with the 10/8 conjecture.

The argument can be repeated with $\Sigma$ replaced by any of its multiples, which proves the theorem.

For a complete proof of Corollary 1 formulated in the introduction we refer the reader to [S1], and only prove it here in the simplest case $p = 2$. If $q \equiv 3 \mod 4$, then $\Sigma(2, q, 2qk+1)$, $k$ odd, is the boundary of the 4-manifold obtained by plumbing according to the following diagram:

\[
\begin{array}{cccccccc}
(q+1)/2 & 2 & 2 & 2 & 2 & 2 & \cdots & k+1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\end{array}
\]

The intersection form of this manifold is isomorphic to $\frac{q+1}{4} \cdot E_8 \oplus H$. If $q \equiv 1 \mod 4$, the homology sphere $\Sigma(2, q, 2qk+1)$ bounds plumbing with intersection form $\frac{q-1}{4} \cdot E_8 \oplus 3 \cdot H$. To deal with this case, we need to reduce the rank by surgering out two "hyperbolics" $H$. This can be done as follows. First, redraw the plumbing diagram:

\[
\begin{array}{cccccccc}
(q-1)/2 & -2 & 0 & 2 & 2 & 2 & \cdots & k+1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\end{array}
\]

After two obvious blow-downs we get the manifold in Figure 1 with the intersection form $\frac{q-1}{4} \cdot E_8 \oplus 2 \cdot H$. An equivalent link description is shown in Figure 2. After two more blow-downs we get rid of another "hyperbolic" $H$. The final link description is shown in Figure 3.
The corresponding intersection form is isomorphic to $\frac{q-1}{4} \cdot E_8 \oplus H$, and an argument similar to that in Theorem 2 can be used to show that $\Sigma(2, q, 2qk+1)$, $k$ odd, has infinite order in $\Theta^3_2$.

**Example.** In addition to those listed in Corollary 1, there are of course many homology spheres having infinite order in $\Theta^3_2$ due to Theorem 2. For example, $\Sigma(8, 13, 21)$ is of this sort because it has $\bar{\mu}$-invariant equal 4 and can be obtained by plumbing on an even weighted graph of rank 34.

**References**


Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109  
E-mail address: saveliev@math.lsa.umich.edu