A TOPOLOGY ON LATTICE ORDERED GROUPS

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Abstract. We show that a lattice ordered group can be topologized in a natural way. The topology depends on the choice of a set $C$ of admissible elements ($C$-topology). If a lattice ordered group is 2-divisible and satisfies a version of Archimedes' axiom ($C$-group), then we show that the $C$-topology is Hausdorff. Moreover, we show that a $C$-group with the $C$-topology is a topological group.

In section 1 we recall the definition as well as the elementary properties of lattice ordered groups, especially the properties of the norm $N$ on such groups.

In section 2 we introduce the notion of a set of admissible elements $C$ (definition 2) in a lattice ordered 2-divisible group $A$. It is proved (lemma 2) that the sets $U_{x_0, r} = \{ x \in A : r - N(x - x_0) \in C \}$ constitute a base of a topology on $A$ (the $C$-topology).

In section 3 we describe the properties of the $C$-topology on lattice ordered 2-divisible $C$-Archimedean groups (such groups are called $C$-groups). It is proved that $C$-groups are Hausdorff topological groups (theorem 1 and theorem 2).

1. Lattice ordered groups

A lattice ordered group [1, Ch. VI, §§8, 9] is an ordered group $A$ such that there exist $\text{sup}(x, y)$ and $\text{inf}(x, y)$, for every $x, y \in A$. Note that:

\[
\text{inf}(x, y) = -\text{sup}(-x, -y)
\]

Definition 1 ([1, VI, definition 4]). The norm $N$ on a lattice ordered group $A$ is the function $N : A \rightarrow A$ defined by $N(x) = \text{sup}(x, -x)$.

Lemma 1. Let $N$ be the norm on a lattice ordered group $A$. Then:

(i) $N(x) = \text{sup}(x, 0) - \text{inf}(x, 0)$ for every $x, y \in A$.
(ii) $N(x) = x$ if and only if $x \geq 0$. In particular, $N(N(x)) = N(x)$, for every $x \in A$.
(iii) $N(x) \geq 0$, for every $x \in A$.
(iv) $(N(x) = 0) \Leftrightarrow (x = 0)$, for every $x \in A$.
(v) $N(mx) = |m|N(x)$, for every $x \in A$ and for every $m \in \mathbb{Z}$.
(vi) $N(x + y) \leq N(x) + N(y)$, for all $x, y \in A$.

Proof. See [1, VI, Proposition 9 and corollary 4 of Proposition 11].
Directly from definition 1 we conclude that:

\[ N(x) \leq \epsilon \iff -\epsilon \leq x \leq \epsilon \]

for all \( \epsilon \geq 0, x \in A \). In particular, \( -N(x) \leq x \leq N(x) \), for every \( x \in A \).

2. The C-topology

In this section we suppose that \( A \) is a lattice ordered 2-divisible group. Note that in a lattice ordered group there is no nontrivial torsion element. Namely, \( nx = 0 \), for \( n \in \mathbb{N} \), implies \( nN(x) = 0 \) or equivalently \( (n-1)N(x) = -N(x) \); hence \( N(x) = 0 \), and so \( x = 0 \) (lemma 1 (iii), (iv) and (v)). Therefore, \( \frac{1}{x} \) is uniquely determined, for every \( x \in A \). It is easily seen (by lemma 1, (ii) and (v)) that \( x \geq 0 \) implies \( \frac{x}{2} \geq 0 \).

**Definition 2.** A set of admissible elements in a lattice ordered 2-divisible group \( A \) is any nonempty subset \( C \) of the set \( A^+ \) of all positive elements having the following properties:

(i) \( 0 \notin C \),

(ii) \( (x \in C \land y \geq x) \Rightarrow (y \in C) \),

(iii) \( (x, y \in C) \Rightarrow (\inf(x, y) \in C) \),

(iv) \( (x \in C) \Rightarrow (\frac{x}{2} \in C) \).

It is obvious that \( C \subseteq A^+ \setminus \{0\} \). If there exist at least two coprime elements in the group, then the given inclusion is strict (by (iii), (i) of definition 2 and the fact that coprime elements are necessarily positive). Recall that \( x, y \in A \) are coprime if \( \inf(x, y) = 0 \) ([1, V, definition 5]).

**Remark.** Suppose that \( a \) is a strictly positive element of \( A \). Put \( A_a = \{ x \in A : x \geq a \} \). Then \( A_a \) satisfies (i), (ii), (iii) of definition 2. Denote \( A_{a, n} = \frac{1}{n} A_a \), for \( n \in \mathbb{N} \). Then we have \( A_{a, n+1} \supseteq A_{a, n} \), for every \( n \in \mathbb{N} \). Put \( C = \bigcup_{n \in \mathbb{N}} A_{a, n} \). Then \( C \) is a set of admissible elements. This is the minimal set of admissible elements containing \( a \).

Recall that the open ball in a normed space is defined by the relation \( r - |x - x_0| > 0 \). In our case we have

**Definition 3.** Let \( A \) be a lattice ordered 2-divisible group and let \( C \) be a set of admissible elements of \( A \). The open \( C \)-ball of radius \( r \in C \), with the centre \( x_0 \in A \), is the set of all \( x \in A \) such that \( r - N(x - x_0) \in C \). We denote this set by \( U_{x_0,r} \).

**Lemma 2.** Let \( A \) be a lattice ordered 2-divisible group. Then open \( C \)-balls constitute a base of a topology on \( A \) (we called this topology the \( C \)-topology).

**Proof.** Since \( x_0 \in U_{x_0,r} \), for all \( r \in C \), we get that open \( C \)-balls form an open cover of the space \( A \). Let \( z_0 \in U_{x_0,r} \cap U_{y_0,R} \) be arbitrary. By the definition of \( C \)-balls, we have \( r - N(z_0 - x_0) = c_1 \) and \( R - N(z_0 - y_0) = c_2 \), for some \( c_1, c_2 \in C \). Then \( U_{x_0,r} \subseteq U_{x_0,r} \cap U_{y_0,R} \), for \( \epsilon = \inf(c_1, c_2) \). Namely, for \( x \in U_{x_0,r} \), we have \( \epsilon - N(x-x_0) = c_3 \), for some \( c_3 \in C \). Applying the triangle inequality (lemma 1, (vi)), we get \( r - N(x-x_0) \geq r - N(x_0 - z_0) + \epsilon - N(z_0 - x) - \epsilon = c_1 + c_3 - \epsilon \in C \). Similar reasoning holds for the second ball. This completes the proof.

**Remark.** Lemma 2 is valid even if \( A \) is not necessarily 2-divisible. Hence \( C \) does not need to satisfy (iv) of definition 2.

It is easy to see that if \( V_{x_0,r} = \{ x \in A : N(x-x_0) < r \} \) and \( F_{x_0,r} = \{ x \in A : N(x-x_0) \leq r \} \), then we have \( U_{x_0,r} \subseteq V_{x_0,r} \subseteq F_{x_0,r} \).
Example 1. Let $A = \mathbb{R}^2$ with the relation of order defined by $(a, b) \leq (c, d) \iff (a \leq c \land b \leq d)$. Then $A$ is a divisible lattice ordered group such that

$$\sup ((a, b), (c, d)) = (\sup(a, c), \sup(b, d)), \quad A^+ = \{(a, b) : a \geq 0 \land b \geq 0\},$$

and $C = \{(a, b) : a > 0 \land b > 0\}$ is the set of admissible elements. It is easy to see that:

$$U_{(x_0, y_0), r} = \{(x, y) \in \mathbb{R}^2 : -r < x - x_0 < r \land -r < y - y_0 < r\};$$

$$F_{(x_0, y_0), r} = \{(x, y) \in \mathbb{R}^2 : -r \leq x - x_0 \leq r \land -r \leq y - y_0 \leq r\};$$

$$V_{(x_0, y_0), r} = F_{(x_0, y_0), r} \setminus \{(x_0 + r, y_0 + r), (x_0 + r, y_0 - r), (x_0 - r, y_0 + r), (x_0 - r, y_0 - r)\}.$$

Thus, the $C$-topology on $A$ is equivalent to the standard topology on $\mathbb{R}^2$.

It can be shown that the set $\{(x, y) \in \mathbb{R}^2 : (x \geq 0) \land (y > 0)\}$ is a set of admissible elements, too. Of course, in this case the corresponding $C$-topology is not equivalent to the previous one.

Note that the norm $N$ is a continuous function with respect to the $C$-topology. Namely, by the triangle inequality, we have $-N(x - y) \leq N x - N y \leq N(x - y)$, so by (2), $N(N x - N y) \leq N(x - y)$.

Lemma 3. Let $A$ be a lattice ordered 2-divisible group, and let $C$ be a set of admissible elements. Then:

(i) $C$ is an open set in the $C$-topology.

(ii) $A = C - C$.

Proof. (i) The inequality $N(x - c) \leq \frac{c}{2}$ is, by (2), equivalent to the inequalities $\frac{c}{2} \leq x \leq \frac{3c}{2}$, for every $c \in C$. Hence, $U_{c, \frac{c}{2}} \subseteq C$.

(ii) By lemma 1, (i) and (1), we have

$$x = \sup(x, 0) - \sup(-x, 0) = (\sup(x, 0) + c) - (\sup(-x, 0) + c)$$

for all $x \in A, c \in C$. 

3. $C$-groups

In this section we will assume that $A$ is a lattice ordered 2-divisible $C$-Archimedean group. This means that in the group $A$ the following version of Archimedes’ axiom holds:

$$\forall x \in C \forall y \in C \exists n \in \mathbb{N} (n \cdot y > x).$$

One can see that (3) is equivalent to

$$\forall x \geq 0 \forall y \in C (\exists n \in \mathbb{N} (n \cdot y > x)).$$

A consequence of the given assumption is that $U \cap C \neq \emptyset$, for every neighbourhood $U$ of zero. Namely, if $U$ is the open $C$-disc around zero of radius $y$ and if $x \in C$ is arbitrary, then there exists $n \in \mathbb{N}$ such that $y - \frac{y}{2n} \in C$. Hence $\frac{y}{2n} \in U \cap C$.

Definition 4. We say that $A$ is a $C$-group if $A$ is a lattice ordered, 2-divisible, $C$-Archimedean group.

Example 2. Let $A$ be $\mathbb{R}^2$ as in example 1. If we choose $C = \{(x, y) : x > 0, y > 0\}$, then $A$ becomes a $C$-group. If we choose $C = \{(x, y) : x \geq 0, y > 0\}$, then $A$ is not a $C$-group.
Example 3. The group $\mathbb{Q}_2$ of all dyadic numbers, with the standard ordering, is a $C$-group ($C$ is the set of strictly positive dyadic numbers). The closure $\text{Cl}\mathbb{Q}_2$ is the additive group of real numbers. Note that every $C$-group is a module over the dyadic numbers.

Lemma 4. Let $A$ be a $C$-group. Then for all $x \in A$ and $c \in C$ there exists $n \in \mathbb{N}$ such that $\frac{1}{2^n} + c \in C$.

Proof. Let $c \in C$, $x \in A$. Then, by lemma 3 (ii), there exist $c_1, c_2 \in C$ such that $x = c_1 - c_2$. Hence, $\frac{c}{2^n} = \frac{c_1}{2^n} - \frac{c_2}{2^n}$, for every $n \in \mathbb{N}$. If we choose $n$ such that $c - \frac{c_2}{2^n} > 0$ (this is possible because the group $A$ is $C$-Archimedean), then we have $\frac{1}{2^n} + c = \frac{c_1}{2^n} + (c - \frac{c_2}{2^n}) \in C$.

Recall that, by the definition, $x \in A$ is a limit of the sequence $(x_n)$ if the following holds: $(\forall c \in C)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) (c - N(x - x_n) \in C)$.

Lemma 5. (i) Let $F \subseteq A$. Then the closure of $F$ is $\text{Cl}F = \{\lim x_n : x_n \in F\}$.

(ii) $A^+ = \text{Cl}C$.

Proof. (i) Let the set of all limits of the sequences with terms from $F$ be denoted by $X$.

Suppose that $x \in X$. Then every open neighbourhood of $x$ cuts $F$. Therefore, $X \subseteq \text{Cl}F$. Suppose that $x \in \text{Cl}F$ and that $c \in C$. Let $U_n$ be an open $C$-disc around $x$ with radius $\frac{1}{2^n}$. By definition, $U_n \cap F \neq \emptyset$, for every $n \in \mathbb{N}$. Choose $x_n \in U_n \cap F$, for every $n \in \mathbb{N}$. Then $x = \lim x_n$. Therefore $\text{Cl}F \subseteq X$. (ii) Suppose that $x \in A^+$. Choose an arbitrary $c \in C$. Then $\lim(x + \frac{1}{2^n}) = x$ and $x + \frac{1}{2^n} \in C$ for all $n \in \mathbb{N}$. Hence, $x \in \text{Cl}C$.

Suppose that $x \in \text{Cl}C$. Then $x = \lim x_n$, for $x_n \in C$. Applying the continuity of the norm $N$ and lemma 1 (ii), we get $N(x_n) = \lim(Nx_n) = \lim x_n = x$. Hence, $x \geq 0$.

Lemma 6. Suppose that $x \geq 0$ and that $x < c$, for every $c \in C$. Then $x = 0$.

Proof. Let $x = \lim x_n, x_n \in C$ for all $n \in \mathbb{N}$ (this is possible by lemma 5, (ii)). Therefore, $N(x - x_n) < \epsilon$, for every sufficiently large $n \in \mathbb{N}$, and for arbitrary $\epsilon \in C$. Hence, $-\epsilon < x - x_n < \epsilon$, and so, $x_n < x + \epsilon < 2\epsilon$, for every sufficiently large $n \in \mathbb{N}$. Thus, $(x_n)$ converges to zero; hence, $x = 0$.

Theorem 1. $A$ is a Hausdorff space.

Proof. Suppose that $x, y \in A$ and $x \neq y$. If $N(x - y) = \epsilon \in C$, then $\frac{1}{\epsilon}$-neighbourhoods around $x$ and around $y$ are disjoint. Suppose that $N(x - y) \in A^+ \setminus C$. After a translation we can assume that $y = 0$ and $Nx \in A^+ \setminus C$ (note that $Nx \neq 0$). If we show that $0$ and $Nx$ can be separated by open $C$-balls, then we can conclude that $0$ and $x$ can be separated, too. Namely, if $U$ is an open $C$-ball around $0$ and $W$ an open $C$-ball around $Nx$ which are disjoint, then $U$ is disjoint from an open $C$-ball $V$ around $x$ such that $NV \subseteq W$ ($V$ exists because $N$ is a continuous function). If not, there exists $z \in U \cap V$. Then we get $Nz \in N(U \cap V) \subseteq NU \cap NV \subseteq U \cap V$, a contradiction (note that by the definition of $C$-balls we have $NU \subseteq U$).

Therefore, we can suppose that $x \in A^+ \setminus C$ and $x \neq 0$. It can be easily seen that $0$ can be separated from $x$. If not, we get that $x \in U$, for every open neighbourhood $U$ around zero. Applying lemma 6, we conclude that $x = 0$ (a contradiction). Let’s
prove that $x$ can be separated from $0$. Choose $U = U_\epsilon$ an open $C$-ball of radius $\epsilon$ around zero, $\epsilon \in C$, such that $x \notin U_{2\epsilon}$. Then $x \in V = U_{x+\frac{\epsilon}{2}}$. We claim that $U \cap V = \emptyset$. If not, there exists $z \in U \cap V$; hence, $Nz < \epsilon$ and $N(z - x - \frac{\epsilon}{2}) < \epsilon$. Therefore, $x + \frac{\epsilon}{2} = N(x + \frac{\epsilon}{2}) \leq N(x + \frac{\epsilon}{2} - z) + Nz < 2\epsilon$; hence, $x < \frac{4\epsilon}{2}$. This contradicts the assumption $x \notin U_{2\epsilon}$. The theorem is proved.

**Theorem 2.** Every $C$-group is a topological group.

**Proof.** It is easy to see that the mapping $A \to A$, $x \mapsto -x$ is continuous. We have to show that the mapping $f : A \times A \to A$, $(x, y) \mapsto x + y$ is continuous, too. Let $U$ be an open $C$-ball of radius $\epsilon$ around $x_0 + y_0$, and let $V$ and $W$ be open $C$-balls of radius $\frac{\epsilon}{2}$ around $x_0$ and $y_0$, respectively. Then $V \times W$ is an open neighbourhood around $(x_0, y_0)$. Take $(x, y) = z \in V \times W$. Then $\epsilon - N(x_0 + y_0 - (x + y)) \geq \frac{\epsilon}{2} - N(x_0 - x) + \frac{\epsilon}{2} - N(y_0 - y) \in C$. Since $(x, y)$ can be chosen arbitrarily, we have $f(U \times V) \subset U$, so the continuity is proved. According to theorem 1, $A$ is a Hausdorff space. Therefore, $A$ is a topological group. □

**Concluding remarks**

One can define an analogue of Cauchy sequence in a $C$-group ($C$-Cauchy sequence). It can be shown in a standard manner (but not so easily) that every $C$-group can be “completed”. The “completion” $\hat{A}$ of a $C$-group $A$ is a $C$-group with the properties:

(i) $A$ is dense in $\hat{A}$,

(ii) $\hat{A}$ is $C$-complete (every $C$-Cauchy sequence with terms from $\hat{A}$ has limit in $\hat{A}$).

Moreover, it can be shown that $\hat{A}$ has a structure of ordered linear real space with semilinear topology. Such spaces are of special interest (see, for example [2]).

**References**


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