CONTINUITY OF LIE MAPPINGS OF THE SKEW ELEMENTS OF BANACH ALGEBRAS WITH INVOLUTION

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Abstract. Let $A$ and $B$ be centrally closed prime complex Banach algebras with linear involution. If $A$ is semisimple, then any Lie derivation of the skew elements of $A$ is continuous and any Lie isomorphism from the skew elements of $B$ onto the skew elements of $A$ is continuous.

The Lie product $[a, b] = ab - ba$ induces on any Banach algebra $A$ a Lie structure of great interest for their intimate connections with the geometry of manifolds modeled on Banach spaces. In case $A$ has a linear involution $^*$, then the skew elements are the linear subspace $K_A = \{a \in A : a^* = -a\}$ which is a Lie subalgebra of $A$. A Lie derivation of $K_A$ is a linear mapping $d$ from $K_A$ to itself which satisfies $d([a, b]) = [d(a), b] + [a, d(b)]$ for all $a, b \in K_A$. If $B$ is another Banach algebra with linear involution, then a Lie isomorphism from $K_B$ onto $K_A$ is a linear bijection $\phi$ from $K_B$ onto $K_A$ satisfying $\phi([a, b]) = [\phi(a), \phi(b)]$ for all $a, b \in K_B$.

Examples 1. Let $H$ be a complex Hilbert space. Let $L(H)$ denote the primitive $C^*$-algebra of all continuous linear operators on $H$, and for each $a \in L(H)$, let $a^*$ denote the usual adjoint operator of $a$.

1. If $J$ is a conjugation of $H$, then it is easy to check that the mapping $^*$ from $L(H)$ to itself defined by $a^* = Ja^*J$ is a linear involution on $L(H)$. If $J$ is an anticonjugation of $H$, then the mapping $a^* = -Ja^*J$ is a linear involution on $L(H)$. The skew elements relative to the preceding involutions are classical complex Banach-Lie algebras of bounded operators (see [3]).

2. Let us denote by $C_\infty$ the set of all compact linear operators on $H$ and let $\|\cdot\|_\infty$ be the usual operator norm. For $1 \leq p < \infty$, let $C_p$ denote the usual class of those compact linear operators $a$ on $H$ for which $\|a\|_p = (\sum_{n=1}^{\infty} \mu_n^p)^{1/p} < \infty$, where $\{\mu_n\}$ is the sequence of eigenvalues of the operator $(a^*a)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. According to [2, Lemmas XI.9, XI.10, and XI.14], $C_p$ is a two-sided ideal of $L(H)$ which becomes a complex Banach algebra for the norm $\|\cdot\|_p$. Since $C_p$ contains all the continuous linear operators with finite-dimensional range, we deduce that $C_p$ is primitive. The involutions introduced in the preceding example leave invariant $C_p$, and their skew elements are classical complex Banach-Lie algebras of compact operators (see [3]).

It was proved in [3] that Lie derivations and Lie $^*$-automorphisms of all the preceding Banach-Lie algebras are continuous.

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The purpose of this paper is to prove the following results.

**Theorem 1.** Let $A$ be a centrally closed prime semisimple complex Banach algebra with linear involution, and let $d$ be a Lie derivation of $K_A$. Then $d$ is continuous.

**Theorem 2.** Let $A$ and $B$ be centrally closed prime complex Banach algebras with linear involution, and assume that $A$ is in addition semisimple. If $\phi$ is a Lie isomorphism from $K_B$ onto $K_A$, then $\phi$ is continuous.

We recall that a prime algebra $A$ is centrally closed if any linear mapping $f$ from a two-sided ideal $I$ of $A$ to $A$ satisfying $f(ab) = af(b)$ and $f(ba) = f(b)a$ for all $a \in A$ and $b \in I$ is a multiple of the identity operator.

**Examples 2.**
1. Any primitive complex Banach algebra is centrally closed (see [6, Theorem 12]).
2. [7, Proposition 2.5] shows that prime $C^*$-algebras are centrally closed.

From now on we assume $A$ and $B$ to be Banach algebras satisfying the requirements in Theorems 1 and 2, $d$ stands for a Lie derivation of $K_A$, and $\phi$ denotes a Lie isomorphism from $K_B$ onto $K_A$.

It should be noted that the involutions are the identity on the center of $A$ and $B$ and consequently they are involutions of the first kind. On account of either [1, Remark 3] or [9, Remark 1.3] (both of them based on the proof of [4, Theorem 2.2]), we deduce that the subalgebras $\langle K_A \rangle$ of $A$ and $\langle K_B \rangle$ of $B$ generated by $K_A$ and $K_B$, respectively, contain nonzero two-sided ideals $I_A$ of $A$ and $I_B$ of $B$, respectively.

**Proof of Theorem 1.** If $A$ has finite dimension, then $d$ is continuous.

Assume that $A$ has infinite dimension. From [9, Theorem 1.1] we deduce that $d$ can be extended to an ordinary linear derivation $D$ of $\langle K_A \rangle$. Since $A$ is prime, it follows that $I_A$ is an essential ideal of $A$, i.e. $I_A \cap J \neq 0$ whenever $J$ is a nonzero two-sided ideal of $A$. Consequently, the restriction of $D$ to $I_A$ is an essentially defined derivation of $A$ in the sense of [10] and therefore it is closable. Let $\{a_n\}$ be a sequence in $K_A$ satisfying $\lim a_n = 0$ and $\lim d(a_n) = a$ for some $a \in K$. For any $b \in I_A$, $\{a_nb\}$ is a sequence in $I_A$ converging to zero and

$$\lim D(a_nb) = \lim (d(a_n)b + a_nD(b)) = ab.$$ 

Hence $ab = 0$ for all $b \in I_A$. Since $A$ is prime, we conclude that $a = 0$. From Johnson's uniqueness-of-norm theorem [5] we deduce that the involution of $A$ is continuous. Accordingly, $K_A$ is closed in $A$. The closed graph theorem now shows that $d$ is continuous.

**Proof of Theorem 2.** If $K_B$ has finite dimensions, then $\phi$ is continuous.

If $K_B$ is infinite-dimensional, then so is $K_A$ and therefore $A$ and $B$ have infinite dimension. From [1, Theorem 3] we deduce that $\phi$ can be extended to an ordinary linear isomorphism $\Phi$ from $\langle K_B \rangle$ onto $\langle K_A \rangle$.

We claim that $B$ is semisimple. Let $\text{Rad}(C)$ stand for the Jacobson radical of any subalgebra $C$ of either $A$ or $B$. Since $I_A$ is a two-sided ideal of both $A$ and $\langle K_A \rangle$, it follows that

$$0 = \text{Rad}(A) \cap I_A = \text{Rad}(I_A) = \text{Rad}(\langle K_A \rangle) \cap I_A.$$ 

Since $A$ is prime we conclude that $\text{Rad}(\langle K_A \rangle) = 0$. As $\langle K_B \rangle$ is isomorphic to $\langle K_A \rangle$ we deduce that $\text{Rad}(\langle K_B \rangle) = 0$. On the other hand,

$$0 = \text{Rad}(\langle K_B \rangle) \cap I_B = \text{Rad}(I_B) = \text{Rad}(B) \cap K_B.$$
which shows that $\text{Rad}(B) = 0$, as claimed.

As in the preceding proof we deduce that $K_B$ is closed in $B$.

Let $\{a_n\}$ be a sequence in $I_B$ such that $\lim a_n = 0$ and $\lim \Phi a_n = \Phi a$ for some $a \in I_B$. We claim that $r(\Phi a) = 0$. To this end we follow the pattern established in [8]. For any element $c$ in either $A$ or $B$ let $r(c)$ denote its spectral radius. Moreover, if $C$ is a subalgebra of either $A$ or $B$ containing $c$, then we will denote by $\text{Sp}(c, C)$ the spectrum of $c$ in $C$. For all $n \in \mathbb{N}$ and $z \in \mathbb{C}$ set $p_n(z) = z\Phi(a_n) + (\Phi(a) - \Phi(a_n))$ and note that

$$r(p_n(z)) \leq ||p_n(z)|| \leq |z| ||\Phi(a_n)|| + ||\Phi(a) - \Phi(a_n)||.$$ 

On the other hand, it is immediate that

$$\text{Sp}(\Phi(za_n + (a - a_n)), A) \subset \text{Sp}(za_n + (a - a_n), I_B).$$

Since $I_B$ is a two-sided ideal of $B$ we see that

$$\text{Sp}(za_n + (a - a_n), I_B) = \text{Sp}(za_n + (a - a_n), B)$$

and consequently

$$r(p_n(z)) = r(\Phi(za_n + (a - a_n))) \leq r(za_n + (a - a_n))$$

$$\leq |z| ||a_n|| + |a - a_n|.$$ 

The rest of the proof of our claim runs as in [8].

Let $\{a_n\}$ be a sequence in $K_B$ such that $\lim a_n = 0$ and $\lim \phi(a_n) = b$ for some $b \in K_A$. Choose $a \in K_B$ such that $\phi(a) = b$. For all $a' \in I_B$ and $c \in I_A$, $\{a_n a' \Phi^{-1} c\}$ is a sequence in $I_B$ converging to zero and $\lim \Phi(a_n a' \Phi^{-1} c) = \Phi(aa' \Phi^{-1} c)$. According to the above claim, we have $r(\Phi(aa')c) = 0$. Consequently, $\Phi(aa')I_A$ lies in the radical of $A$ and hence $\Phi(aa') = 0$. From this we conclude that $aI_B = 0$, which shows that $a = 0$ and therefore $b = \phi(a) = 0$. The closed graph theorem shows that $\phi$ is continuous. 

On account of Examples 2.1 and 2.2, our theorems now yield the following results.

**Corollary 1.** Lie derivations and Lie automorphisms of the skew elements of a primitive complex Banach algebra with linear involution are continuous.

**Corollary 2.** Lie derivations and Lie automorphisms of the skew elements of a prime $C^*$-algebra with linear involution are continuous.

**References**


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