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UNIQUENESS IN THE CAUCHY PROBLEMS FOR HIGHER ORDER ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract. In this note, we study the uniqueness in Cauchy problems for a class of higher order elliptic differential operators with Lipschitz coefficients. In particular, we prove the uniqueness under assuming the potentials being $L^{r_j}_{\text{loc}}$ with certain correct numbers r_j's.

Notation. Let Ω be a domain in R^d. Suppose $P(x,D) = \sum_{|\alpha|=m} a_\alpha(x)D^\alpha$ is a differential operator of degree m with real functions $a_\alpha(x)$ on Ω. We denote by $P = P(x, \cdot + ik)$ the symbol of $P(x,D)$ and by $N^P_{(x,k)}$ the zero set of $P(x, \cdot + ik)$ for any $(x,k) \in \Omega \times R^d$. Let's define a subset in $\Omega \times S^{d-1}$:

$$\Sigma_P = \{(x,k) \in \Omega \times S^{d-1} : \sum \frac{dP}{dz_j}(x, \xi + ik) \cdot k_j \neq 0, \qquad \det\mathrm{Hess}_C P(x, \xi + ik) \neq 0 \ \forall \xi \in N^P_{(x,k)}\}$$

where $\mathrm{Hess}_C P = \left(\frac{d^2P}{dz_j dz_l}\right)$ is the complex Hessian matrix of P, and $z = \xi + ik \in C^d$.

If u is a function on Ω, we define its normal support $N(\mathrm{supp} u)$ as a subset of $\Omega \times S^{d-1}$. Say $(x,k) \in N(\mathrm{supp} u)$ if there is a neighborhood V of x such that $\psi(y) \leq \psi(x)$ for all $y \in V \cap \mathrm{supp} u$ and $d\psi(x) = \pm k$, where ψ is some smooth function.

Let $s = \frac{2(d+1)}{d+3}$ be the restriction number and s' be its conjugate number. We let $W^{m,2}$ be the Sobolev space of functions whose derivatives up to order m belong to L^2. We have the following theorem.

Theorem. *Suppose $P(x,D)$ is an elliptic differential operator with real Lipschitz functions a_α as coefficients on Ω and is of order $m < \frac{d}{s}$. If a function $u \in W^{m,2}_{\text{loc}}(\Omega)$ satisfies*

$$|Pu(x)| \leq \sum_{0<\mu\leq m} V_\mu | \bigtriangledown^{m-\mu} u| \tag{1}$$

with $V_\mu \in L^{\frac{d}{\mu}}_{\text{loc}}(\Omega)$, then $N(\mathrm{supp} u) \subset \Sigma^c_P$.

Remarks. (1) Actually we will prove that $N(\mathrm{supp} u) \subset \Lambda^c_P$ where Λ_P is the set of $(x,k) \in \Omega \times S^{d-1}$ such that $N^P_{(x,k)}$ is locally contained in a smooth hypersurface with nonzero Gaussian curvature, which is smaller than Σ_P. In other words, we

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may replace the assumptions in Σ_P by directly assuming some curvature condition for $N^P_{x,k}$. Σ_P is a natural condition and is easy to verify. But the proof of $\Sigma_P \subset \Lambda_P$ is nontrivial which is essentially shown in Lemma 1 below. For more details, see [3].

(2) When coefficients are constants, this theorem was proved by the author in [3]. When $P(x,D)$ is hyperbolic, under some other curvature assumption for $N^P_{x,k}$, Sogge proves the same result in the case where $V_\mu = 0$ for all $\mu \le m-1$; see [2]. In general, if we don't care about the optimal condition for the potentials, this is an old theorem by Calderon. See [1], [4].

Calderon's theorem is actually equivalent to the following uniqueness theorem in the Cauchy problem.

Theorem 1. *Suppose $P(x,D)$ is an elliptic differential operator with real Lipschitz functions a_α as coefficients on a domain Ω which contains $R^d \setminus B(-e_d, \frac{1}{2})$ and satisfies the conditions*

$$\frac{dP}{dz_d}(0, \xi + ie_d) \neq 0,$$

$$\det \mathrm{Hess}_C P(0, \xi + ie_d) \neq 0$$

for all $\xi \in N^P_{(0,e_d)}$, where $\mathrm{Hess}_C P$ is the complex Hessian matrix of P. Then for any function $u \in W^{m,2}_{\mathrm{loc}}(\Omega)$ satisfying (1) for some $V_\mu \in L^{\frac{d}{\mu}}_{\mathrm{loc}}(\Omega)$, u vanishes in a neighborhood of 0 if u vanishes outside $B(-e_d, 1)$.

Let's first prove our Theorem by assuming Theorem 1.

Proof of the Theorem. Let $(x^0, k^0) \in N(\mathrm{supp} u)$. Suppose $(x^0, k^0) \in \Sigma_P$. By the definition of $N(\mathrm{supp} u)$, there is a little ball B such that $x^0 \in \partial B$ and $u = 0$ in B. Then there is a map F which is the composition of translation, rotation, dilation and Kelvin transformation with respect to x^0 and B such that $F(x^0) = 0$ and $F(k^0) = e_d$. Moreover $u \circ F^{-1} = 0$ outside $B(-e_d, 1)$ and $u \circ F^{-1}$ is defined on a domain Ω which contains $R^d \setminus B(-e_d, \frac{1}{2})$. Let $v(y) = u \circ F^{-1}(y)$. Then v satisfies the following differential inequality by (1):

$$|Q(y,D)v(y)| \le \sum_{0<\mu\le m} V^1_\mu(y) | \bigtriangledown^{m-\mu} v(y)|$$

where $Q(y,\eta) = P(F^{-1}(y), ({}^tDF^{-1}(y))^{-1}\eta)$ and $V^1_\mu(y) = V_\mu \circ F^{-1}(y)$ plus some bounded functions. So one may check that $(0, e_d) \in \Sigma_Q$ which means the assumptions in Theorem 1 are satisfied. So applying Theorem 1 to Q and v, we have $v = 0$ in a neighborhood of 0. Pull back v to u by F. We have $u = 0$ in a neighborhood of x^0. This is a contradiction with $x^0 \in \mathrm{supp} u$.

In order to prove Theorem 1, we need several lemmas. Let's first study the differential operator with real constants coefficients. We denote by A the vector $(a_\alpha)_{|\alpha|=m} \in R^M$ for some number M determined by m and $P_A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$, and denote by $N_{(A,k)}$ the zero set of $P_A(\cdot + ik)$. We are always interested in the case that P_A is elliptic. Let's introduce some functions as follows:

$$S(A, \xi, k) = \left| \sum_j \frac{dP_A}{dz_j}(\xi + ik)k_j \right|,$$

$$H(A,\xi,k) = \left|\det\left(\frac{d^2P_A}{dz_jdz_l}(\xi+ik)\right)\right|,$$

$$L(A,\xi,k) = \sum_{(j,l)}\left|\det\begin{pmatrix} \frac{d\mathrm{re}P_A}{d\xi_j}(\xi+ik) & \frac{d\mathrm{re}P_A}{d\xi_l}(\xi+ik) \\ \frac{d\mathrm{im}P_A}{d\xi_j}(\xi+ik) & \frac{d\mathrm{im}P_A}{d\xi_l}(\xi+ik)\end{pmatrix}\right|.$$

We notice that the assumption in Theorem 1 says that when $A = (a_\alpha(0))$ and $k = e_d$, the first two of the above functions are positive on $N^P_{(0,e_d)}$. By the Cauchy-Riemann equation and the transversality theorem, we proved that $L(A,\xi,k)$ is also positive on $N^P_{(0,e_d)}$. See [3].

Lemma 1. *Suppose for some $A \in R^M$ and $k_0 \in S^{d-1}$ the above three functions are positive on $N_{(A,k_0)}$. Then there are some positive numbers c_0, b, ϵ, an integer J, a neighborhood K of k_0 in S^{d-1} and finite small balls $\{B_j(\epsilon)\}_{j=1}^J$ such that for any $B \in R^M$ with $||B-A|| \le b$ and any $k \in K$ there are finite hypersurfaces $\{S_j\}_{j=1}^J$ for which the following properties hold:*

(1) $N_{(B,k)} \cap B_j(\epsilon) \subset S_j \cap B_j(\epsilon)$;

(2) $N_{(B,k)} \subset \bigcup_{j=1}^J B_j(\frac{\epsilon}{2})$;

(3) $S_j \cap B_j(\epsilon)$ is a piece of hypersurface with nonzero Gaussian curvature which is bounded by c_0 from below for all j.

Moreover for each such (B,k), there is a diffeomorphism $G_{(B,k)} : \bigcup_{j=1}^J B_j(\epsilon) \to D(\epsilon)\times N_{(B,k)}$ such that $|G'_{(B,k)}|$ is bounded by c_0 from below.

Proof. We will prove this lemma in several steps as follows.

Step 1: There are positive constants c, b and a neighborhood K of k_0 in S^{d-1} and an ϵ neighborhood U of $N_{(A,k_0)}$ such that for any $B \in R^M$ with $||B-A|| \le b$ and any $k \in K$,

$$N_{(B,k)} \subset \frac{1}{2}U,$$
$$\min\left(S(B,\xi,k), H(B,\xi,k), L(B,\xi,k)\right) \ge c$$

for all $\xi \in U$.

Proof of Step 1. Since P_A is an elliptic polynomial, the set $N_{(A,k_0)}$ is a compact boundaryless submanifold of codim 2 by assumption. Functions S, H and L are continuous in three variables A, ξ and k. So by assumption and compact argument and the ϵ neighborhood theorem, Step 1 is proved.

Step 2: There are ϵ and finite small balls such that for any B and k as in Step 1 there are finite hypersurfaces as in Lemma 1. (1), (2) and (3) of Lemma 1 hold.

Proof of Step 2. Since $S(A,\xi,k_0)$ and $H(A,\xi,k_0)$ are positive functions, Proposition 0.1 of [3] implies that there are finite ϵ balls $\{B_j(\epsilon)\}_{j=1}^J$ with centers $\{\xi_j\} \subset N_{(A,k_0)}$ such that

$$N_{(A,k_0)} \subset \bigcup_{j=1}^J B_j(\frac{\epsilon}{4}).$$

Moreover there are also finite real numbers t_j and vectors $\{x_j\}_{j=1}^J \subset R^d$ such that if we define functions $f_j(A,\xi,k_0)$ by

$$\mathrm{re}P_A(\xi+ik_0) + t_j\mathrm{im}P_A(\xi+ik) + \langle x_j, \xi-\xi_j\rangle\left(t_j\mathrm{re}P_A(\xi+ik_0) - \mathrm{im}P_A(\xi+ik)\right)$$
$$-\langle x_j, \xi-\xi_j\rangle\left(\mathrm{re}P_A(\xi+ik_0) + t_j\mathrm{im}P_A(\xi+ik)\right)$$
$$\times\frac{\langle t_j\bigtriangledown\mathrm{re}P_A(\xi_j+ik_0) - \bigtriangledown\mathrm{im}P_A(\xi_j+ik_0), \bigtriangledown\mathrm{re}P_A(\xi_j+ik_0)+t_j\bigtriangledown\mathrm{im}P_A(\xi_j+ik_0)\rangle}{\langle\bigtriangledown\mathrm{re}P_A(\xi_j+ik_0)+t_j\bigtriangledown\mathrm{im}P_A(\xi_j+ik_0), \bigtriangledown\mathrm{re}P_A(\xi_j+ik_0)+t_j\bigtriangledown\mathrm{im}P_A(\xi_j+ik_0)\rangle},$$

then $f_j(A,\cdot,k_0)^{-1}(0)$ is a hypersurface with Gaussian curvature bounded by $2c_0$ from below in $B_j(\epsilon)$ for some constant c_0 which depends only on A and k_0. Now let's fix a B and a k as in Step 1. When b and K are small enough, $N_{(B,k)} \subset \bigcup_{j=1}^{J} B_j(\frac{\epsilon}{2})$. Choose $\eta_j \in B_j(\frac{\epsilon}{2})$ with $P_B(\eta_j + ik) = 0$. Replace A, k_0 and ξ_j by B, k and η_j in the function f_j for each j. Then once again when b and K are small enough, $f_j(B,\cdot,k)^{-1}(0) \cap B_j(\epsilon)$ is a piece of hypersurface with Gaussian curvature bounded by c_0 from below for all j. This proves Step 2 with $S_j = f_j(B,\cdot,k)^{-1}(0)$.

Step 3: The last part in Lemma 1 holds when ϵ is small and J is larger.

Proof of Step 3. By the ϵ neighborhood theorem, when ϵ is small and J is large, there is a diffeomorphism $G_{(A,k_0)} : \bigcup B_j(2\epsilon) \to D(2\epsilon) \times N_{(A,k_0)}$ where $D(2\epsilon)$ is a 2-dimensional ball of radius 2ϵ. In fact $G_{(A,k_0)}$ may be defined by extending $N_{(B,k)}$ along the normal directions, which we may choose as $\bigtriangledown \mathrm{re} P_A(\xi + ik_0) + t_j \bigtriangledown \mathrm{im} P_A(\xi + ik_0)$ and $t_j \bigtriangledown \mathrm{re} P_A(\xi + ik_0) - \bigtriangledown \mathrm{im} P_A(\xi + ik_0) - v$ where v is the projection of $t_j \bigtriangledown \mathrm{re} P_A(\xi + ik_0) - \bigtriangledown \mathrm{im} P_A(\xi + ik_0)$ in the $\bigtriangledown \mathrm{re} P_A(\xi + ik_0) + t_j \bigtriangledown \mathrm{im} P_A(\xi + ik_0)$ direction in each $B_j(\frac{1}{2}\epsilon)$. Since $P_B(\xi + ik)$ are smooth in (B,ξ,k) and $L(B,\xi,k) \geq c$ by the assumption, for each B closing A and each k closing k_0, there is a diffeomorphism $G_{(B,k)} : \bigcup B_j(\epsilon) \to D(\epsilon) \times N_{(B,k)}$ such that $|G'_{(B,k)}|$ is bounded by $\frac{1}{2}|G'_{(A,k_0)}|$ from below. This proves Step 3.

Finally if we let c_0 be a new constant decided by Step 2 and Step 3, we prove Lemma 1.

Let Γ be the open cone such that $\Gamma \cap S^{d-1} = K$ which is as in Lemma 1. If E is a compact convex set with interior, then we define

$$g_E(x) = \min(T \geq 1 : x \in TE).$$

Fix once and for all $t > d$, and define $||u||_{p,E} = ||g_E^t u||_p$. Then by the Holder inequality we have

$$||u||_p \leq C||u||_{q,E}|E|^{\frac{1}{p}-\frac{1}{q}} \tag{2}$$

for any $q \geq p$, where C depends only on t and d.

Lemma 2. *Suppose P_A is as in Lemma 1 and is of order $m < \frac{d}{s}$. Let b and Γ be as before or as in Lemma 1. Then there is a constant C_A such that for all $B \in R^M$ with $||B - A|| \leq b$ and any $k \in \Gamma$ and all compact convex subsets $E \subset R^d$ with $|E| \geq |k|^{-d}$, we have*

$$||e^{k\cdot x} \bigtriangledown^{m-\mu} f||_{q_\mu} \leq C_A(|k|^d|E|)^{\frac{\mu}{d}}||e^{k\cdot x}P_B(D)f||_{2,E} \tag{3}$$

for all $f \in W^{m,2}$ with compact support and all integers $0 < \mu \leq m$, where q_μ are the real numbers satisfying $\frac{1}{2} - \frac{1}{q_\mu} = \frac{\mu}{d}$. When $\mu = 0$, we have the following inequality:

$$||e^{k\cdot x} \bigtriangledown^{m} f||_{2} \leq C_A(|k|\mathrm{diam}E)||e^{k\cdot x}P_B(D)f||_{2,E}. \tag{4}$$

Proof. Let $a = (\frac{1}{s}, 0)$, $b = (1,0)$, $c = (1,\frac{1}{2})$ and $d = (\frac{1}{s},\frac{1}{s'})$. Let Q be a subset of R^2 consisting of the quadrilateral $abcd$ and two sides ad and bc. Let B and k with $|k| = 1$ be as in Lemma 2. So the conclusions of Lemma 1 hold for this (B,k). First let $0 < \mu \leq m$.

The inequality (3) is equivalent to

$$\|(m\hat{v})^\vee\|_{q_\mu} \leq C_A(|k|^d|E|)^{\frac{\mu}{d}}||v||_{2,E} \tag{5}$$

with $m(\xi) = \frac{|\xi+ik|^{m-\mu}}{P_B(\xi+ik)}$ for all $v \in C_0^\infty$.

Let $U_{\frac{1}{2}} = \bigcup_{j=1}^{J} B_j(\frac{\epsilon}{2})$ and $U_1 = \bigcup_{j=1}^{J} B_j(\epsilon)$ which are in Lemma 1. Let ϕ be a smooth cutoff function taking 1 on $U_{\frac{1}{2}}$, and 0 on U_1^c. Write $m = m_1 + m_2$ with $m_1 = m\phi$ and $m_2 = m(1-\phi)$. By Lemma 1, the exact proof of Lemma 2.1 in [3] shows that

$$\|(m_1\hat{v})^\vee\|_q \le C_A||v||_p \tag{6}$$

for all $(\frac{1}{p}, \frac{1}{q}) \in Q$, where C_A is some constant which depends only on A, k_0 and d. Since $m_2(\xi) \le (1+|\xi|)^{-\mu}$, by the Bessel potential theory, we have

$$\|(m_2\hat{v})^\vee\|_q \le C_A||v||_p \tag{7}$$

for all $\frac{1}{p} - \frac{1}{q} = \frac{\mu}{d}$. Let q_μ be such that $\frac{1}{2} - \frac{1}{q_\mu} = \frac{\mu}{d}$, and let q_μ^1 be such that $\frac{1}{s} - \frac{1}{q_\mu^1} = \frac{\mu}{d}$ if $\mu \ge 2$, q_1^1 is sufficiently close to s' and is bigger than s'. Then for any compact convex set $|E| \ge 1$, since $q_\mu^1 < q_\mu$ and m_1 has compact support, we have by using (6) and (2)

$$\|(m_1\hat{v})^\vee\|_{q_\mu} \le \|(m_1\hat{v})^\vee\|_{q_\mu^1} \le C_A||v||_s \le C_A|E|^{\frac{1}{s}-\frac{1}{2}}||v||_{2,E} \tag{8}$$

which is bounded by $C_A|E|^{\frac{\mu}{d}}||v||_{2,E}$ since $|E| \ge 1$. Combining (8) and (7) we prove (5) and hence (3) with $|k| = 1$. After a scaling we prove Lemma 2 with $\mu \ge 1$.

Finally when $\mu = 0$, the inequality (4) was already showed in [4] without using any curvature property in Lemma 1. So this proves Lemma 2.

Lemma 3. *Suppose f is supported in a ball B. Let $D(a, N)$ be a fixed ball in R^d. Then there is a pairwise disjoint compact convex subset $\{E_{k_j}\}$ with $\{k_j\} \subset D(a, N)$ such that*

$$||e^{k_j\cdot x}f \cdot g_{E_{k_j}}||_{1,E_{k_j}} \le C_0^2||e^{k_j\cdot x}f||_{L^1(E_{k_j})}, \tag{9}$$

$$\sum |E_{k_j}|^{-1} \ge C^{-1}N^d, \ \forall s \ge 1, \tag{10}$$

$$\operatorname{diam} E_{k_j} \le C_0 N^{-\frac{1}{2}}, \tag{11}$$

$$E_{k_j} \text{ contains a ball of radius } (C_0N)^{-1},$$

$$E_{k_j} \subset 2B$$

where C_0 is a universal constant depending only on d.

Proof. This is a special case of Wolff's measure lemma in [4].

Now let's start to prove Theorem 1. First we claim that we may assume the Lipschitz norm of $a_\alpha(x)$ is less than a small number ρ which will be chosen later. In fact let $F^1(x) = \delta^{-1}x$, $F_2(x) = (\bar{x}, -x_d)$, $F_3(x) = \frac{x+e_d}{|x+e_d|^2} - e_d$ and let $F = F_3 \circ F_2 \circ F_1$. Then if δ is small enough, the function $v = u \circ F^{-1}$ is defined on a domain which contains $R^d \setminus B(-e_d, \frac{1}{2})$ and $v = 0$ outside $B(-e_d, 1)$. Moreover v satisfies the following differential inequality:

$$|P_\delta(y, D)v(y)| \le \sum_{0<\mu\le m} V_\mu(y)| \bigtriangledown^{m-\mu} v(y)| \tag{12}$$

where $V_\mu(y)$ has the same properties as before, $P_\delta(y, D) = \sum_{|\alpha|=m} a_\alpha^\delta(y)D^\alpha$ with $a_\alpha^\delta(0) = a_\alpha(0)$ and $||a_\alpha||_{\text{Lip}} \le \delta||a_\alpha||_{\text{Lip}}$. Let ρ be this number. On the other hand, if we let $A = (a_\alpha^\delta(0)) = (a_\alpha(0))$ and b, Γ be as in Lemma 2 or Lemma 1 with

$k_0 = e_d$, then when δ is small enough for any $y \in B(0, \frac{1}{2})$ with $B = (a_\alpha^\delta(y))$ the inequalities (3) and (4) hold for all small δ.

Let's assume $0 \in \text{supp}v$. Let S be the convex hull of $\text{supp}v \cap \{y \in R^d : y_d \geq -\frac{1}{16}\}$ and ϕ be a smooth cutoff function such that $\phi = 0$ when $y_d \leq -\frac{1}{8}$, $\chi = 1$ in a neighborhood of ∂S and $\sum_{0<\mu m} ||V_\mu||^{\frac{d}{\mu}}_{L^{\frac{d}{\mu}}(\text{supp}\phi)} \leq \beta$ with a small constant β to be chosen later. Let $w = v\phi$. Then by (4)

$$|P_\delta(y, D)w(y)| \leq \sum_{0<\mu\leq m} V_\mu(y)| \bigtriangledown^{m-\mu} w(y)| + \chi \tag{13}$$

where $\chi \in L^2$ and $\text{supp}\chi \subset A^1 \cup A_2$; here $A_2 = \{y \in B(-e_d, 1) : -\frac{1}{16} \geq y_d \geq -\frac{1}{8}\}$ and A_1 is a compact subset of S. Let $r \leq \frac{1}{32}$ be a fixed small number so that the cone $\Gamma_r = \{k \in R^d : k_d > r^{-1}\sqrt{|k|^2 - k_d^2}\}$ is contained in Γ which is as in Lemma 2 for P_A. So r is independent of ρ.

Lemma 4. *If $\tau > 0$, then there is an L_0 such that if $k \in \Gamma_r$ and $|k| \geq L_0$, then*

$$||e^{k\cdot y}\chi \cdot g_E||_{2,E} \leq ||e^{k\cdot y} \sum_{0<\mu\leq m} V_\mu| \bigtriangledown^{m-\mu} w|||_2 \tag{14}$$

for all $E \subset B(0, \frac{1}{2})$ with E containing a ball of radius $\tau|k|^{-1}$.

Proof. Since Γ_r has conjugate cone $\{k \in R^d : \langle k, k'\rangle \leq 0 \ \forall k' \in \Gamma_r\}$ which contains $B(-e_d, 1) \cap \{y : y_d \leq \frac{1}{6}\} \supset A_2$, the rest of the proof is exactly the same as the proof of Lemma 7.1 of [4]. So we are done.

Proof of Theorem 1. Let $L \geq L_0$ be a large number. We will apply Lemma 3 to the function

$$f = \left(\sum_{0<\mu\leq m} V_\mu| \bigtriangledown^{m-\mu} w| + \rho L^{-\frac{1}{2}}| \bigtriangledown^m w|\right)^2$$

and the ball $B(Le_d, \frac{1}{2}rL)$ with $a = Le_d$ and $N = \frac{1}{2}rL$. So $\frac{1}{2}L \leq |k_j| \leq 2L$. Let $Y_j = E_{k_j} \cap \text{supp}w$, let y_j be the center of the convex set E_{k_j} and let $B_j = (a_\alpha^\delta(y_j))$. So we have $||B_j - A|| \leq b$ and the inequalities (3) and (2) in Lemma 2. Then by using Holder's inequality, (3), (4), and (11)

$$\begin{aligned}
&||e^{k_j\cdot y}\left(\sum_{0<\mu\leq m} V_\mu| \bigtriangledown^{m-\mu} w| + \rho L^{-\frac{1}{2}}| \bigtriangledown^m w|\right)||_{L^2(E_{k_j})} \\
&\leq \sum_{0<\mu\leq m} ||V_\mu||_{L^{\frac{d}{\mu}}(Y_j)}||e^{k_j\cdot y}w||_{q_\mu} + \rho L^{-\frac{1}{2}}||e^{k_j\cdot y} \bigtriangledown^m w||_2 \\
&\leq C_A\left(\sum_{0<\mu\leq m} (|k_j|^d|E_{k_j}|)^{\frac{\mu}{d}}||V_\mu||_{L^{\frac{d}{\mu}}(Y_j)} + \rho L^{-\frac{1}{2}}|k_j|\text{diam}E_{k_j}\right)||e^{k_j\cdot y}P_{B_j}(D)w||_{2,E_{k_j}} \\
&\leq 2C_A\left(\sum_{0<\mu\leq m} (L^d|E_{k_j}|)^{\frac{\mu}{d}}||V_\mu||_{L^{\frac{d}{\mu}}(Y_j)} + C_0r^{-1}\rho\right)||e^{k_j\cdot y}P_{B_j}(D)w||_{2,E_{k_j}}.
\end{aligned} \tag{15}$$

On the other hand, since a^δ_α is Lipschitz continuous it follows that $|a^\delta_\alpha(y_j)-a^\delta_\alpha(y)| \leq \rho \cdot |y_j - y| \leq \rho \mathrm{diam} E_{k_j} g_{E_{k_j}} \leq C_0 r_{-1} \rho L^{-\frac{1}{2}} g_{E_{k_j}}$ by (11). So

$$|P_{B_j}(D)w(y)| \leq |P_\delta(y,D)w(y)| + C_0 r^{-1}\rho L^{-\frac{1}{2}} g_{E_{k_j}} | \bigtriangledown^m w|$$

and hence by (13)

$$|P_{B_j}(D)w(y)| \leq \sum_{0<\mu\leq m} V_\mu | \bigtriangledown^{m-\mu} w| + C_0 r^{-1}\rho L^{-\frac{1}{2}} g_{E_{k_j}} | \bigtriangledown^m w| + \chi.$$

Because of (14), we may ignore the term χ in the following process. Now by using (9) we have

(16)

$$\begin{aligned} &||e^{k_j\cdot y} P_{B_j}(D)w||_{2,E_{k_j}} \\ &\quad\leq 2C_0 r^{-1} ||e^{k_j\cdot y}\left(\sum_{0<\mu\leq m} V_\mu | \bigtriangledown^{m-\mu} w| + \rho L^{-\frac{1}{2}} | \bigtriangledown^m w|\right) g_{E_{k_j}} ||_{2,E_{k_j}} \\ &\quad\leq 2C_0^2 r^{-1} ||e^{k_j\cdot y}\left(\sum_{0<\mu\leq m} V_\mu | \bigtriangledown^{m-\mu} w| + \rho L^{-\frac{1}{2}} | \bigtriangledown^m w|\right) ||_{L^2(E_{k_j})}. \end{aligned}$$

So combining (15) and (16), we have

$$1 \leq 2C_0^2 r^{-1} \cdot 2C_A \left(\sum_{0<\mu\leq m} (L^d|E_{k_j}|)^{\frac{\mu}{d}} ||V_\mu||_{L^{\frac{d}{\mu}}(Y_j)} + C_0 r^{-1}\rho\right). \tag{17}$$

Remember the constants r, C_0 and C_A are independent of ρ, i.e., δ. So after making δ and hence ρ small, (17) implies

$$\sum_{0<\mu\leq m} (L^d|E_{k_j}|)^{\frac{\mu}{d}} ||V_\mu||_{L^{\frac{d}{\mu}}(Y_j)} \geq C$$

and hence

$$\max_{0<\mu\leq m}\{||V_\mu||^{\frac{d}{\mu}}_{L^{\frac{d}{\mu}}(Y_j)}\} \geq C(L^d|E_{k_j}|)^{-1} \tag{18}$$

for some constant C depending only on d and A. Summing up over j for (18), (10) implies that

$$\sum_{0<\mu\leq m} ||V_\mu||^{\frac{d}{\mu}}_{L^{\frac{d}{\mu}}(\mathrm{supp} w)} \geq C_0^{-1} C,$$

which is a contradiction if β is small enough. This proves Theorem 1. □

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