UNIQUENESS IN THE CAUCHY PROBLEMS FOR HIGHER ORDER ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract. In this note, we study the uniqueness in Cauchy problems for a class of higher order elliptic differential operators with Lipschitz coefficients. In particular, we prove the uniqueness under assuming the potentials being $L^{r_j}_{loc}$ with certain correct numbers $r_j$'s.

Notation. Let $\Omega$ be a domain in $\mathbb{R}^d$. Suppose $P(x, D) = \sum_{|\alpha|=m} a_\alpha(x)D^\alpha$ is a differential operator of degree $m$ with real functions $a_\alpha(x)$ on $\Omega$. We denote by $P = P(x, \cdot + ik)$ the symbol of $P(x, D)$ and by $N^P(x, k)$ the zero set of $P(x, \cdot + ik)$ for any $(x, k) \in \Omega \times \mathbb{R}^d$. Let's define a subset in $\Omega \times S^{d-1}$:

$$\Sigma_P = \{(x, k) \in \Omega \times S^{d-1} : \sum \frac{dP}{dz_j}(x, \xi + ik) \cdot k_j \neq 0, \det \text{Hess}_C P(x, \xi + ik) \neq 0 \forall \xi \in N^P(x, k)\}$$

where $\text{Hess}_C P = \left(\frac{d^2 P}{dz_j dz_l}\right)$ is the complex Hessian matrix of $P$, and $z = \xi + ik \in \mathbb{C}^d$.

If $u$ is a function on $\Omega$, we define its normal support $N(\text{supp} u)$ as a subset of $\Omega \times S^{d-1}$. Say $(x, k) \in N(\text{supp} u)$ if there is a neighborhood $V$ of $x$ such that $\psi(y) \leq \psi(x)$ for all $y \in V \cap \text{supp} u$ and $d\psi(x) = \pm k$, where $\psi$ is some smooth function.

Let $s = \frac{2(d+1)}{d+3}$ be the restriction number and $s'$ be its conjugate number. We let $W^{m, 2}_{loc}$ be the Sobolev space of functions whose derivatives up to order $m$ belong to $L^2$. We have the following theorem.

Theorem. Suppose $P(x, D)$ is an elliptic differential operator with real Lipschitz functions $a_\alpha$ as coefficients on $\Omega$ and is of order $m < \frac{d}{2}$. If a function $u \in W^{m, 2}_{loc}(\Omega)$ satisfies

$$|Pu(x)| \leq \sum_{0<\mu\leq m} V_\mu |\nabla^{m-\mu} u|$$

with $V_\mu \in L^{\frac{d}{\mu}}_{loc}(\Omega)$, then $N(\text{supp} u) \subset \Sigma_P$.

Remarks. (1) Actually we will prove that $N(\text{supp} u) \subset \Lambda_P$ where $\Lambda_P$ is the set of $(x, k) \in \Omega \times S^{d-1}$ such that $N^P(x, k)$ is locally contained in a smooth hypersurface with nonzero Gaussian curvature, which is smaller than $\Sigma_P$. In other words, we

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may replace the assumptions in $\Sigma_P$ by directly assuming some curvature condition for $N^P_{x,k}$. $\Sigma_P$ is a natural condition and is easy to verify. But the proof of $\Sigma_P \subset \Lambda_P$ is nontrivial which is essentially shown in Lemma 1 below. For more details, see [3].

(2) When coefficients are constants, this theorem was proved by the author in [3]. When $P(x, D)$ is hyperbolic, under some other curvature assumption for $N^P_{x,k}$, Sogge proves the same result in the case where $V_\mu = 0$ for all $\mu \leq m - 1$; see [2]. In general, if we don’t care about the optimal condition for the potentials, this is an old theorem by Calderon. See [1], [4].

Calderon’s theorem is actually equivalent to the following uniqueness theorem in the Cauchy problem.

**Theorem 1.** Suppose $P(x, D)$ is an elliptic differential operator with real Lipschitz functions $a_\alpha$ as coefficients on a domain $\Omega$ which contains $\mathbb{R}^d \setminus \overline{B(-e_d, \frac{1}{2})}$ and satisfies the conditions

\[
\frac{dP}{dz_d}(0, \xi + ie_d) \neq 0,
\]

\[
\det \text{Hess}_P(0, \xi + ie_d) \neq 0
\]

for all $\xi \in N^P_{(0, e_d)}$, where $\text{Hess}_P$ is the complex Hessian matrix of $P$. Then for any function $u \in W^{m,2}_{\text{loc}}(\Omega)$ satisfying (1) for some $V_\mu \in L^\frac{m}{2}_{\text{loc}}(\Omega)$, $u$ vanishes in a neighborhood of 0 if $u$ vanishes outside $B(-e_d, 1)$.

Let’s first prove our Theorem by assuming Theorem 1.

**Proof of the Theorem.** Let $(x^0, k^0) \in N(\suppu)$. Suppose $(x^0, k^0) \in \Sigma_P$. By the definition of $N(\suppu)$, there is a little ball $B$ such that $x^0 \in \partial B$ and $u = 0$ in $B$. Then there is a map $F$ which is the composition of translation, rotation, dilation and Kelvin transformation with respect to $x^0$ and $B$ such that $F(x^0) = 0$ and $F(k^0) = e_d$. Moreover $u \circ F^{-1} = 0$ outside $B(-e_d, 1)$ and $u \circ F^{-1}$ is defined on a domain $\Omega$ which contains $\mathbb{R}^d \setminus \overline{B(-e_d, \frac{1}{2})}$. Let $v(y) = u \circ F^{-1}(y)$. Then $v$ satisfies the following differential inequality by (1):

\[
|Q(y, D)v(y)| \leq \sum_{0 < \mu \leq m} V^1_\mu(y)|\nabla^{m-\mu} v(y)|
\]

where $Q(y, \eta) = P(F^{-1}(y), (DF^{-1}(y))^{-1}\eta)$ and $V^1_\mu(y) = V_\mu \circ F^{-1}(y)$ plus some bounded functions. So one may check that $(0, e_d) \in \Sigma_Q$ which means the assumptions in Theorem 1 are satisfied. So applying Theorem 1 to $Q$ and $v$, we have $v = 0$ in a neighborhood of 0. Pull back $v$ to $u$ by $F$. We have $u = 0$ in a neighborhood of $x^0$. This is a contradiction with $x^0 \in \suppu$.

In order to prove Theorem 1, we need several lemmas. Let’s first study the differential operator with real constants coefficients. We denote by $A$ the vector $(a_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$ for some number $M$ determined by $m$ and $P_A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$, and denote by $N(A, k)$ the zero set of $P_A(\cdot + ik)$. We are always interested in the case that $P_A$ is elliptic. Let’s introduce some functions as follows:

\[
S(A, \xi, k) = \sum_j \frac{dP_A}{dz_j}(\xi + ik)k_j,
\]
$$H(A, \xi, k) = \det \left( \frac{d^2 P_A}{dz_jdz_l}(\xi + ik) \right),$$
$$L(A, \xi, k) = \sum_{(j,l)} \left| \det \left( \frac{det P_A}{\dim P_A}(\xi + ik) \frac{det P_A}{\dim P_A}(\xi + ik) \right) \right|.$$

We notice that the assumption in Theorem 1 says that when \(A = (a_\alpha(0))\) and \(k = e_d\), the first two of the above functions are positive on \(N_{(0,e_d)}^F\). By the Cauchy-Riemann equation and the transversality theorem, we proved that \(L(A, \xi, k)\) is also positive on \(N_{(0,e_d)}^P\). See [3].

**Lemma 1.** Suppose for some \(A \in R^M\) and \(k_0 \in S^{d-1}\) the above three functions are positive on \(N_{(A,k_0)}\). Then there are some positive numbers \(c_0, b, \epsilon,\) an integer \(J,\) a neighborhood \(K\) of \(k_0\) in \(S^{d-1}\) and any finite small balls \(\{B_j(\epsilon)\}_{j=1}^J\) such that for any \(B \in R^M\) with \(\|B - A\| \leq b\) and any \(k \in K\) there are finite hypersurfaces \(\{S_j\}_{j=1}^J\) for which the following properties hold:

1. \(N_{(B,k)} \cap B_j(\epsilon) \subset S_j \cap B_j(\epsilon)\);
2. \(N_{(B,k)} \subset \bigcup_{j=1}^J B_j(\frac{\epsilon}{2})\);
3. \(S_j \cap B_j(\epsilon)\) is a piece of hypersurface with nonzero Gaussian curvature which is bounded by \(c_0\) from below for all \(j\).

Moreover for each such \((B, k)\), there is a diffeomorphism \(G_{(B,k)} : \bigcup_{j=1}^J B_j(\epsilon) \rightarrow D(\epsilon) \times N_{(B,k)}\) such that \(|G_{(B,k)}|\) is bounded by \(c_0\) from below.

**Proof.** We will prove this lemma in several steps as follows.

**Step 1:** There are positive constants \(c, b\) and a neighborhood \(K\) of \(k_0\) in \(S^{d-1}\) and an \(\epsilon\) neighborhood \(U\) of \(N_{(A,k_0)}\) such that for any \(B \in R^M\) with \(\|B - A\| \leq b\) and any \(k \in K\),
$$N_{(B,k)} \subset \frac{1}{2} U,$$
$$\min (S(B, \xi, k), H(B, \xi, k), L(B, \xi, k)) \geq c$$
for all \(\xi \in U\).

**Proof of Step 1.** Since \(P_A\) is an elliptic polynomial, the set \(N_{(A,k_0)}\) is a compact boundaryless submanifold of codim 2 by assumption. Functions \(S, H\) and \(L\) are continuous in three variables \(A, \xi\) and \(k\). So by assumption and compact argument and the \(\epsilon\) neighborhood theorem, Step 1 is proved.

**Step 2:** There are \(\epsilon\) and finite small balls such that for any \(B\) and \(k\) as in Step 1 there are finite hypersurfaces as in Lemma 1. (1), (2) and (3) of Lemma 1 hold.

**Proof of Step 2.** Since \(S(A, \xi, k_0)\) and \(H(A, \xi, k_0)\) are positive functions, Proposition 0.1 of [3] implies that there are finite \(\epsilon\) balls \(\{B_j(\epsilon)\}_{j=1}^J\) with centers \(\{\xi_j\} \subset N_{(A,k_0)}\) such that
$$N_{(A,k_0)} \subset \bigcup_{j=1}^J B_j(\frac{\xi_j}{4}).$$

Moreover there are also finite real numbers \(t_j\) and vectors \(\{x_j\}_{j=1}^J \subset R^d\) such that if we define functions \(f_j(A, \xi, k_0)\) by
$$\text{re} P_A(\xi + ik_0) + t_j \text{im} P_A(\xi + ik) + (x_j, \xi - \xi_j)(t_j \text{re} P_A(\xi + ik_0) - \text{im} P_A(\xi + ik))$$
$$- (x_j, \xi - \xi_j) (\text{re} P_A(\xi + ik_0) + t_j \text{im} P_A(\xi + ik))$$
$$\times \frac{(t_j \nabla \text{re} P_A(\xi_j + ik_0) - \nabla \text{im} P_A(\xi_j + ik_0), \nabla \text{re} P_A(\xi_j + ik_0) + t_j \nabla \text{im} P_A(\xi_j + ik_0))}{(\nabla \text{re} P_A(\xi_j + ik_0) + t_j \nabla \text{im} P_A(\xi_j + ik_0), \nabla \text{re} P_A(\xi_j + ik_0) + t_j \nabla \text{im} P_A(\xi_j + ik_0))},$$

and
then \( f_j(A,\cdot, k_0)^{-1}(0) \) is a hypersurface with Gaussian curvature bounded by \( 2c_0 \) from below in \( B_j(\epsilon) \) for some constant \( c_0 \) which depends only on \( A \) and \( k_0 \). Now let’s fix \( A \) and \( k \) as in Step 1. When \( b \) and \( K \) are small enough, \( N_{(B,k)} \subset \bigcup_{j=1}^t B_j(\frac{\epsilon}{2}). \) Choose \( \eta_j \in B_j(\frac{\epsilon}{2}) \) with \( P_0(\eta_j + ik) = 0 \). Replace \( A, k_0 \) and \( \xi_j \) by \( B, k \) and \( \eta_j \) in the function \( f_j \) for each \( j \). Then once again when \( b \) and \( K \) are small enough, \( f_j(B,\cdot,k)^{-1}(0) \) is a piece of hypersurface with Gaussian curvature bounded by \( c_0 \) from below for all \( j \). This proves Step 2 with \( S_j = f_j(B,\cdot,k)^{-1}(0) \).

Step 3: The last part in Lemma 1 holds when \( \epsilon \) is small and \( J \) is larger.

Proof of Step 3. By the \( \epsilon \) neighborhood theorem, when \( \epsilon \) is small and \( J \) is large, there is a diffeomorphism \( G_{(A,k_0)} : \bigcup B_j(2\epsilon) \to D(2\epsilon) \times N_{(A,k_0)} \) where \( D(2\epsilon) \) is a 2-dimensional ball of radius \( 2\epsilon \). In fact \( G_{(A,k_0)} \) may be defined by extending \( N_{(B,k)} \) along the normal directions, which we may choose as \( \nabla \text{re} P_A(\xi + ik_0) + t_j \nabla \text{im} P_A(\xi + ik_0) \) and \( t_j \nabla \text{re} P_A(\xi + ik_0) - \nabla \text{im} P_A(\xi + ik_0) \). Vertical is \( \xi \) is in the direction of each \( B_j(\frac{\epsilon}{2}) \). Since \( P_B(\xi + ik) \) are smooth in \((B,\xi,k)\) and \( L(B,\xi,k) \geq c \) by the assumption, for each \( B \) closing \( A \) and each \( k \) closing \( k_0 \), there is a diffeomorphism \( G_{(B,k)} : \bigcup B_j(\epsilon) \to D(\epsilon) \times N_{(B,k)} \) such that \( |G_{(B,k)}| \) is bounded by \( \frac{1}{2} |G_{(A,k_0)}| \) from below. This proves Step 3.

Finally if we let \( c_0 \) be a new constant decided by Step 2 and Step 3, we prove Lemma 1.

Let \( \Gamma \) be the open cone such that \( \Gamma \cap S^{d-1} = K \) which is as in Lemma 1. If \( E \) is a compact convex set with interior, then we define

\[
g_E(x) = \min(T \geq 1 : x \in TE)
\]

Fix once and for all \( t > d \), and define \( \|u\|_{p,E} = \|\gamma_p u\|_p \). Then by the Holder inequality we have

\[
\|u\|_p \leq C \|u\|_{q,E} |E|^\frac{1}{p} - \frac{1}{q}
\]

for any \( q \geq p \), where \( C \) depends only on \( t \) and \( d \).

Lemma 2. Suppose \( P_A \) is as in Lemma 1 and is of order \( m < \frac{d}{s} \). Let \( b \) and \( \Gamma \) be as before or as in Lemma 1. Then there is a constant \( C_A \) such that for all \( B \in R^M \) with \( |B - A| \leq b \) and any \( k \in \Gamma \) and all compact convex subsets \( E \subset R^d \) with \( |E| \geq |k|^{-d} \), we have

\[
\|e^{k} \cdot v^\mu f\|_{q_m} \leq C_A(|k|^d |E|)^{\frac{1}{s}} \|e^{k}P_B(D)f\|_{2,E}
\]

for all \( f \in W^{m,2} \) with compact support and all integers \( 0 < \mu \leq m \), where \( q_m \) are the real numbers satisfying \( \frac{1}{2} - \frac{1}{q_m} = \frac{d}{s} \). When \( \mu = 0 \), we have the following inequality:

\[
\|e^{k} \cdot v^\mu f\|_2 \leq C_A(|k|\text{diam}E)|e^{k}P_B(D)f\|_{2,E}
\]

Proof. Let \( a = (\frac{1}{2},0) \), \( b = (1,0) \), \( c = (1, \frac{1}{2}) \) and \( d = (\frac{1}{2}, \frac{1}{2}) \). Let \( Q \) be a subset of \( R^2 \) consisting of the quadrilateral \( abcd \) and two sides \( ad \) and \( bc \). Let \( B \) and \( k \) with \( |k| = 1 \) be as in Lemma 2. So the conclusions of Lemma 1 hold for this \((B,k)\). First let \( 0 < \mu \leq m \).

The inequality (3) is equivalent to

\[
\|(|m\hat{v})^\mu\|_{q_m} \leq C_A(|k|^d |E|)^{\frac{d}{s}} \|v\|_{2,E}
\]

with \( m(\xi) = \frac{|\xi + ik|^{m-\mu}}{P_{\mu}(\xi + ik)} \) for all \( v \in C_0^\infty \).
Let $U_{\frac{1}{2}} = \bigcup_{j=1}^{J} B_j(\frac{1}{2})$ and $U_1 = \bigcup_{j=1}^{J} B_j(\epsilon)$ which are in Lemma 1. Let $\phi$ be a smooth cutoff function taking 1 on $U_{\frac{1}{2}}$, and 0 on $U_1$. Write $m = a_1 + m_2$ with $m_1 = m\phi$ and $m_2 = m(1 - \phi)$. By Lemma 1, the exact proof of Lemma 2.1 in [3] shows that

$$
\|(m_1\hat{v})^\vee\|_q \leq C_A \|v\|_p
$$

for all $(\frac{1}{p}, \frac{1}{q}) \in Q$, where $C_A$ is some constant which depends only on $A, k_0$ and $d$. Since $m_2(\xi) \leq (1 + |\xi|)^{-\mu}$, by the Bessel potential theory, we have

$$
\|(m_2\hat{v})^\vee\|_q \leq C_A \|v\|_p
$$

for all $(\frac{1}{p}, \frac{1}{q}) = \frac{\mu}{2d}$. Let $q_\mu$ be such that $\frac{1}{s} - \frac{1}{q_\mu} = \frac{\mu}{2d}$, and let $q_1^s$ be such that $\frac{1}{s} - \frac{1}{q_1^s} = \frac{s}{d}$ if $s \geq 2$, $q_1^s$ is sufficiently close to $s'$ and is bigger than $s'$. Then for any compact convex set $|E| \geq 1$, since $q_1^s < q_\mu$ and $m_1$ has compact support, we have by using (6) and (2)

$$
\|(m_1\hat{v})^\vee\|_{q_\mu} \leq \|(m_1\hat{v})^\vee\|_{q_1^s} \leq C_A \|v\|_{s} \leq C_A |E|^\frac{1}{2} - \frac{1}{p} \|v\|_{2,E}
$$

which is bounded by $C_A |E|^\frac{1}{2} \|v\|_{2,E}$ since $|E| \geq 1$. Combining (8) and (7) we prove (5) and hence (3) with $|k| = 1$. After a scaling we prove Lemma 2 with $\mu \geq 1$.

Finally when $\mu = 0$, the inequality (4) was already showed in [4] without using any curvature property in Lemma 1. So this proves Lemma 2.

**Lemma 3.** Suppose $f$ is supported in a ball $B$. Let $D(a, N)$ be a fixed ball in $R^d$. Then there is a pairwise disjoint compact convex subset $\{E_{k_j}\}$ with $\{k_j\} \subset D(a, N)$ such that

\[
\left\|e^{k_j \cdot x} f \cdot g_{E_{k_j}} \right\|_{L^1(E_{k_j})} \leq \frac{C_0^2 \|e^{k_j \cdot x} f\|_{L^1(E_{k_j})}}{|E_{k_j}|^{-1}} \geq C^{-1} N^{-d}, \forall s \geq 1,
\]

\[
\text{diam} E_{k_j} \leq C_0 N^{-\frac{1}{2}},
\]

$$
E_{k_j} \text{ contains a ball of radius } (C_0 N)^{-1},
$$

$$
E_{k_j} \subset 2B
$$

where $C_0$ is a universal constant depending only on $d$.

**Proof.** This is a special case of Wolff’s measure lemma in [4].

Now let’s start to prove Theorem 1. First we claim that we may assume the Lipschitz norm of $a_\alpha(x)$ is less than a small number $\rho$ which will be chosen later. In fact let $F_1(x) = \delta^{-1} x$, $F_2(x) = (x, -x_d)$, $F_3(x) = \frac{x + \delta e_d}{|x + \delta e_d|} - e_d$ and let $F = F_3 \circ F_2 \circ F_1$. Then if $\delta$ is small enough, the function $v = u \circ F^{-1}$ is defined on a domain which contains $R^d \setminus B(-e_d, \frac{1}{2})$ and $v = 0$ outside $B(-e_d, 1)$. Moreover $v$ satisfies the following differential inequality:

$$
|P_\delta(y, D)v(y)| \leq \sum_{0 < \mu \leq m} V_\mu(y) \|v\|_\mu^{-\mu} v(y)
$$

where $V_\mu(y)$ has the same properties as before, $P_\delta(y, D) = \sum_{|\alpha|=m} a_\alpha^\delta(y) D^\alpha$ with $a_\alpha^\delta(0) = a_\alpha(0)$ and $\|a_\alpha\|_{\text{lip}} \leq \delta \|a_\alpha\|_{\text{lip}}$. Let $\rho$ be this number. On the other hand, if we let $A = (a_\alpha^\delta(0)) = (a_\alpha(0))$ and $b, \Gamma$ be as in Lemma 2 or Lemma 1 with
\(k_0 = c_d\), then when \(\delta\) is small enough for any \(y \in B(0, \frac{1}{2})\) with \(B = (a_\delta^d(y))\) the inequalities (3) and (4) hold for all small \(\delta\).

Let’s assume \(0 \in \text{suppv}\). Let \(S\) be the convex hull of \(\text{suppv} \cap \{y \in \mathbb{R}^d : y_d \geq -\frac{1}{10}\}\) and \(\phi\) be a smooth cutoff function such that \(\phi = 0\) when \(y_d \leq -\frac{1}{8}\), \(\chi = 1\) in a neighborhood of \(\partial S\) and \(\sum_{0 < \mu \leq m} \|V_\mu\|_{L^\infty(\text{supp}\phi)} \leq \beta\) with a small constant \(\beta\) to be chosen later. Let \(w = v\phi\). Then by (4)

\[
|P_0(y, D)w(y)| \leq \sum_{0 < \mu \leq m} V_\mu(y)|\nabla^{m-\mu} w(y)| + \chi
\]

where \(\chi \in L^2\) and \(\text{supp}\chi \subset A_1 \cup A_2\); here \(A_2 = \{y \in B(-e_d, 1) : -\frac{1}{10} \geq y_d \geq -\frac{1}{8}\}\) and \(A_1\) is a compact subset of \(S\). Let \(r \leq \frac{1}{32}\) be a fixed small number so that the cone \(\Gamma_r = \{k \in \mathbb{R}^d : k_d > r^{-1} \sqrt{k_1^2 - k_2^2}\}\) is contained in \(\Gamma\) which is as in Lemma 2 for \(P_A\). So \(r\) is independent of \(\rho\).

**Lemma 4.** If \(\tau > 0\), then there is an \(L_0\) such that if \(k \in \Gamma_r\) and \(|k| \geq L_0\), then

\[
\|e^{k \cdot y} \cdot g_E\|_{2, E} \leq \|e^{k \cdot y} \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w|\|_2
\]

for all \(E \subset B(0, \frac{1}{2})\) with \(E\) containing a ball of radius \(|k|^{-1}\).

**Proof.** Since \(\Gamma_r\) has conjugate cone \(\{k \in \mathbb{R}^d : (k, k') \leq 0 \forall k' \in \Gamma_r\}\) which contains \(B(-e_d, 1) \cap \{y : y_d \leq \frac{1}{8}\} \supset A_2\), the rest of the proof is exactly the same as the proof of Lemma 7.1 of [4]. So we are done.

**Proof of Theorem 1.** Let \(L \geq L_0\) be a large number. We will apply Lemma 3 to the function

\[
f = \left( \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w| + \rho L^{-\frac{1}{2}} |\nabla^m w| \right)^2
\]

and the ball \(B(L e_d, \frac{1}{2}r L)\) with \(a = L e_d\) and \(N = \frac{1}{2}r L\). So \(\frac{1}{2}L \leq |k_j| \leq 2L\). Let \(Y_j = E_{k_j} \cap \text{suppv}\), let \(y_j\) be the center of the convex set \(E_{k_j}\) and let \(B_j = (a_\delta^d(y_j))\). So we have \(|B_j - A_j| \leq b\) and the inequalities (3) and (2) in Lemma 2. Then by using Holder’s inequality, (3), (4), and (11)

\[
\|e^{k_j \cdot y} \left( \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w| + \rho L^{-\frac{1}{2}} |\nabla^m w| \right)\|_{L^2(E_{k_j})}
\leq \sum_{0 < \mu \leq m} \|V_\mu\|_{L^{\frac{d}{d-2}}(Y_j)} \|e^{k_j \cdot y} w\|_{d, \text{supp}(E_{k_j})} + \rho L^{-\frac{1}{2}} \|e^{k_j \cdot y} \nabla^m w\|_2
\leq C_A \left( \sum_{0 < \mu \leq m} (|k_j|^d |E_{k_j}|)^{\frac{d}{2}} \|V_\mu\|_{L^{\frac{d}{d-2}}(Y_j)} + \rho L^{-\frac{1}{2}} |k_j| \text{diam} E_{k_j} \right) \|e^{k_j \cdot y} P_{B_j}(D)w\|_{2, E_{k_j}}
\leq 2C_A \left( \sum_{0 < \mu \leq m} (L^d |E_{k_j}|)^{\frac{d}{2}} \|V_\mu\|_{L^{\frac{d}{d-2}}(Y_j)} + C_0 r^{-1} \rho \right) \|e^{k_j \cdot y} P_{B_j}(D)w\|_{2, E_{k_j}}.
\]
On the other hand, since $a_{\alpha}^\delta$ is Lipschitz continuous it follows that $|a_{\alpha}^\delta(y_j) - a_{\alpha}^\delta(y)| \leq \rho \cdot |y_j - y| \leq \rho \text{diam}E_{k_j}g_{E_{k_j}} \leq C_0r^{-\frac{1}{2}}L^{-\frac{1}{2}}g_{E_{k_j}}$ by (11). So

$$|P_{B_j}(D)w(y)| \leq |P_\delta(y, D)w(y)| + C_0r^{-\frac{1}{2}}L^{-\frac{1}{2}}g_{E_{k_j}}|\nabla^m w|$$

and hence by (13)

$$|P_{B_j}(D)w(y)| \leq \sum_{0<\mu \leq m} V_\mu |\nabla^{m-\mu} w| + C_0r^{-\frac{1}{2}}L^{-\frac{1}{2}}g_{E_{k_j}}|\nabla^m w| + \chi.$$

Because of (14), we may ignore the term $\chi$ in the following process. Now by using (9) we have

$$\leq 2C_0r^{-\frac{1}{2}}|e_{k_j}^j \cdot P_{B_j}(D)w||_{2,E_{k_j}}$$

$$\leq 2C_0^2r^{-\frac{1}{2}}|e_{k_j}^j \cdot \left( \sum_{0<\mu \leq m} V_\mu |\nabla^{m-\mu} w| + \rho L^{-\frac{1}{2}}|\nabla^m w| \right) g_{E_{k_j}}||_{2,E_{k_j}}$$

So combining (15) and (16), we have

$$1 \leq 2C_0^2r^{-\frac{1}{2}} \cdot 2C_A \left( \sum_{0<\mu \leq m} (L^d|E_{k_j}|)\frac{d}{d^\mu} ||V_\mu||_{L^{\frac{d}{d^\mu}}(Y_j)} + C_0r^{-\frac{1}{2}}L^{-\frac{1}{2}}g_{E_{k_j}} |\nabla^m w| \right).$$

Remember the constants $r$, $C_0$ and $C_A$ are independent of $\rho$, i.e., $\delta$. So after making $\delta$ and hence $\rho$ small, (17) implies

$$\sum_{0<\mu \leq m} (L^d|E_{k_j}|)\frac{d}{d^\mu} ||V_\mu||_{L^{\frac{d}{d^\mu}}(Y_j)} \geq C$$

and hence

$$\max_{0<\mu \leq m} \{||V_\mu||_{L^{\frac{d}{d^\mu}}(Y_j)}\} \geq C(L^d|E_{k_j}|)^{-1}$$

for some constant $C$ depending only on $d$ and $A$. Summing up over $j$ for (18), (10) implies that

$$\sum_{0<\mu \leq m} ||V_\mu||_{L^{\frac{d}{d^\mu}}(\text{supp} w)} \geq C_0^{-1}C,$$

which is a contradiction if $\beta$ is small enough. This proves Theorem 1.

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