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# EIGENVALUE PINCHING THEOREMS ON COMPACT SYMMETRIC SPACES

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Abstract. We prove two first eigenvalue pinching theorems for Riemannian symmetric spaces (Theorems 1 and 2). As their application, we answer negatively a question raised by Elworthy and Rosenberg, who proposed to show that for every compact simple Lie group $G$ with a bi-invariant Riemannian metric $h$ on $G$ with respect to $-\frac{1}{2}B$, $B$ being the Killing form of the Lie algebra $\mathfrak{g}$, the first eigenvalue $\lambda_1(h)$ would satisfy

$$\sum_{j=1}^{2}\sum_{\ell=3}^{n}|[v_j, v_\ell]|^2 > n(2\lambda_1(h)-1),$$

for all orthonormal bases $\{v_j\}_{j=1}^n$ of tangent spaces of $G$ (cf. Corollary 3). This problem arose in an attempt to give a spectral geometric proof that $\pi_2(G)=0$ for a Lie group $G$.

## §1. Eigenvalue pinching theorems

Lichnerowicz and Obata's celebrated theorem says (cf. [2]) that if the Ricci curvature of an $n$-dimensional compact Riemannian manifold $(M,g)$ satisfies

$$\mathrm{Ric}_g \geq n-1,$$

then the first eigenvalue $\lambda_1(g)$ of the Laplacian $\Delta_g$ satisfies

$$\lambda_1(g) \geq n,$$

and equality holds if and only if $(M,g)$ is the standard unit sphere $(S^n, g_0)$ of curvature 1.

It is known (cf. [8], [7], [3], [1]) that the following eigenvalue pinching theorem holds:

**Theorem.** *Let $(M,g)$ be a compact $n$-dimensional Riemannian manifold with sectional curvature $K_g \geq 1$. Then, there exists a constant $C(n)>1$, depending only on $n$, such that, if $C(n)n \geq \lambda_1(g) \geq n$, $M$ is homeomorphic to $S^n$.*

It would be a difficult problem to determine such a constant $C(n)$ explicitly. We would propose the conjecture that

$$C(n) \geq \left(\frac{4}{n}+1\right)\frac{n-1}{n} > 1.$$

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Indeed, we prove under a strong symmetry condition,

**Theorem 1.** *Let $(M,g)$ be a compact $n$-dimensional Riemannian manifold with Ricci curvature*

$$\mathrm{Ric}_g \geq n-1.$$

*Assume that*

$$\left(\frac{4}{n}+1\right)(n-1) \geq \lambda_1(g) \geq n,$$

*and $(M,g)$ is a simply connected, irreducible, Riemannian symmetric space. Then $(M,g)$ is the standard sphere $(S^n, g_0)$.*

*Proof.* All simply connected compact irreducible Riemannian symmetric spaces are divided into two classes: type I or type II (cf. [5]). The spaces of type II are of the form $G/K$ with a compact simple Lie group $G$ and some closed subgroup $K$, and the invariant metric is given by $g = \frac{1}{2(n-1)}h$, where $h$ is a $G$-invariant Riemannian metric on $M = G/K$ induced from the Killing form of the Lie algebra $\mathfrak{g}$ of $G$. Then the Ricci tensor $\mathrm{Ric}_h$ satisfies

$$\mathrm{Ric}_h = \frac{1}{2}h.$$

All the first eigenvalues of $(M,h)$ are listed in the tables in [9]. A space of type I is a compact simply connected simple Lie group $G$, and $g = \frac{1}{4(n-1)}g'$. Here $g'$ is a bi-invariant metric on $G$ induced from the Killing form of $\mathfrak{g}$ whose Ricci tensor satisfies

$$\mathrm{Ric}_{g'} = \frac{1}{4}g',$$

and one can also find in [9] a list of the first eigenvalues of $(G,g')$. Due to these tables, one can see that all the first eigenvalues of $(M,g)$, except when $(M,g)$ is the unit sphere $(S^n,g_0)$, are bigger than $\left(\frac{4}{n}+1\right)(n-1)$. □

## §2. Isometric immersions

In [6], Lawson and Simons proved the following theorem.

**Theorem.** *Let $(M,g)$ be an $n$-dimensional compact Riemannian manifold isometrically immersed in the unit sphere $S^N(1)$ with second fundamental form $\beta$. Assume that*

$$\sum_{j=1}^{q}\sum_{\ell=q+1}^{n}\left(2|\beta(v_j,v_\ell)|^2 - \langle\beta(v_j,v_j),\beta(v_\ell,v_\ell)\rangle\right) < q(n-q),$$

*for every orthonormal base $\{v_j\}_{j=1}^n$ of $T_xM$ $(x\in M)$. Then the homology groups of $M$ vanish, $H_q(M,\mathbf{Z}) = H_{n-q}(M,\mathbf{Z}) = 0$.*

In [4], Elworthy and Rosenberg proved

**Theorem.** *Let $(M,g)$ be an $n$-dimensional compact Riemannian manifold isometrically minimally immersed in $S^N(1)$. Assume that*

$$\sum_{j=1}^{q}\sum_{\ell=q+1}^{n} K_g(v_j,v_\ell) > \frac{q(n-q)}{2},$$

*for every orthonormal base* $\{v_j\}_{j=1}^n$ *of* $T_xM$, $x \in M$. *Then*

$$H_q(M, \mathbf{Z}) = H_{n-q}(M, \mathbf{Z}) = 0.$$

*In particular, if* $K_g(v_j, v_\ell) > \frac{1}{2}$ *for all* $v_j$, $v_\ell \in T_xM$ $(x \in M)$, $j < \ell$, *then* $M$ *is a homotopy sphere. Here* $K_g(X, Y)$, *for an orthonormal pair* $X$, $Y \in T_xM$, *means the sectional curvature of* $(M, g)$ *for a plane in* $T_xM$ *defined by* $X$ *and* $Y$.

We show:

**Theorem 2.** *Let* $(M, h)$ *be a compact simply connected irreducible Riemannian symmetric space and* $h$ *an invariant Riemannian metric induced from the Killing form of the Lie algebra of the isometry group of* $(M, h)$. *Assume that*

$$(\#) \qquad \sum_{j=1}^{2} \sum_{\ell=3}^{n} K_h(v_j, v_\ell) > \frac{n}{4}(2\lambda_1(h) - 1),$$

*for all orthonormal bases* $\{v_j\}_{j=1}^n$ *of* $T_xM$, $x \in M$. *Then* $(M, h)$ *is the unit sphere* $(S^n, 2(n-1)g_0)$ *of constant curvature* $\frac{1}{2(n-1)}$.

*Proof.* In general, for a Riemannian manifold $(M, g)$,

$$\begin{aligned}\sum_{j=1}^{2} \sum_{\ell=3}^{n} K_g(v_j, v_\ell) &= \sum_{j=1}^{2} \sum_{\ell=1}^{n} K_g(v_j, v_\ell) - \sum_{j,\ell=1}^{2} K_g(v_j, v_\ell) \\ &= \mathrm{Ric}_g(v_1) + \mathrm{Ric}_g(v_2) - 2K_g(v_1, v_2) \\ &\geq 2(\inf \mathrm{Ric}_g - \max K_g).\end{aligned}$$

For our case, since $(M, h)$ satisfies $\mathrm{Ric}_h = \frac{1}{2}h$,

$$\sum_{j=1}^{2} \sum_{\ell=3}^{n} K_h(v_j, v_\ell) = 1 - 2K_h(v_1, v_2),$$

which yields

$$0 \leq \inf \sum_{j=1}^{2} \sum_{\ell=3}^{n} K_h(v_j, v_\ell) = 1 - 2 \max K_h \leq 1.$$

Thus, the inequality (#) implies $1 \geq \frac{n}{4}(2\lambda_1(h) - 1)$.

But, if we put $g = \frac{1}{2(n-1)}h$, then $\mathrm{Ric}_g = (n-1)g$, and the first eigenvalue $\lambda_1(g)$ of $g$ satisfies

$$\left(\frac{4}{n} + 1\right)(n-1) \geq \lambda_1(g).$$

Therefore, we have Theorem 2, because of Theorem 1. □

Theorem 2 immediately yields

**Corollary 3.** *Let* $G$ *be a compact simply connected simple Lie group, and* $h$ *a bi-invariant Riemannian metric* $G$ *induced from the Killing form* $B$ *of the Lie algebra* $\mathfrak{g}$ *of* $G$ *by identifying* $G \cong G \times G/\Delta G$ *as a symmetric space. That is,*

$$h_x(X_x, Y_x) = -\frac{1}{2}B(X, Y), \qquad X,\, Y \in \mathfrak{g},$$

*where* $X_x, Y_x \in T_xG$, $x \in G$. *Assume that*

$$\sum_{j=1}^{2}\sum_{\ell=3}^{n}|[v_j, v_\ell]|^2 > n(2\lambda_1(h) - 1),$$

*for every orthonormal base* $\{v_j\}_{j=1}^n$ *of* $\mathfrak{g}$ *with respect to* $-\frac{1}{2}B$. *Then* $(M, h)$ *is* $(SU(2), h)$.

*Remark 4.* (1) In [4, pp. 74-75], Elworthy and Rosenberg proposed to give a spectral geometric proof that $\pi_2(G) = 0$ for a Lie group $G$. The problem has been reduced to a question as to whether the above inequality in Corollary 3 would hold for all compact simply connected simple Lie group. But our Corollary 3 answers their question negatively.

(2) Let $(M, h)$ be a compact simply connected irreducible Riemannian symmetric space. Let

$$\Phi_1 : (M, \frac{\lambda_1}{n}h) \to S^N(1)$$

be an isometric minimal immersion into the unit sphere defined by the first eigenfunctons of $(M, h)$. Let $\beta$ be the second fundamental form and $\widetilde{h} = \frac{\lambda_1}{n}h$. The inequality (#) in Theorem 2 is equivalent to

$$\sum_{j=1}^{2}\sum_{\ell=3}^{n}K_{\widetilde{h}}(v'_j, v'_\ell) > \frac{|\beta|^2}{2} + \frac{n}{2},$$

for every orthonormal base $\{v'_j\}_{j=1}^n$ of $T_xM$ $(x \in M)$ relative to $\widetilde{h}$.

(3) Note that

$$\max K_h = \max\{|\alpha|^2; \alpha \in \Sigma^+(G, K)\},$$

where $\Sigma^+(G, K)$ is the set of all positive roots $\alpha$ of $G$ relative to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ whose restrictions $\alpha|_{\mathfrak{a}}$ to $\mathfrak{a}$ do not vanish. Here $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition of $\mathfrak{g}$, and the Cartan subalgebra $\mathfrak{t}$ is by definition a maximal abelian subalgebra of $\mathfrak{g}$ containing a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ contained in $\mathfrak{m}$.

(4) If a Riemannian symmetric space $(M, g)$ satisfies the inequality (#), then there might exist a $q$-th moment stable equation for $q \geq \frac{\dim M}{2}$ (cf. [4, p. 59]).

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