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SUBSYSTEMS OF THE SCHAUDER SYSTEM THAT ARE QUASIBASES FOR $L^p[0,1]$, $1 \le p < +\infty$

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Abstract. We show that if $\Phi = \{\varphi_i\colon\ i = 1, 2, \dots\}$ is a subsystem of the Faber-Schauder system, and if Φ is complete in $L^2[0,1]$, then Φ is a quasibasis for each space $L^p[0,1]$, $1 \le p < +\infty$. Although it follows from the work of Ul'yanov that each element of $L^p[0,1]$ can be represented by a Schauder series that converges unconditionally to the function, in the metric of the space, it proves to be the case that none of the aforementioned systems is an unconditional quasibasis for any of the L^p-spaces herein considered.

1. A question posed by Lusin [6], in 1915, asks whether it be possible to find, for every measurable function on $[0, 2\pi]$, a trigonometric series, with coefficient sequence converging to zero, that converges almost everywhere to the function. In the case of a.e. real-valued functions Menshov [7], [8] resolved this problem in the affirmative, and then proved that the restriction to real-valued functions can be removed if one replaces pointwise convergence by convergence in measure [9]. In a beautiful generalization of the latter theorem, Talalyan proved that the same conclusion obtains if the trigonometric system is replaced by any normal Schauder basis for $L^p[a,b]$, $p > 1$. (See, for example, [13].) Moreover, in [14], Talalyan showed that the analogue of Lusin's problem, in which the role of the trigonometric system is played by the Schauder system, also has a positive resolution.

Subsequently, this last result was rediscovered by Goffman [3], who observed that not all of the Schauder functions are required for this purpose, and a characterization of those Schauder subsystems for which such representations are possible was given in [16].

In a closely related investigation, Ul'yanov [15] has described those Orlicz classes, the members of which can be represented by Schauder series that converge in the metric of the space. In the present work, a measure of synthesis of these last two studies is accomplished by showing that the Schauder subsystems considered in [16] are, in fact, quasibases for each space $L^p[0,1]$, $1 \le p < +\infty$. The arguments given below, mutatis mutandis, suffice to show that this result holds for the separable Orlicz spaces as well.

Although it follows from the work of Ul'yanov that each element of $L^p[0,1]$ can be represented by a Schauder series that converges unconditionally to the function,

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in the metric of the space, it proves to be the case that none of the aforementioned systems is an unconditional quasibasis for any of the L^p-spaces herein considered.

2. Let B be a Banach space, let B^* be the conjugate space associated thereunto, and let $X = \{x_n : n \in \mathbb{N}\}$ be a subset of B. Then, X is a quasibasis for B iff there is a system $\{y_n^* : n \in \mathbb{N}\}$, contained in B^*, such that, for each x in B, $\sum_{n=1}^{\infty} y_n^*(x)x_n$ converges to x in the norm of B. This generalization of the Schauder-basis concept, introduced by Gelbaum [2], is a weaker notion, since the associated coefficient functionals need not be uniquely determined [12, pp. 278, 766]. If, for some allowable system Y^*, each expansion $\sum_{n=1}^{\infty} y_n^*(x)x_n$ converges unconditionally in B, then X is termed an unconditional quasibasis for B.

The Schauder, or, more properly, Faber-Schauder, system herein considered is the familiar collection of spike functions whose supports are dyadic subintervals of $[0,1]$. Except for the first of them, the elements of this system are indefinite integrals of the Haar functions, normalized so as to have maximum value 1. (As the proof of the following Lemma will make clear, this special system may be replaced by any member of the class of systems introduced by Schauder (see, for example [4, p. 50]).) As for the Haar functions, one has

$$h_1(t) = 1, \quad \forall t \in [0,1];$$

for $k = 0, 1, \dots, j$; $j = 1, 2, \dots 2^k$,

$$h_k^{(j)}(t) = \begin{cases} 2^{\frac{k}{2}}, & \text{if } \frac{2j-2}{2^{k+1}} < t < \frac{2j-1}{2^{k+1}}; \\ -2^{\frac{k}{2}}, & \text{if } \frac{2j-1}{2^{k+1}} < t < \frac{2j}{2^{k+1}}; \\ 0, & \text{otherwise;} \end{cases}$$

and, for $n = 2^k + j$, $h_n = h_k^{(j)}$.

As it happens, each Haar function can be expanded in an almost everywhere convergent Schauder series; indeed, relatively sparse subsets of the Schauder system are sufficient for this purpose, and, from this fact, it follows that these subsystems are quasibases for the L^p-spaces.

Lemma. *Let $\Phi = \{\varphi_n; n \in \mathbb{N}\}$ be any subsystem of the Faber-Schauder system for which*

$$(*) \qquad \mu(\limsup_n E_n) = 1,$$

where E_n represents the support of φ_n and μ is the Lebesgue measure, let $p \in [1, +\infty)$, and let h be a (nonzero) scalar multiple of some Haar function. Then, there exists a sequence of Φ-polynomials, $\{P_j\}_{j=1}^{\infty}$,

$$P_j = \sum_{i=n(j)+1}^{n(j+1)} b_i \varphi_i,$$

with

$$0 = n(1) < n(2) < \dots,$$

such that
(α) $\|h - \sum_{j=1}^{\ell} P_j\|_p \leq 2^{-\ell}, \quad \forall \ell \geq 1$;
(β) $\max\{\| \sum_{i=n(\ell)+1}^{s} b_i\varphi_i\|_p : s \leq n(\ell+1)\} \leq 2^{-\ell+1}, \quad \forall \ell > 1$;
and for every measurable subset, F, of $[0,1]$,
(γ) $\max\{\| \sum_{i=1}^{s} b_i\varphi_i\|_{L^p(F)} : s \in \mathbb{N}\} \leq \|h\|_{L^p(F)}$.

Proof. Let $E^+ = \{t : h(t) > 0\}$, let $E^- = \{t : h(t) < 0\}$, let $E = E^+ \cup E^-$, and let $m = \|h\|_\infty$. Because $\mathcal{V} = \{E_n : E_n \subset E\}$ is a Vitali covering of E, it is possible to choose pairwise disjoint members of $\mathcal{V}$, $E_{n_1}, \ldots, E_{n_{k(1)}}$, such that $\mu(E \setminus \bigcup_{i=1}^{k(1)} E_{n_i}) < \frac{1}{4m}$, and each E_{n_i} is contained in E. Denote the midpoint of E_{n_i} by τ_i, let

$$\varphi^{(1)} = f_1 = \sum_{i=1}^{k(1)} h(\tau_i)\varphi_{n_i},$$

let $E^{(1)} = \bigcup_{i=1}^{k(1)} E_{n_i}$, and let $F^{(1)} = E \setminus E^{(1)}$. Then $(h - f_1)\chi_{E^+} \geq 0$, $(f_1 - h)\chi_{E^-} \geq 0$, and

$$\begin{aligned}
\int_0^1 |h - f_1| d\mu &= \int_{F^{(1)}} |h - f_1| d\mu + \int_{E^{(1)}} |h - f_1| d\mu \\
&< \frac{1}{4} + \int_{E^{(1)} \cap E^+} (h - f_1) d\mu + \int_{E^{(1)} \cap E^-} (f_1 - h) d\mu \\
&= \frac{1}{4} + \frac{1}{2} \int_{E^{(1)} \cap E^+} h \, d\mu - \frac{1}{2} \int_{E^{(1)} \cap E^-} h \, d\mu \\
&= \frac{1}{4} + \frac{1}{2} \int_{E^{(1)}} |h| d\mu < \frac{1}{2}(1 + J),
\end{aligned}$$

where $J = \displaystyle\int_E |h| d\mu = \|h\|_1$.

Let $E^{(1)} = G_1^{(1)} \cup \cdots \cup G_{N_1}^{(1)}$, where the $G_k^{(1)}$ are nonoverlapping (dyadic) intervals, such that, for each k, f_1 is linear over $G_k^{(1)}$ and vanishes at one end of $G_k^{(1)}$. From an application of the Vitali theorem to $\mathcal{V}_1 = \{E_n : n > n_{k(1)}\}$ follows the existence of a finite subfamily of disjoint members of $\mathcal{V}_1$, each element of which is contained in some $G_k^{(1)}$, whose union covers $E^{(1)}$ save for a set whose measure does not exceed $\frac{1}{32\,m}$. Proceeding in the same manner, let $F^{(1)} = H_1^{(1)} \cup \cdots \cup H_{M_1}^{(1)}$, where the $H_k^{(1)}$ are nonoverlapping (dyadic) intervals, such that $h - f_1 = h$ is linear over each of them. Again, there are finitely-many, pairwise-disjoint elements of $\mathcal{V}_1$, each of which lies entirely within some $H_k^{(1)}$, whose union covers $F^{(1)}$ save for a set whose measure does not exceed $\frac{1}{32\,m}$. Let $\{E_{n_{k(1)+1}}, \ldots, E_{n_{k(2)}}\}$ be the union of the two subsets of $\mathcal{V}_1$ thus selected, let $E^{(2)} = \bigcup_{i=k(1)+1}^{k(2)} E_{n_i}$, and let $F^{(2)} = E \setminus E^{(2)}$. Again, denoting the center of E_{n_i} by τ_i, let

$$\varphi^{(2)} = \sum_{i=k(1)+1}^{k(2)} (h - f_1)(\tau_i)\varphi_{n_i},$$

and let $f_2 = \varphi^{(1)} + \varphi^{(2)}$. Then

$$\begin{aligned}
0 \leq f_1 \leq f_2 \leq h, &\quad \text{on } E^+, \\
h \leq f_2 \leq f_1 \leq 0, &\quad \text{on } E^-,
\end{aligned}$$

and

$$\begin{aligned}\int_0^1 |h-f_2|d\mu &= \int_{F^{(2)}} + \int_{E^{(2)}} \\ &< \frac{1}{16} + \int_{E^{(2)}\cap E^+} (h-f_2)d\mu + \int_{E^{(2)}\cap E^-} (f_2-h)d\mu \\ &\le \frac{1}{16} + \frac{1}{2}\int_{E^{(2)}\cap E^+} (h-f_1)d\mu + \frac{1}{2}\int_{E^{(2)}\cap E^-} (f_1-h)d\mu \\ &\le \frac{1}{16} + \frac{1}{2}\int_{E^+} (h-f_1)d\mu + \frac{1}{2}\int_{E^-} (f_1-h)d\mu \\ &< \frac{1}{16} + \frac{1}{2}(\frac{1}{4} + \frac{1}{2}J) < \frac{1}{2^2}(1+J).\end{aligned}$$

Proceeding thus, inductively, one constructs, for every natural number n, a finite Schauder series

$$f_n = \sum_{i=1}^{k(n)} a_i \varphi_{n_i},$$

such that, for all j,

$$\begin{aligned} 0 \le \sum_{i=1}^{j} a_i\varphi_{n_i} \le \sum_{i=1}^{j+1} a_i\varphi_{n_i} \le h, \text{ on } E^+, \\ h \le \sum_{i=1}^{j+1} a_i\varphi_{n_i} \le \sum_{i=1}^{j} a_i\varphi_{n_i} \le 0, \text{ on } E^-,\end{aligned}$$

and

$$\int_0^1 |h-f_n|d\mu < \frac{1}{2^n}(1+J), \quad \forall n.$$

From two applications of the Lebesgue theorem of dominated convergence, it follows that

$$h(t) = \sum_{i=1}^{\infty} a_i\varphi_{n_i}(t), \text{ a.e.},$$

and that

$$\lim_n \int_0^1 |h-f_n|^p d\mu = 0.$$

The sequence $\{P_j\}_{j=1}^{\infty}$ is defined by means of the following inductive scheme. Let $n(1)=0$, let $n(2)$ be the least element of $\{n : \|h-f_n\|_p < 2^{-1}\}$, and let

$$P_1 = f_{n(2)}.$$

If $n(j+1)$ and P_j have been defined, let $n(j+2)$ be the least element of $\{n > n(j+1) : \|h-f_n\|_p < 2^{-j-1}\}$, and let

$$P_{j+1} = f_{n(j+2)} - f_{n(j+1)}.$$

The satisfaction of condition (α) is immediate, and the verification of conditions (β) and (γ) involves nothing more than the monotonicity, on E^+ and on E^-, of the partial sums of $\sum_{i=n(\ell)+1}^{n(\ell+1)} a_i\varphi_{n_i}$. □

Theorem 1. *If $\Phi = \{\varphi_i : i = 1, 2, \dots\}$ is a subsystem of the Faber-Schauder system, and if Φ satisfies the condition $(*)$, then Φ is a quasibasis for each space $L^p[0,1]$, $1 \le p < +\infty$.*

Proof. The following argument is an abbreviation of one given in [5].

Let $\mathcal{H} = \{f_k : k = 1, 2, \dots\}$ be the system of Haar functions normalized with respect to the L^p-norm, and let $\{g_k : k = 1, 2, \dots\}$ be the corresponding conjugate system. By virtue of the Lemma, and the fact that every truncate of Φ also satisfies (*), one may construct a double sequence of Φ-polynomials, $\{P_{kj}\}_{k=1, j=k}^{\infty, \infty}$,

$$P_{kj} = \sum_{i=n_{k-1}(j)+1}^{n_k(j)} a_i \varphi_i,$$

with

$$0 = n_0(1) < n_1(1) = n_0(2) < n_1(2) < n_2(2) = n_0(3) < \dots$$
$$\dots < n_0(j) < n_1(j) < \dots < n_j(j) = n_0(j+1) < \dots,$$

such that
(i) $\|f_k - \sum_{j=k}^{\ell} P_{kj}\|_p \le 2^{-\ell}, \forall k, \forall \ell \ge k$;
(ii) $\sup\{\| \sum_{i=n_{k-1}(\ell)+1}^{s} a_i \varphi_i \|_p : s \le n_k(\ell)\} \le 2^{-\ell+1}$, if $\ell > k$;
and, for every measurable subset, F, of $[0,1]$,
(iii) $\sup\{\| \sum_{i=n_{\ell-1}(\ell)+1}^{s} a_i \varphi_i \|_{L^p(F)} : s \le n_\ell(\ell)\} \le \|f_\ell\|_{L^p(F)}$.

One associates with Φ the system $\Psi = \{\psi_i : i \in \mathbb{N}\}$, where

$$\psi_i = a_i g_k, \text{ for } n_{k-1}(\ell) < i \le n_k(\ell).$$

Then, with the coefficient functionals defined on $L^p[0,1]$ by setting

$$b_i(\cdot) = \int_0^1 (\cdot) \psi_i d\mu, \forall i,$$

Φ proves to be a quasibasis for $L^p[0,1]$.

The following estimates suffice for the demonstration.

Let f be an arbitrary element of $L^p[0,1]$, and let $\sum_{k=1}^{\infty} c_k(f) f_k$ be the expansion of f in the system $\mathcal{H}$. Since $\mathcal{H}$ is a Schauder basis for $L^p[0,1]$, it follows that

$$\lim_k c_k(f) = 0.$$

The partial sums, $S_n(f)$, of the series

$$\sum_{i=1}^{\infty} b_i \varphi_i$$

fall into two disjoint classes, according as

$$n = n_k(\ell), \text{ or } n_{k-1}(\ell) < n < n_k(\ell),$$

for some natural numbers k and ℓ (with $\ell \ge k$). One estimates the error made in approximating f by $S_n(f)$, for each variety of n.

In the first case, if $\ell > k$, then one has

$$\begin{aligned}\|f - \sum_{i=1}^{n} b_i(f)\varphi_i\|_p &= \|f - \sum_{j=1}^{k}\sum_{i=j}^{\ell} c_j P_{ji} - \sum_{j=k+1}^{\ell-1}\sum_{i=j}^{\ell-1} c_j P_{ji}\|_p \\ &\le \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_p + \|\sum_{j=1}^{k} c_j(f_j - \sum_{i=j}^{\ell} P_{ji}) + \sum_{j=k+1}^{\ell-1} c_j(f_j - \sum_{i=j}^{\ell-1} P_{ji})\|_p \\ &\le \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_p + \sum_{j=1}^{k} |c_j|\|f_j - \sum_{i=j}^{\ell} P_{ji}\|_p + \sum_{j=k+1}^{\ell-1} |c_j|\|f_j - \sum_{i=j}^{\ell-1} P_{ji}\|_p \\ &\le \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_p + 2^{-\ell+1}\sum_{j=1}^{\ell-1}|c_j| \\ &\le \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_p + 2^{-\ell+1}(\ell)\|f\|_p,\end{aligned}$$

since

$$|c_j| = |\int_0^1 f g_j d\mu| \le \|f\|_p\|g_j\|_q = \|f\|_p\|h_j\|_p\|h_j\|_q = \|f\|_p\|h_j\|_\infty^2\mu(\Delta_j) = \|f\|_p,$$

where Δ_j is the support of h_j. If, on the other hand, $\ell = k$, then a simpler computation yields

$$\|f - \sum_{i=1}^{n} b_i(f)\varphi_i\|_p \le \|f - \sum_{j=1}^{\ell} c_j f_j\|_p + 2^{-\ell}(\ell)\|f\|_p.$$

As for an n that satisfies $n_{k-1}(\ell) < n < n_k(\ell)$, with $\ell \ge k$, suppose, first, that $\ell > k$. Then

$$\begin{aligned}\|f - S_n(f)\|_p &\le \|f - \sum_{i=1}^{n_{k-1}(\ell)} b_i(f)\varphi_i\|_p + \|\sum_{i=n_{k-1}(\ell)+1}^{n} b_i(f)\varphi_i\|_p \\ &\le \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_p + 2^{-\ell+1}(\ell-1)\|f\|_p + \|c_k \sum_{i=n_{k-1}(\ell)+1}^{n} a_i\varphi_i\|_p \\ &\le \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_p + 2^{-\ell+1}(\ell)\|f\|_p.\end{aligned}$$

On the other hand, if $\ell = k$, then

$$\|f - S_n(f)\|_p \le \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_p + 2^{-\ell+1}(\ell)\|f\|_p + \|c_\ell \sum_{i=n_{\ell-1}(\ell)+1}^{n} a_i\varphi_i\|_p.$$

Setting

$$\sigma_{\ell n} = \sum_{i=n_{\ell-1}(\ell)+1}^{n} a_i\varphi_i,$$

one has

$$\begin{aligned}\|\sigma_{\ell n}\|_p^p &= \|\sigma_{\ell n}\|^p_{L^p(\Delta_\ell)} + \|\sigma_{\ell n}\|^p_{L^p([0,1]\setminus\Delta_\ell)}\\ &\le \|f_\ell\|^p_{L^p(\Delta_\ell)} + \|f_\ell\|^p_{L^p([0,1]\setminus\Delta_\ell)}, \text{ by virtue of condition (iii),}\\ &= 1.\end{aligned}$$

Thus,

$$\|c_\ell \sigma_{\ell n}\|_p \le |c_\ell| = o(1), \text{ as } \ell \to \infty,$$

and it follows that

$$\lim_n \|S_n(f) - f\|_p = 0.$$

□

Theorem 2. *The Faber-Schauder system is not an unconditional quasibasis for any of the spaces $L^p[0,1]$, $1 \le p < +\infty$.*

Proof. Three preliminary comments are in order: (i) If some subsystem of a Schauder system were an unconditional quasibasis for some space $L^p[0,1]$, then the entire system would be an unconditional quasibasis for that space. Thus, Theorem 2 assures one that no subsystem of the Faber-Schauder system can be an unconditional quasibasis for any of the L^p-spaces cited. (ii) It is known [11] that there is no unconditional Schauder basis for $L^1[0,1]$, and the proof of this result can be modified so as to apply to quasibases. Thus, the following argument treats only the cases $1 < p < +\infty$. (iii) If $X = \{x_n : n \in \mathbb{N}\}$, together with the conjugate system $Y^* = \{y_n^* : n \in \mathbb{N}\}$, is an unconditional quasibasis for a Banach space, B, then the family of operators

$$\{S_{n,\varepsilon}(\cdot) = \sum_{k=1}^{n} \varepsilon_k y_k^*(\cdot) x_k : n \in \mathbb{N}, \varepsilon \in \{-1,0,1\}^{\aleph_0}\}$$

is a bounded subset of $L(B,B)$. Here, again, the proof of this fact is very nearly identical to the proof of the corresponding result for Schauder bases.

Now suppose that the Faber-Schauder system $\Phi = \{\varphi_n : n = 0,1,\dots\}$ were an unconditional quasibasis for some space $L^p[0,1]$, with $p \in (1,+\infty)$. Let $\Psi^* = \{\psi_n^* : n = 0,1,\dots\}$ be a corresponding subset of the conjugate space such that, for each $f \in L^p[0,1]$, the expansion

$$\sum_{n=0}^{\infty} b_n(f)\varphi_n, \quad b_n(f) = \int_0^1 f\psi_n^* d\mu,$$

converges unconditionally to f in the L^p-norm.

For each $n \in \mathbb{N}$, let r_n denote the n^{th} Rademacher function:

$$r_n(t) = \begin{cases} +1, & \frac{2k-2}{2^n} < t < \frac{2k-1}{2^n};\\ -1, & \frac{2k-1}{2^n} < t < \frac{2k}{2^n}; \quad k = 1,\dots,2^{n-1};\end{cases}$$

$r_n(\frac{j}{2^n}) = 0$, $j = 0,1,\dots,2^n$; and let

$$b_k^n = \int_0^1 r_n \psi_k^* d\mu, \quad k = 0,1,\dots.$$

Then, for each n,

$$r_n = \sum_{k=0}^{\infty} b_k^n \varphi_k,$$

in $L^p[0,1]$, and these series converge unconditionally there.

Let Δ be an interval of the form $(\frac{j-1}{2^n}, \frac{j}{2^n})$, for some $j \in \{1, \dots, 2^n\}$. In such an interval, r_{n+1} will complete one full cycle. Let $\sum_\Delta b_k^{n+1}\varphi_k$ be obtained by deleting from the series $\sum_{k=2^n+1}^{\infty} b_k^{n+1}\varphi_k$ all of those terms, $b_j^{n+1}\varphi_j$, whose supports are not contained in Δ. Because $\sum_{k=1}^{\infty} b_k^{n+1}\varphi_k$ converges unconditionally in $L^p[0,1]$, it follows that $\sum_\Delta b_k^{n+1}\varphi_k$ converges there as well; thus, for almost all t in Δ, one has

$$r_{n+1}(t) - \sum_{k=0}^{2^n} b_k^{n+1}\varphi_k(t) = \sum_\Delta b_k^{n+1}\varphi_k(t),$$

and, as a consequence,

$$\sum_\Delta |b_k^{n+1}|\varphi_k \geq |r_{n+1} - \sum_{k=0}^{2^n} b_k^{n+1}\varphi_k|, \quad \text{a.e. in } \Delta.$$

Because $r_{n+1} - \sum_{k=0}^{2^n} b_k^{n+1}\varphi_k$ is piecewise linear on Δ, relatively simple geometric arguments can be used to demonstrate the existence of a positive constant, K, such that

$$\int_\Delta \sum_\Delta |b_k^{n+1}|\varphi_k d\mu \geq K|\Delta|.$$

(Crude, but elementary, estimates assure one that $K \geq \frac{1}{32}$.) It follows that

$$\int_0^1 \sum_{k=2^n+1}^{\infty} |b_k^{n+1}|\varphi_k d\mu \geq K, \quad \forall n \in \mathbb{N}.$$

Let $n(0) = 1$, and let $m(1)$ be chosen so that

$$\int_0^1 \sum_{k=2^{n(0)}+1}^{2^{m(1)}} |b_k^{n(0)+1}|\varphi_k d\mu \geq \frac{2}{3}K.$$

Because the sequence of expansion coefficients in the Fourier-Rademacher series of any integrable function converges to zero, one can find a natural number $n(1) > m(1)$ such that

$$\int_0^1 \sum_{k=2^{n(1)}+1}^{\infty} |b_k^{n(0)+1}|\varphi_k d\mu < \frac{K}{2^4},$$

and

$$|b_k^{n(1)+1}| < \frac{1}{2^3}|b_k^{n(0)+1}|, \quad \forall k \in [2^{n(0)}+1, 2^{m(1)}].$$

Similarly, there exists a natural number, $m(2)$, greater than $n(1)$, such that

$$\int_0^1 \sum_{k=2^{n(1)}+1}^{2^{m(2)}} |b_k^{n(1)+1}|\varphi_k d\mu \geq \frac{2}{3}K,$$

and there is a natural number, $n(2)$, that exceeds $m(2)$, such that

$$\int_0^1 \sum_{k=2^{n(2)}+1}^{\infty} |b_k^{n(0)+1}| \varphi_k d\mu < \frac{K}{2^5},$$
$$\int_0^1 \sum_{k=2^{n(2)}+1}^{\infty} |b_k^{n(1)+1}| \varphi_k d\mu < \frac{K}{2^5},$$
$$|b_k^{n(2)+1}| < \frac{1}{2^4} |b_k^{n(0)+1}|, \quad \forall k \in [2^{n(0)}+1, 2^{m(1)}],$$

and

$$|b_k^{n(2)+1}| < \frac{1}{2^4} |b_k^{n(1)+1}|, \quad \forall k \in [2^{n(1)}+1, 2^{m(2)}].$$

Proceeding thus, inductively, one constructs two sequences of elements of $\mathbb{N}$, $\{n(j)\}_{j=0}^{\infty}$, $\{m(j)\}_{j=1}^{\infty}$, such that

$$1 = n(0) < m(1) < n(1) < m(2) < \cdots < m(j) < n(j) < \dots,$$

and, for every j in $\mathbb{N}$,

$$\int_0^1 \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |b_k^{n(j)+1}| \varphi_k d\mu \geq \frac{2}{3} K,$$
$$\int_0^1 \sum_{k=2^{n(j+1)}+1}^{\infty} |b_k^{n(\ell)+1}| \varphi_k d\mu < \frac{K}{2^{j+4}},$$

and

$$|b_k^{n(j+1)+1}| < \frac{1}{2^{j+3}} |b_k^{n(\ell)+1}|, \quad \forall k \in [2^{n(\ell)}+1, 2^{m(\ell+1)}], \quad \ell = 0, 1, \dots, j.$$

Now consider the series $\sum_{j=1}^{\infty} \frac{1}{j} r_{n(j)+1}$. By virtue of Khinchin's inequality (see, for example, [4, p. 130ff], there is a positive constant, C_p, that depends only upon p, such that

$$\left\| \sum_{j=m+1}^{n} \frac{1}{j} r_{n(j)+1} \right\|_p \leq C_p \left(\sum_{j=m+1}^{n} \frac{1}{j^2} \right)^{\frac{1}{2}}.$$

Accordingly, the foregoing series converges, in the L^p-norm, to an element $f \in L^p[0,1]$. Since Φ, together with the conjugate system Ψ^*, is an unconditional quasibasis for $L^p[0,1]$,

$$f = \sum_{k=0}^{\infty} b_k(f) \varphi_k,$$

and the series converges unconditionally in $L^p[0,1]$. Since

$$\left|b_k(f) - \sum_{i=1}^{j} \frac{1}{i} b_k^{n(i)+1}\right| = \left|b_k(f) - \sum_{i=1}^{j} \frac{1}{i} \int_0^1 r_{n(i)+1}\psi_k^* d\mu\right|$$
$$\leq \left|\int_0^1 \left(f - \sum_{i=1}^{j} \frac{1}{i} r_{n(i)+1}\right) \psi_k^* d\mu\right|$$
$$\leq \left\|f - \sum_{i=1}^{j} \frac{1}{i} r_{n(i)+1}\right\|_p \|\psi_k^*\|_q \to 0, \text{ as } j \to +\infty,$$

one has

$$b_k(f) = \sum_{i=1}^{\infty} \frac{1}{i} b_k^{n(i)+1}.$$

Thus, for all $k \in [2^{n(j)}+1, 2^{m(j+1)}]$, and for every $j \geq 1$,

$$|b_k(f)| \geq \frac{1}{j}|b_k^{n(j)+1}| - \sum_{i \neq j} \frac{1}{i}|b_k^{n(i)+1}|$$
$$= \frac{1}{j}|b_k^{n(j)+1}| - \sum_{i=1}^{j-1} \frac{1}{i}|b_k^{n(i)+1}| - \sum_{i=j+1}^{\infty} \frac{1}{i}|b_k^{n(i)+1}|$$
$$\geq \frac{1}{j}|b_k^{n(j)+1}| - \sum_{i=1}^{j-1} \frac{1}{i}|b_k^{n(i)+1}| - \sum_{i=j+1}^{\infty} \frac{1}{i}\frac{1}{2^{i+2}}|b_k^{n(j)+1}|$$
$$\geq \frac{7}{8j}|b_k^{n(j)+1}| - \sum_{i=1}^{j-1} \frac{1}{i}|b_k^{n(i)+1}|.$$

Hence, for every $j \geq 1$,

$$\int_0^1 \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |b_k(f)|\varphi_k d\mu \geq \frac{7}{8j} \int_0^1 \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |b_k^{n(j)+1}|\varphi_k d\mu$$
$$- \sum_{i=1}^{j-1} \frac{1}{i} \int_0^1 \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |b_k^{n(i)+1}|\varphi_k d\mu$$
$$\geq \frac{7K}{12j} - \sum_{i=1}^{j-1} \frac{1}{i} \frac{K}{2^{j+1}} \geq \frac{7K}{12j} - \frac{K}{16j} > \frac{K}{2j}.$$

Let the sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ be defined in the following manner:

$$\varepsilon_k = \begin{cases} \operatorname{sgn} b_k(f), & \text{if } 2^{n(j)}+1 \leq k \leq 2^{m(j+1)}, \quad j = 0, 1, \ldots; \\ 0, & \text{otherwise;} \end{cases}$$

and let

$$f_\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k b_k(f)\varphi_k = \sum_{j=0}^{\infty} \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |b_k(f)|\varphi_k.$$

By virtue of preliminary observation (iii), f_ε should be an element of $L^p[0,1]$, but

$$\|f_\varepsilon\|_p \geq \|f_\varepsilon\|_1 \geq \frac{K}{2}\sum_{j=1}^{\infty}\frac{1}{j} = +\infty,$$

and, from this contradiction, it follows that Φ is not an unconditional quasibasis for $L^p[0,1]$. □

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