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GLASNER SETS AND POLYNOMIALS IN PRIMES

R. NAIR AND S. L. VELANI

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Abstract. A set of integers S is said to be Glasner if for every infinite subset A of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\varepsilon > 0$ there exists some $n \in S$ such that the dilation $nA = \{nx\colon\ x \in A\}$ intersects every integral of length ε in $\mathbb{T}$. In this paper we show that if p_n denotes the nth prime integer and f is any non-constant polynomial mapping the natural numbers to themselves, then $(f(p_n))_{n\geq 1}$ is Glasner. The theorem is proved in a quantitative form and generalizes a result of Alon and Peres (1992).

1. Introduction

Following D. Berend and Y. Peres [BP] we say a set S of integers is Glasner if for every infinite set A contained in the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\varepsilon > 0$, some dilation $nA = \{nx\colon\ x \in A\}$ with $n \in S$ is ε-dense (that is, nA intersects every interval of length ε). This definition is motivated by the 1979 result of S. Glasner [G] in which he showed that given an infinite set $A \subset \mathbb{T}$ there exists a natural number n such that nA is ε-dense in $\mathbb{T}$.

In [BP] and also [AP] it is shown that sequences other than the natural numbers are also Glasner. For instance in [BP] it is shown that if P is a non-constant polynomial with integer coefficients, then $\{P(n)\colon\ n \in \mathbb{N}\}$ is Glasner, and in [AP] it is shown that $\{p_n\colon\ n \in \mathbb{N}\}$, where p_n denotes the nth rational prime, is also Glasner. In [AP] too there is a greater emphasis on the quantitative forms of results in [BP]. The methods of [AP] are Fourier analytic and it is this which allows the more quantitative forms of the result. This means that given $\varepsilon > 0$ a lower bound for $\#A$ can be given in terms of ε for an arbitrary finite set A to ensure that there is some element $n \in S$ such that the dilation nA is ε-dense. In the opposite direction it is known that finite unions of sequences $(k_n)_{n\geq 1}$ such that

$$\liminf_{n\to\infty} \tfrac{k_{n+1}}{k_n} \geq g > 1$$

for some q are not Glasner [BP]. In this paper the following theorem is proved

Theorem 1. *Let f be a non-constant polynomial of degree $L \geq 1$ mapping the natural numbers to themselves and suppose $\delta > 0$. There exists a positive real number $\varepsilon(f, \delta)$ such that if $0 < \varepsilon < \varepsilon(f, \delta)$, then any set X contained in $\mathbb{T}$ of*

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cardinality

$$s > \left(\frac{1}{\varepsilon}\right)^{2L+\delta}$$

has an ε-dense dilation of the form $f(p)X$, for some rational prime integer p.

The theorem is easily seen to be a generalization of Theorem 6.3 [AP], in particular it allows us to combine the two separate statements of the theorem in [AP] by considering polynomials in primes.

The paper is organized as follows. In Section 2, we group together various concepts and results required in proving Theorem 1. In Section 3 we give the proof of Theorem 1.

2. Some preliminary lemmas

In this section we state some technical results required in the sequel. The first is abstracted from special cases dealt with in [AP].

Lemma 2. *Given $\varepsilon > 0$, let $X = \{x_1, \ldots, x_s\}$ be any set of finitely many points contained in $\mathbb{T}$ such that for every natural number n indexing the sequence $(k_n)_{n\geq 1}$ the dilation $k_n X$ is not ε-dense. Then there is an absolute constant $C > 0$ such that if*

$$M = \left[\left(\frac{1}{\varepsilon}\right)\log^2\left(\frac{1}{\varepsilon}\right)\right],$$

we have for any $N > 1$

$$s^2 \leq \left(\frac{C}{\varepsilon}\right)\sum_{m=1}^{M}\sum_{j=1}^{s}\sum_{l=1}^{s}\frac{1}{N}\sum_{n=1}^{N} e_m(k_n(x_j - x_l)),$$

where $e_m(t) = \exp(2\pi\ imt)$.

The following result is an immediate consequence of Theorem 4 in [Na].

Lemma 3. *If f is a non-constant polynomial mapping the natural numbers to themselves, α is an irrational real number and p_n is the nth rational prime number, then the sequence $(f(p_n)\alpha)_{n\geq 1}$ is uniformly distributed modulo one. In view of Weyl's criterion, this is equivalent to*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e_m(f(p_n)\alpha) = 0,$$

for every integer m other than zero.

We next need the following classical estimate for rational exponential sums [V].

Lemma 4. *Let θ denote a polynomial of degree L mapping the natural numbers to themselves. Then for any positive ε, there is a constant $C_1 = C_1(\varepsilon, L) > 0$ such that*

$$\left|\frac{1}{\phi(b)}\sum_{\substack{a=1\\(a,b)=1}}^{b} e_1\left(\frac{\theta(a)}{b}\right)\right| \leq \frac{C_1}{b^{\frac{1}{L}-\varepsilon}},$$

where ϕ denotes Euler's totient function and (a, b) denotes the highest common factor of a and b.

The following is also taken from [AP].

Lemma 5. *Let $\{x_1, \dots, x_s\}$ be an arbitrary set of s distinct points in the unit interval $[0,1)$. For $m \in \mathbb{N}$, denote by h_m the number of pairs (i,j) with $1 \leq i < j \leq s$, such that $m(x_i - x_j)$ is an integer. Suppose $\beta > 0$. Then if s is sufficiently large, for any $m \geq 1$ the partial sum*

$$H_m = \sum_{l=1}^{m} h_l$$

satisfies the inequality

$$H_m \leq (sm)^{\beta+1}.$$

In fact, if $m > \exp(10)$, then the estimate can be sharpened to

$$H_m \leq 3(sm)^{(\log\log m)^{-1}+1}.$$

The trivial upper bound for H_m in the above lemma is sm^2.

3. Proof of Theorem 1

Let $\varepsilon > 0$ and suppose that $X = \{x_1, \dots, x_s\}$ is a set of s points contained in $\mathbb{T}$ such that the dilation $f(p_n)X$ is not ε-dense for any $n \in \mathbb{N}$. Then on setting

$$M = \left[\left(\frac{1}{\varepsilon}\right)\log^2\left(\frac{1}{\varepsilon}\right)\right],$$

Lemma 2 implies that for any $N > 1$

$$s^2 \leq \left(\frac{C}{\varepsilon}\right)\sum_{m=1}^{M}\sum_{j=1}^{s}\sum_{l=1}^{s}\frac{1}{N}\sum_{n=1}^{N} e_m(f(p_n)(x_j - x_l)). \tag{3.1}$$

We now obtain an upper bound for the right hand side of the above expression which in turn will imply the theorem.

As a consequence of Lemma 3, if for a particular j and l the difference $x_j - x_l$ is irrational, then the average furthermost to the right in (3.1) tends to zero as N tends to infinity. This means that in estimating the right hand side of (3.1) we need only consider the contribution of the terms in the double sum in j and l for which the corresponding $x_j - x_l$ is rational.

Note that because f maps the natural numbers to themselves it must have rational coefficients. Hence for any rational $\frac{a}{b}$ in reduced form, we may write

$$mf(n)\frac{a}{b} = \frac{(a'_L n^L + \cdots + a'_1 n)}{b'} + r = \frac{\theta(n)}{b'} + r,$$

where r is a rational, b' depends on b, m and the coefficients of f, and the highest common factor of the integers $a'_L, \dots, a'_1$ and b' is one. As a consequence of Dirichlet's theorem on arithmetic progressions, primes are uniformly distributed among the reduced residue classes modulo b'. This together with the fact that

$$\varepsilon_1\left(mf(p_n)\frac{a}{b}\right) = e_1\left(\frac{\theta(c)}{b'} + r\right) \quad \text{whenever } p_n \equiv c \pmod{b'},$$

implies that if $x_j - x_l = \frac{a}{b}$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e_m(f(p_n)(x_j - x_l)) = \frac{\lambda}{\phi(b')} \sum_{\substack{a=1 \\ (a,b')=1}}^{b'} e_1\left(\frac{\theta(a)}{b'}\right), \tag{3.2}$$

where λ has absolute value one.

From Lemma 4 we have that for any positive ε_1, there exists a positive constant $C_1 = C_1(\varepsilon_1, L)$ and $C_1^* = C_1^*(\varepsilon_1, f)$ such that

$$\left| \frac{\lambda}{\phi(b')} \sum_{\substack{a=1 \\ (a,b')=1}}^{b'} e_1\left(\frac{\theta(a)}{b'}\right) \right| < \frac{C_1}{(b')^{\frac{1}{L}-\varepsilon_1}} \leq C_1^* \left(\frac{(m,b)}{b}\right)^{\frac{1}{L}-\varepsilon_1}.$$

Also because there are at most $\frac{M}{r}$ multiples of r less than M that divide b we know that

$$\sum_{m=1}^{M} (m,b)^{\frac{1}{L}-\varepsilon_1} \leq \sum_{\substack{r|b \\ r\leq M}} \left(\frac{M}{r}\right) r^{\frac{1}{L}-\varepsilon_1}.$$

We also note that

$$\sum_{\substack{r|b \\ r\leq M}} \left(\frac{M}{r}\right) r^{\frac{1}{L}-\varepsilon_1} \leq M \sum_{\substack{r|b \\ r\leq M}} 1 \leq M d(b) \leq C_2 M b^{\varepsilon_1},$$

where C_2 is a positive constant depending only on ε_1 and as usual $d(n)$ denotes the number of integers between one and n inclusive that divide n.

It follows from (3.2) and the above estimates that for any distinct $x_j, x_l \in X$ for which $x_j - x_l = \frac{a}{b}$ we have the inequality

$$\sum_{m=1}^{M} \left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e_m(f(p_n)(x_j - x_l)) \right| \leq C_3 M b^{-\frac{1}{L}+\varepsilon_0}, \tag{3.3}$$

where $C_3 = C_1^* C_2$ and $\varepsilon_0 = 2\varepsilon_1$.

Inequality (3.3) is the crucial estimate. To complete the proof we proceed as in [AP] (proof of Theorem 6.3). Let

$$g_b = \#\{(j,l)\colon\ 1 \leq j < l \leq s \text{ and } x_j - x_l = \frac{a}{b} \text{ for some } a \text{ with } (a,b) = 1\}$$

and let

$$G_b = \sum_{i=1}^{b} g_i.$$

We can bound the right hand side of (3.1) by the sum over all $1 \leq j, l \leq s$ of the left hand side of (3.3). By partial summation we obtain that for some positive constant

C_4 depending only on ε_0 and L,

$$\begin{aligned} s^2 &\le C_4 M\varepsilon^{-1}\left(s+\sum_{b\ge 2} g_b b^{-\frac{1}{L}+\varepsilon_0}\right) \\ &= C_4 M\varepsilon^{-1}\left(s+\sum_{b\ge 2} G_b\big(b^{-\frac{1}{L}+\varepsilon_0}-(b+1)^{-\frac{1}{L}+\varepsilon_0}\big)\right). \end{aligned} \tag{3.4}$$

In order to estimate the sum on the right hand side we use the trivial inequality $G_b \le s^2$ when $b > s$ and for $b \le s$ we make use of Lemma 5 which implies that $G_b \le H_b \le (sb)^{1+\varepsilon_0}$, assuming that s is sufficiently large. It follows that for ε_0 sufficiently small (depending only on the degree L),

$$\begin{aligned} &\sum_{b\ge 2} G_b\left(b^{-\frac{1}{L}+\varepsilon_0}-(b+1)^{-\frac{1}{L}+\varepsilon_0}\right) \\ &\qquad\le s^{1+\varepsilon_0}\left(\sum_{b=2}^{s} b^{2\varepsilon_0-\frac{1}{L}}+s^{1-\frac{1}{L}}\right) \\ &\qquad\le s^{2+3\varepsilon_0-\frac{1}{L}}(1+s^{-2\varepsilon_0}). \end{aligned}$$

This together with (3.4) implies that

$$s^2 \le C_5 M\varepsilon^{-1} s^{2+3\varepsilon_0-\frac{1}{L}},$$

where $C_5 = C_5(\varepsilon_0, L)$ is a positive constant. Finally, on substituting the value of M into the above expression we obtain that for any positive δ and for any sufficiently small (depending on δ and L) positive ε the inequality

$$s \le \left(\frac{1}{\varepsilon}\right)^{2L+\delta}$$

is satisfied, under the assumption that $f(p_n)X$ is never ε-dense. This completes the proof of the theorem.

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Department of Pure Mathematics, University of Liverpool, P.O. Box 147, Liverpool L69 3BX, United Kingdom

E-mail address: nair@liv.ac.uk

Department of Mathematics, Imperial College, University of London, Huxley Building, 180 Queen's Gate, London SW7 2BZ, United Kingdom

E-mail address: s.velani@ic.ac.uk