PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 10, October 1998, Pages 3083–3087
S 0002-9939(98)04399-8

THE FUNDAMENTAL GROUP OF A COMPACT METRIC SPACE

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(Communicated by Andreas R. Blass)

ABSTRACT. We give a forcing free proof of a conjecture of Mycielski that the fundamental group of a connected locally connected compact metric space is either finitely generated or has the power of the continuum.

Shelah [S], using models, absoluteness and Cohen's forcing method, gives a proof of the following conjecture of Mycielski.

Theorem. *Suppose that X is a compact metric space, which is connected and locally connected. Then the fundamental group of X is either finitely generated or has the power of the continuum.*

Thus in particular the group of rationals can't be the fundamental group of a connected locally connected compact metric space.

We can't drop 'compact' in the theorem — any countably generated group can be realized as the fundamental group of a Polish space, see [Sp]. We also can't drop 'locally connected' — the fundamental group of the 'tail of the peacock' is free with a countable infinity of generators. It seems to be open, however, whether a finitely generated group can be realized as the fundamental group of a compact metric space.

We present a forcing free proof of Mycielski's conjecture.

Definitions. Let X be a metric space. A path from x_0 to x_1 inside $V \subseteq X$ is a continuous function $f : [0,1] \mapsto V$ with $f(0) = x_0$ and $f(1) = x_1$. f is a loop at x if $f(0) = f(1) = x$. The reversal of f, denoted by f^{-1}, is a path from $f(1)$ to $f(0)$ defined by $f^{-1}(t) = f(1-t)$. The diameter of f is the diameter of the set $\{f(t) : t \in [0,1]\}$.

A path f is homotopic to another path g, $f \sim g$, if there is a homotopy from f to g, i.e., a continuous function $F : [0,1] \times [0,1] \mapsto X$ such that $F(t,0) = f(t)$, $F(t,1) = g(t)$, $F(0,s) = f(0) = g(0)$ and $F(1,s) = f(1) = g(1)$ (thus endpoints are kept constant; this is usually called 'a homotopy relative to $\{0,1\}$'). If f is a path from x_0 to x_1 and g a path from x_1 to x_2, then $f*g$ is the concatenation of f and g, i.e., a path from x_0 to x_2 defined by $(f*g)(t) = f(2t)$ for $t \in [0,1/2]$ and $(f*g)(t) =$

Received by the editors August 17, 1996 and, in revised form, February 26, 1997.
1991 *Mathematics Subject Classification.* Primary 03E15, 55Q05; Secondary 04A20, 55Q52.
Key words and phrases. Baire category, fundamental group, perfect set.
The author was partially supported by KBN grant 2 P03A 011 09. The author thanks J. Mycielski for introducing him to [S].

$g(2t-1)$ for $t \in [1/2, 1]$. The iterated concatenation $f_0 * (f_1 * (\dots (f_{n-1} * f_n) \dots))$ is written as $f_0 * \cdots * f_n$.

The relation $\sim$ is an equivalence relation. If X is path connected, the set of equivalence classes of loops at a given point $x \in X$, with multiplication and inverse defined from $*$ and $^{-1}$, is a group called the fundamental group of X. This group doesn't depend on the choice of x.

Let $\mathbb{N}$ be the set of nonnegative integers. Let $x \in X$ and suppose that f_n ($n \in \mathbb{N}$) are loops at x whose diameters converge to 0. Then $(f_0 * f_1 * \cdots)$ is the pointwise limit of $f_0 * \cdots * f_n$. The limit exists and is continuous, so it is a loop at x. In fact, if $t \in [1 - 2^{-n}, 1 - 2^{-(n+1)}]$, then

$$(f_0 * f_1 * \cdots)(t) = f_n((t - (1 - 2^{-n})) \cdot 2^{n+1}).$$

Note also that

$$(f_0 * f_1 * \cdots) = (f_0 * \cdots * f_n) * (f_{n+1} * \cdots).$$

From now on we assume that X is a connected locally connected compact metric space. Then X is also path connected and locally path connected (a theorem of Mazurkiewicz, see [K]).

Lemma 1. *Suppose that the fundamental group of X is not finitely generated. Then there exists $x \in X$ such that for each $n \in \mathbb{N}$ there exists a loop f_n at x which is of diameter $< 2^{-n}$ and which is not homotopic to the constant loop at x.*

Proof. Suppose otherwise. Then for each $x \in X$ there exists $n(x) \in \mathbb{N}$ such that every loop at x which has diameter $< 2^{-n(x)}$ is homotopic to the constant loop at x. By compactness there exists a cover of X by path connected sets V_i ($i < k$) and there exist points $x_i \in V_i$ ($i < k$) such that for every i the diameter of V_i is $< 2^{-(n(x_i)+1)}$ and any loop at x_i which has diameter $< 2^{-n(x_i)}$ is homotopic to the constant loop at x_i.

Fix a path g_i from x_0 to x_i ($i < k$); $g_0 =$ the constant loop at x_0. For i and j such that $V_i \cap V_j \neq \emptyset$ fix a path h_{ij} in $V_i \cup V_j$ going from x_i to x_j. Note that any path s from x_i to x_j which is contained in $V_i \cup V_j$ is homotopic to h_{ij}. Indeed, suppose that $n(x_i) \leq n(x_j)$. Then $s * h_{ij}^{-1}$ is a loop at x_i and has diameter $< 2^{-n(x_i)}$, so $s * h_{ij}^{-1}$ is homotopic to the constant loop at x_i. An elementary manipulation of this homotopy gives a homotopy from s to h_{ij}. If $n(x_i) > n(x_j)$, consider $h_{ij}^{-1} * s$, a loop at x_j.

We shall show that the fundamental group of X is generated by the (homotopy classes of) loops $\widetilde{h}_{ij} = g_i * h_{ij} * g_j^{-1}$. To this end, suppose that a loop s at x_0 is given. By a change of scale $s \sim s_0 * \cdots * s_l$, where each s_i is a path inside one piece of our cover. Say s_i goes from y_i to y_{i+1} inside $V_{\phi(i)}$; $y_0 = y_{l+1} = x_0$, $\phi(0) = \phi(l) = 0$. For $i = 0, \dots, l$, fix inside $V_{\phi(i)}$ a path t_i from y_{i+1} to $x_{\phi(i)}$; $t_0 = s_0^{-1}$, $t_l =$ the constant loop at x_0. Let $\widetilde{s}_i = t_{i-1}^{-1} * s_i * t_i$ ($i = 1, \dots, l$). Clearly

$$\widetilde{s}_1 * \cdots * \widetilde{s}_l \sim s.$$

Also, each $\widetilde{s}_i$, being a path from $x_{\phi(i-1)}$ to $x_{\phi(i)}$ inside $V_{\phi(i-1)} \cup V_{\phi(i)}$, must be homotopic to $h_{\phi(i-1)\phi(i)}$. Thus

$$s \sim h_{\phi(0)\phi(1)} * \cdots * h_{\phi(l-1)\phi(l)},$$

and hence also

$$s \sim \widetilde{h}_{\phi(0)\phi(1)} * \cdots * \widetilde{h}_{\phi(l-1)\phi(l)}. \qquad \square$$

For the sequel suppose that the fundamental group of X is not finitely generated and let x and f_n ($n \in \mathbb{N}$) be as claimed by Lemma 1. We shall find a set of size of the continuum of mutually non-homotopic loops. For $\alpha \in \{0,1\}^{\mathbb{N}}$ let $f_n^\alpha =$ the constant loop at x if $\alpha(n) = 0$, and let $f_n^\alpha = f_n$ otherwise. Define a loop f_α at x as $(f_0^\alpha * f_1^\alpha * \cdots)$. Write $\alpha \approx \beta$ if $f_\alpha \sim f_\beta$. Then $\approx$ is an equivalence relation in $\{0,1\}^{\mathbb{N}}$. It is enough to prove that $\approx$ has continuum many equivalence classes.

Lemma 2. *Suppose that α and β from $\{0,1\}^{\mathbb{N}}$ differ exactly at one point. Then $\alpha \not\approx \beta$.*

Proof. Suppose that $f_\alpha \sim f_\beta$. Let n be the unique point at which α and β are different. Then, for $m \neq n$ we have $f_m^\alpha = f_m^\beta$, hence $(f_0^\alpha * \cdots * f_{n-1}^\alpha) = (f_0^\beta * \cdots * f_{n-1}^\beta)$ and $(f_{n+1}^\alpha * \cdots) = (f_{n+1}^\beta * \cdots)$. Thus from

$$(f_0^\alpha * \cdots * f_{n-1}^\alpha * f_n^\alpha * f_{n+1}^\alpha * \cdots) \sim (f_0^\beta * \cdots * f_{n-1}^\beta * f_n^\beta * f_{n+1}^\beta * \cdots),$$

we get $f_n^\alpha \sim f_n^\beta$, which is a contradiction. □

More definitions. We recall some basic facts about Polish spaces (see [K]). A Polish space is a completely metrizable separable space. Let Y be a Polish space. For a subset A of Y: A is nowhere dense if its closure has empty interior, A is meager if it is a countable union of nowheredense sets, A is comeager in an open set U if $U \setminus A$ is meager, A has the Baire property if its symmetric difference with some open set is meager. A nonmeager set with the Baire property is comeager in some nonempty open set. The Baire category theorem implies that a set which is comeager in a nonempty open set is nonmeager. The Kuratowski-Ulam theorem implies that if $A \subseteq Y \times Z$ is comeager in $U \times V$, where U and V are open subsets of Polish spaces Y and Z, then

$$\{y \in U : \{z \in V : \langle y, z\rangle \in A\} \text{ is comeager in } V\}$$

is comeager in U.

A subset A of Y is analytic if there exist a Polish space Z and a closed set $D \subseteq Y \times Z$ such that A is the projection of D into Y. Analytic sets have the Baire property. Continuous preimages of analytic sets are analytic.

A subset P of Y is perfect if it is closed, nonempty, and has no isolated points. Perfect sets have the power of the continuum.

The set $\{0,1\}^{\mathbb{N}}$ becomes a Polish space (homeomorphic to the Cantor discontinuum) when viewed as the product of countably many copies of the two-point discrete space $\{0,1\}$. The canonical basis of $\{0,1\}^{\mathbb{N}}$ is the collection of all sets $[\sigma] = \{\alpha : \sigma \subseteq \alpha\}$, where σ is a finite zero-one sequence, i.e., $\sigma : \{0,1,\ldots,n-1\} \mapsto \{0,1\}$ for some n.

Lemma 3. *$\approx$ has the Baire property as a subset of $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.*

Proof. Let H and $\mathbb{H}$ be respectively the spaces of all loops at x and all homotopies between them, endowed with the sup metric. Both spaces are Polish. The homotopy relation $\sim$ restricted to H is an analytic subset of $H \times H$. Indeed, it is the projection onto $H \times H$ of

$$\{\langle\langle f, g\rangle, F\rangle : F \text{ is a homotopy from } f \text{ to } g\},$$

which is a closed subset of $(H \times H) \times \mathbb{H}$. Note also that the function from $\{0,1\}^{\mathbb{N}}$ to H which takes α to f_α is continuous. It follows that $\approx$ is analytic (as a continuous preimage of an analytic set), and thus has the Baire property. $\square$

Lemma 4. *If $E \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ is an equivalence relation which has the Baire property and if $\neg xEy$ whenever x and y differ by one coordinate only, then E is meager.*

Proof. Should E be nonmeager, it would be comeager in some basic neighbourhood $[\sigma] \times [\tau]$. By the Kuratowski-Ulam theorem,

$$A = \{\alpha \in [\sigma] : \{\beta \in [\tau] : \alpha E \beta\} \text{ is comeager in } [\tau]\}$$

is comeager in $[\sigma]$. Let $n >$ the length of σ. Consider a map $\Phi : [\sigma] \mapsto [\sigma]$ defined by $\Phi(\alpha)(n) = 1 - \alpha(n)$ and $\Phi(\alpha)(i) = \alpha(i)$ for $i \neq n$. Φ is a homeomorphism and thus $\Phi[A]$ is comeager in $[\sigma]$. Choose $\alpha \in A \cap \Phi[A]$ and let $\gamma = \Phi(\alpha)$. Then α and γ differ only at n, hence $\neg \alpha E \gamma$. Also, by the definition of A, we have in $[\tau]$ comeagerly many β with $\alpha E \beta$. As $\gamma \in A$, the same is true about γ. Thus there exists β with $\alpha E \beta$ and $\gamma E \beta$. But then $\alpha E \gamma$, which is a contradiction. $\square$

Remark. Another way to see that E is meager might be as follows. Suppose for contradiction that E is nonmeager. Consider $G = \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ as a Polish group with coordinatewise addition mod 2. By the Baire category version of a theorem of Steinhaus (see [O]), if $B \subseteq G$ has the Baire property and is nonmeager, then the difference set $B - B = \{b_0 - b_1 : b_0, b_1 \in B\}$ contains a neighbourhood of the unit element $\langle \overline{0}, \overline{0} \rangle$ (here $\overline{0} = \langle 0, 0, \dots \rangle$). So, for each $\langle \delta, \epsilon \rangle \in G$, which is close enough to $\langle \overline{0}, \overline{0} \rangle$, there exist $\langle \alpha, \beta \rangle \in B$ such that $\langle \alpha + \gamma, \beta + \delta \rangle \in B$. For $n \in \mathbb{N}$ let $\epsilon_n \in \{0,1\}^{\mathbb{N}}$ be the function that takes value 1 at n and 0 elsewhere. Then $\langle \epsilon_n, \overline{0} \rangle \longrightarrow \langle \overline{0}, \overline{0} \rangle$, when $n \longrightarrow \infty$. So, for large enough n there exists $\langle \alpha, \beta \rangle \in B$ with $\langle \alpha + \epsilon_n, \beta \rangle \in B$. Applied to $B = E$ this yields that for large enough n there exist α and β such that $\alpha E \beta$ and $\alpha + \epsilon_n E \beta$, whence $\alpha E \alpha + \epsilon_n$. This contradicts Lemma 2.

Similar arguments show that if E is Lebesgue measurable then it must be null.

Corollary. $\approx$ *is meager.*

Recall now the following theorem of Mycielski [M].

Theorem (Mycielski). *Suppose that Y is a Polish space without isolated points and that $R \subseteq Y \times Y$ is meager. Then there exists a perfect set $P \subseteq Y$ such that if α and β are distinct points of P then $\langle \alpha, \beta \rangle \notin R$.*

Applying this theorem to $\approx$ and $\{0,1\}^{\mathbb{N}}$ we get a perfect set of mutually $\approx$ non-equivalent elements of $\{0,1\}^{\mathbb{N}}$. The proof of Mycielski's conjecture is complete.

A slight modification of the above proof gives the following theorem.

Theorem. *Let $\kappa < 2^{\aleph_0}$ be an infinite cardinal number. Suppose that X is a path connected locally path connected metric space which is κ-Lindelöf (i.e., every open cover of X has a subcover of size $\leq \kappa$). Then the power of the fundamental group of X is either $\leq \kappa$ or $2^{\aleph_0}$.*

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