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# SINGULAR EXTENSIONS OF THE TRACE AND THE RELATIVE DIXMIER PROPERTY IN THE TYPE $\mathrm{II}_1$ FACTORS

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ABSTRACT. If $N \subset M$ is an inclusion of type $\mathrm{II}_1$ factors with $N' \cap M = \mathbf{C}I$, we study the connection between the existence of singular states on $M$ which extend the trace on $N$ and the Dixmier approximation property in $M$ with unitaries in $N$. We also prove the existence of singular conditional expectations from certain free product factors onto irreducible hyperfinite subfactors.

## 1. INTRODUCTION

One of the fundamental approximation properties in von Neumann algebras is the Dixmier property [4], 8.3.6. In the type $\mathrm{II}_1$ factorial case it states that for every element $m$ in a $\mathrm{II}_1$ factor $M$ and for every $\epsilon > 0$, there are unitary operators $u_1, ..., u_n$ in $M$ and positive numbers $\lambda_1, ..., \lambda_n$ of sum 1 such that

$$(1) \qquad || \sum_{i=1}^{n} \lambda_i u_i^* m u_i - \tau(m) I \; || < \epsilon.$$

In other words, $\tau(m)I$ belongs to the norm-closed convex hull of $\{u_i^* m u_i\}$. (It turns out that $\tau(m)I$ is the only element in the center of $M$ with this property.)

A natural question that arises is: if $N \subset M$ is a subalgebra, is (1) still true under the additional requirement that the unitaries $u_i$ belong to $N$? If the answer is yes, then $N$ is called a Dixmier subalgebra of $M$. The case we will be interested in is when $N$ is an irreducible subfactor, that is, a subfactor with trivial relative commutant. The relative Dixmier property has been studied by several authors (see [1] and the references therein), in connection with extensions of pure states, a problem originating in [3].

The only positive result in this direction has been recently obtained by Popa [7]: the Dixmier property (1) remains true for ultrapower factors $M^\omega$ and unitaries $u_i$ in $N^\omega$, where $N \subset M$ is an irreducible inclusion of (strongly) separable $\mathrm{II}_1$ factors.

While no example of a proper, separable Dixmier subfactor is known, in this note we construct examples of irreducible subfactors without the relative Dixmier property. We do this by studying singular states on a factor $M$ which extend the trace on a subfactor $N$. More precisely, we construct singular states on free product factors $M * N$ which extend the trace on $M$. As a consequence, $M$ is not a Dixmier

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subfactor of $M * N$. In particular when $M = R$, the hyperfinite $\mathrm{II}_1$ factor, we obtain singular conditional expectations from $R * N$ onto $R$.

Singular functionals and, more generally, singular maps between operator algebras haven't been used extensively and the existing literature on the subject is rather scarce. We refer the reader to [4], ch.10, for a presentation of the main results.

It should be noted that if in the relative Dixmier property we replace the norm by the Hilbert-Schmidt norm $||x||_2 = \tau(x^*x)^{1/2}$ where $\tau$ is the trace, the situation is quite different: if $N$ is an irreducible subfactor of $M$, then for every $m$ in $M$ and for every $\epsilon > 0$, there are unitary operators $u_1, ..., u_n$ in $N$ and positive numbers $\lambda_1, ..., \lambda_n$ of sum 1 such that

$$|| \sum_{i=1}^{n} \lambda_i u_i^* m u_i - \tau(m)I \; ||_2 < \epsilon. \tag{2}$$

This means that $\tau(m)I$ is ultraweakly (also weakly and strongly) adherent to the convex hull of $\{u_i^* m u_i\}$. Property (2) is a consequence of Popa's noncommutative local Rohlin theorem [5]: with above notation, there are mutually orthogonal projections $e_1, ..., e_n$ in $N$ of sum 1 such that

$$|| \sum_{i=1}^{n} e_i m e_i - \tau(m)I \; ||_2 < \epsilon. \tag{3}$$

Then (2) follows easily from (3) as in [2], 2.2.

## 2. Singular maps

In this section we discuss some properties of singular maps and functionals. We will adopt as the definition of singularity an equivalent property, due to Takesaki [8]. We refer the reader to [4], ch.10, for more information.

**2.1. Definition.** Let $M$ and $N$ be von Neumann algebras. A positive linear map $\phi : M \to N$ is *singular* if for every nonzero projection $f \in M$ there exists a nonzero projection $e \leq f$ in $M$ such that $\phi(e) = 0$. If $N = \mathbf{C}$, then $\phi$ is called a singular positive functional.

2.2. *Remarks.* (i) If $M$ is of countable type (in particular, if $M$ is a factor) then $\phi$ is singular iff there exists an increasing sequence of projections $e_n \nearrow I$ such that $\phi(e_n) = 0$.

(ii) If $(\phi_n)$ is a sequence of singular maps, then under any nonzero projection $f \in M$ there is a nonzero projection $e$ such that $\phi_n(e) = 0$ for all $n$. In particular, finite sums of singular maps and point limits of sequences of singular maps are again singular.

For the following result we refer to [4], 10.1.16, for an alternate proof. We present here a different proof, more in the spirit of our approach.

**2.3. Lemma.** *If $\phi$ is a singular positive functional on $M$ then for every nonzero $m \in M$, $\psi(x) = \phi(m^*xm)$ is singular or zero.*

*Proof.* Write $m$ as a linear combination of four unitaries, $m = \sum_{i=1}^4 \lambda_i u_i$. We use the Cauchy-Schwarz inequality to obtain

$$|\psi(x)| = |\phi(\sum_{i,j} \overline{\lambda_i}\lambda_j \; u_i^* x u_j)| \leq \sum_{i,j} |\phi(\overline{\lambda_i}\lambda_j \; u_i^* x u_j)|$$

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$$\leq \sum_{i,j} |\lambda_i \lambda_j|\ \phi(I)^{1/2} \phi(u_j^* x^* x u_j)^{1/2}.$$

If $f$ is a projection, then

$$0 \leq \psi(f) \leq \sum_{i,j} |\lambda_i \lambda_j|\ ||\phi||^{1/2} \phi(u_j^* f u_j)^{1/2}.$$

Since $\phi$ is singular, choose by 2.2(ii) a projection $0 \neq e \leq f$ such that $\phi(u_j^* f u_j) = 0,\ j = 1, 2, 3, 4$, which shows that $\phi(f) = 0$, so $\phi$ is singular.

Recall that if $N \subset M$ are von Neumann algebras, a map $\Phi : M \to B(H)$ is called $N$- modular if $\Phi(nm) = n\Phi(m)$ and $\Phi(mn) = \Phi(m)n$ for every $m \in M$ and $n \in N$. If $\phi$ is a linear functional on $M$, then $\phi$ is $N$- invariant if $\phi(nm) = \phi(mn)$ for all $m \in M$ and $n \in N$. □

**2.4. Proposition.** *Let $M$ be a type $\mathrm{II}_1$ factor with a hyperfinite subfactor $R$. If there exists a singular state on $M$ extending the trace on $R$, then there exists a singular $R$-invariant state on $M$.*

*Proof.* Let $\psi$ be a singular state extension of the trace $\tau$ on $R$. Let $(R_n)$ be an increasing sequence of type $\mathrm{I}_{2^n}$ subfactors of $R$ whose union is strongly dense in $R$. Define $\psi_n(x) = \int \psi(u^* x u) d\mu_n(u)$ where $\mu_n$ is the Haar measure on the compact unitary group of $R_n$. Since the above integrals are point-norm limits of singular states (see 2.3), $\psi_n$ will be singular states, $R_n$- invariant and extending the trace on $R$. Let $\omega$ be a free ultrafilter on the set of natural numbers $\mathbf{N}$ and define $\phi(x) = \omega\text{-}lim\ \psi_n(x)$. Then $\phi$ is a singular state (this follows from 2.2(ii)) which extends the trace on $R$ and is invariant with respect to $C^*(\cup R_n)$. Our next goal is to show that $\phi$ is in fact $R$- invariant.

Fix $m \in M$ and for every $r_1, r_2 \in R$ define the map $(r_1, r_2) \to \phi(r_2^* m r_1)$. We have

$$|\phi(r_2^* m r_1)| \leq \phi(r_2^* r_2)^{1/2} \phi(r_1^* m^* m r_1)^{1/2}$$

$$\leq \phi(r_2^* r_2)^{1/2} ||m|| \phi(r_1^* r_1)^{1/2} = ||m||.||r_2||_2 ||r_1||_2.$$

This induces a bounded linear map $E : M \to B(L^2(R, \tau))$ by

$$(E(m) r_1, r_2) = \phi(r_2^* m r_1).$$

Clearly $E$ is a positive map, $||E(m)|| \leq ||m||$ and $E(r) = r$ for all $r \in R$. Also, for $r \in R$,

$$(rE(m) r_1, r_2) = (E(m) r_1, r^* r_2) = \phi(r_2^* r m r_1)$$

and

$$(E(rm) r_1, r_2) = \phi(r_2^* r m r_1)$$

which shows that $E$ is left $R$- modular. Similarly, $E$ is right $R$-modular.

If $r_0$ is any operator in $C^*(\cup R_n)$ and $J$ denotes the canonical conjugation on $L^2(R, \tau)$, we have

$$(J r_0 J E(m) r_1, r_2) = (E(m) r_1, J r_0^* J r_2) = (E(m) r_1, r_2 r_0) \phi(r_0^* r_2^* m r_1)$$

and

$$(E(m) J r_0 J r_1, r_2) = (E(m) r_1 r_0^*, r_2) = \phi(r_2^* m r_1 r_0^*).$$

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Invariance of $\phi$ shows that $E(m)$ commutes with $JC^*(\cup R_n)J$, and since $JRJ = R'$, we get $E(M) \subset R$, so $E$ is a conditional expectation. The singularity of $E$ follows easily from

$$\tau(E(m)) = (E(m)I, I) = \phi(m).$$

For every unitary $u \in R$

$$\phi(u^*mu) = \tau(E(u^*mu)) = \tau(u^*E(m)u) = \tau(E(m)) = \phi(m)$$

which shows that $\phi$ is $R$-invariant, and the proof is complete. □

We record the existence of $E$ in the following

**2.5. Corollary.** *If there exists a singular state on $M$ which extends the trace on $R$, then there exists a singular conditional expectation from $M$ onto $R$.*

## 3. The relative Dixmier property

In this section we introduce the relative Dixmier property, following [2], and we establish the connection with singular states.

**3.1. Definition.** Let $M$ be a $\mathrm{II}_1$ factor. A unital subalgebra $B \subset M$ is called a *Dixmier subalgebra* if the norm-closed convex hull of $\{u^*mu;\ u \in B \text{ unitary}\}$ contains $\tau(m)I$ for every $m \in M$.

3.2. *Remark.* If $B \subset M$ is a Dixmier subalgebra then $B' \cap M = \mathbf{C}I$. Indeed, if $p \in B' \cap M$ is a projection, then the relative Dixmier property shows that $p = \tau(p)I$, which implies $p = 0$ or $p = I$.

**3.3. Proposition.** *Let $M$ be a $\mathrm{II}_1$ factor with a separable Dixmier subfactor $N$. Then there is no singular state on $M$ which extends the trace on $N$.*

*Proof.* Let $\tau$ denote the trace on $N$ and suppose there exists a singular state $\phi$ on $M$ such that $\phi(n) = \tau(n)$ for every $n \in N$. We repeat the construction in the proof of 2.4. Fix $m \in M$ and for every $n_1, n_2 \in N$ define the map $(n_1, n_2) \to \phi(n_2^*mn_1)$. We have

$$|\phi(n_2^*mn_1)| \le \phi(n_2^*n_2)^{1/2}\phi(n_1^*m^*mn_1)^{1/2}$$

$$\le \phi(n_2^*n_2)^{1/2}||m||\phi(n_1^*n_1)^{1/2} = ||m||.||n_2||_2||n_1||_2.$$

This induces a bounded linear map $E : M \to B(L^2(N, \tau))$ by

$$(E(m)n_1, n_2) = \phi(n_2^*mn_1).$$

Clearly $||E(m)|| \le ||m||$ and $E(n) = n$ for all $n \in N$. A computation similar to the one in 2.4 shows that $E$ is $N$- modular.

We now show that $E$ is singular. Fix $p$ a nonzero projection in $M$ and $(n_k)_{k\ge1}$ a countable subset of M dense in $L^2(N, \tau)$. Choose, by 2.2(ii), a projection $0 \ne e \le p$ such that $\phi(n_k^*en_k) = 0$ for all $k$. Then for all $k$

$$(E(e)n_k, n_k) = \phi(n_k^*en_k) = 0;$$

therefore $E(e) = 0$. Now $E(u^*mu) = u^*E(m)u$ for every unitary $u \in N$. The relative Dixmier property implies that there are positive numbers $\lambda_i$ of sum 1 and unitaries $u_i \in N$ such that

$$\sum_1^n \lambda_i u_i^* E(m) u_i = E(\sum_1^n \lambda_i u_i^* m u_i)$$

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is within an epsilon from $E(\tau(m)I) = \tau(m)I$. In particular, for $m = e$ we get $\tau(e) = 0$, contradiction.

We will show (4.7) that one cannot remove the separability assumption on $N$. □

## 4. Examples and applications

**4.1.** If $N \subset M$ are separable $\mathrm{II}_1$ factors and $dim\ N' \cap M = \infty$, there exist singular states on $M$ which extend the trace on $N$. To see this, let $(e'_n)$ be a countable family of mutually orthogonal projections in $N' \cap M$, $\sum e'_i = I$. Define the unit vectors in $L^2(M, \tau)$ $\xi_n = e'_n \tau(e'_n)^{-1/2}$. Clearly $\xi_n = e'_n \xi_n$ and define on $M$ $\phi_n(x) = (x\xi_n, \xi_n)$. Then $\phi_n$ are normal states on $M$ and for $x \in N$ we have

$$\phi_n(x) = (x\xi_n, \xi_n) = \tau(x\xi_n\xi_n^*) = \tau(xe'_n/\tau(e'_n)) = \tau(x).$$

The last equality is $\tau(xe'_n) = \tau(x)\tau(e'_n)$ and it follows from the uniqueness of the trace on $N$. Now let $\omega$ be a free ultrafilter on $\mathbf{N}$ and define $\phi(x) = \omega\text{-}lim\ \phi_n(x)$. It follows that $\phi$ is a state on $M$ which extends the trace on $N$. Also, for every $k \geq 1$, $\phi(I - e'_k) = 0$ and 2.2(ii) implies that $\phi$ is singular.

**4.2.** S. Popa proved in [6] that if $M$ and $N$ are $\mathrm{II}_1$ factors, then $M$ has trivial relative commutant in $M * N$. (In fact [6], 4.1, is a more general result. See also [1], 2.5.) For the basics on free product factors we refer to [9].

For any projections $e \in M$ and $f \in N$ the definition of freeness shows that

$$\tau((e - \tau(e)I)(f - \tau(f)I)) = 0$$

which implies that $\tau(ef) = \tau(e)\tau(f)$.

Choose now an arbitrary decreasing sequence of projections $(f_n)$ in $N$ with $so\text{-}lim\ f_n = 0$ and define on $M * N$ the states

$$\phi_n(x) = \tau(xf_n)/\tau(f_n)$$

For any projection $e \in M$ we have $\phi_n(e) = \tau(ef_n)/\tau(f_n) = \tau(e)$ so $\phi_n$ are extensions of the trace on $M$. If $\omega$ is a free ultrafilter on $\mathbf{N}$, define $\phi(x) = \omega\text{-}lim\ \phi_n(x)$. Since $\phi(I - f_n) = 0$ for all $n$, $\phi$ is a singular state on $M * N$ which extends the trace on $M$.

**4.3.** It now follows from 3.3 that no separable $M$ can be a Dixmier subfactor of $M * N$. If $M = R$, the hyperfinite type $\mathrm{II}_1$ factor, then 2.4 and 2.5 show that there exist $R$- invariant singular states on $R * N$, hence singular conditional expectations from $R * N$ onto $R$.

In view of 2.4 and 3.3 we ask the following

**4.4. Question.** Does every separable $\mathrm{II}_1$ factor $M$ have a Dixmier subfactor ? Can this Dixmier subfactor be hyperfinite ?

**4.5.** Yet another situation where the above ideas apply is the commutative case, which we only present for comparison. If $A = B = L^\infty([0,1])$, the constructions in 4.1 and 4.2 can be repeated almost word by word to exhibit a singular state on $A \overline{\otimes} B$ (i.e. a measure on $[0,1] \times [0,1]$ singular with respect to the Lebesgue product measure) which extends the (normal) trace (i.e. the Lebesgue measure) on $A$.

**4.6.** Let $M$ be a separable $\mathrm{II}_1$ factor and $N$ be a separable subfactor of $M^\omega$. By a theorem of Popa [7] there is a separable diffuse abelian algebra $A \subset M^\omega$ which is free with respect with $N$. The construction in 4.2 applies to a decreasing sequence of projections in $A$ to obtain a singular state on $M^\omega$ which extends the trace on

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$N$. Following 3.3, no separable $N \subset M^\omega$ is a Dixmier subfactor. Recall, however, ([7]) that if $Q \subset M$ is an irreducible subfactor then $Q^\omega \subset M^\omega$ is a (nonseparable) Dixmier subfactor.

**4.7.** Since $M$ has trivial relative commutant in $M * N$ ([6]), it follows that $M^\omega$ is a Dixmier subfactor of $(M * N)^\omega$ ([7]). We will exhibit a singular state on the latter which extends the trace on the former, thus showing that the separability assumption in 3.3 cannot be dropped. Using the notation in 4.2 and 4.3, if $f_n$ is a sequence of projections in $N$ decreasing to 0, define the sequence $F_n = (f_n, f_n, ...)$ of projections in $(M * N)^\omega$, decreasing to 0. If $E$ is any projection in $M^\omega$, there are projections $e_n$ in $M$ such that $E = (e_1, e_2, ...)$. Since $\tau(e_i f_n) = \tau(e_i)\tau(f_n)$, we get $\tau(EF_n) = \tau(E)\tau(F_n)$, so the construction in 4.2 applies to obtain a singular state on $(M * N)^\omega$ which extends the trace on $M^\omega$.

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