PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 10, October 1998, Pages 2903–2904
S 0002-9939(98)04406-2

SHARP LOG–SOBOLEV INEQUALITIES

OSCAR S. ROTHAUS

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We show existence of a wide variety of Log–Sobolev inequalities in which the constant is exactly that required by the Poincaré inequality which may be inferred from the Log–Sobolev.

We are given a smooth compact Riemannian manifold M, intrinsic gradient ∇, and volume element $d\mu$ — we assume W.L.O.G. that $\mu(M) = 1$ — and a smooth positive function m, giving rise to a probability measure $md\mu = dm$, for which we have a log–Sobolev inequality (LSI):

$$\rho \int_M |\nabla f|^2 dm \geq \int_M |f|^2 \ln |f|^2 dm - \int_M |f|^2 dm \ln \int_M |f|^2 dm. \tag{1}$$

It is well known that $\rho \geq 2/\lambda$, where λ is the first non–zero eigenvalue of the Schrödinger operator

$$f \to \Delta f + \nabla f \cdot \frac{\nabla m}{m},$$

(Δ the usual Laplacian) attached to the Dirichlet form in (1).

We call the LSI sharp if $\rho = 2/\lambda$. Many examples of sharp inequalities are known, the most familiar arising from M the sphere with the usual metric scaled to give M unit volume, and $m = 1$.

We will show here that for every compact homogeneous Riemannian manifold, there are a continuum of choices of m for which sharp LSI's exist.

We follow the notation and conclusions of [1], which we now briefly review. For every M as described initially there is a least constant $\rho_0(M)$, the hypercontractive constant for M, such that

$$\rho_0(M) \int |\nabla f|^2 d\mu \geq \int |f|^2 \ln |f|^2 d\mu - \int |f|^2 d\mu \ln \int |f|^2 d\mu. \tag{2}$$

For any positive $\rho < \rho_0(M)$ there is a minimum $\alpha(\rho)$, called the defect, such that

$$\rho \int |\nabla f|^2 d\mu \geq \int |f|^2 \ln |f|^2 d\mu - \int |f|^2 d\mu \ln \int |f|^2 d\mu - \alpha(\rho) \int |f|^2 d\mu, \tag{3}$$

with the inequality an equality for a real non–trivial minimizing function $f = J_\rho$ satisfying $\int J_\rho^2 d\mu = 1$ and the non–linear P.D.E.

$$\rho \Delta J_\rho + J_\rho \ln J_\rho^2 - \alpha(\rho) J_\rho = 0. \tag{4}$$

Received by the editors September 27, 1996 and, in revised form, February 24, 1997.
1991 *Mathematics Subject Classification.* Primary 46E35, 46E39.

(If $\rho_0(M) > 2/\lambda$, where λ is the first non–trivial eigenvalue of the Laplacian, then there is known [1] to be a non–trivial ($\not\equiv 1$) minimizer for $\rho = \rho_0(M)$ as well.)

If one replaces f in the defective LSI (3) by fJ_ρ, it is thrown into the non–defective form:

$$\rho \int |\nabla f|^2 J_\rho^2 d\mu \geq \int |f|^2 \ln |f|^2 J_\rho^2 d\mu - \int |f|^2 J_\rho^2 d\mu \ln \int |f|^2 J_p^2 d\mu, \tag{5}$$

i.e., a version of our equation (1).

We know, as noted earlier, that in equation (5), $\rho \geq 2/\tau$ where τ is the first non–trivial eigenvalue of the Schrödinger operator attached to the Dirichlet form in (5); i.e., the operator

$$f \to \Delta f + 2\frac{\nabla J_\rho}{J_\rho}\nabla f. \tag{6}$$

Our principal result is

Theorem. *If M is a compact homogeneous Riemannian manifold, then (5) above is a sharp LSI.*

(Thus we have distinct choices of probability measure with sharp LSI's for M for each $\rho < \rho_0(M)$, and for $\rho = \rho_0(M)$ in some cases.)

Proof. In the P.D.E. for an eigenvector of the operator (6)

$$\Delta f + 2\frac{\nabla J_0}{J_\rho}\nabla f + \theta f = 0,$$

make the Liouville substitution $f = g/J_\rho$, which throws it into the form

$$\Delta g - \frac{\Delta J_\rho}{J_\rho} g + \theta g = 0,$$

which becomes using the differential equation (4) for J_ρ,

$$\Delta g + \frac{1}{\rho}[\ln J_\rho^2 - \alpha(\rho)]g + \theta g = 0. \tag{7}$$

The eigenvector J_ρ belongs to the smallest eigenvalue $\theta = 0$.

Next take the differential equation (4) satisfied by J_ρ and apply a Killing vector X. X commutes with the Laplacian, and so we obtain

$$\Delta(XJ_\rho) + (2/\rho)(1 + \ln J_\rho)(XJ_\rho) - \frac{\alpha(\rho)}{\rho}(XJ_\rho) = 0$$

which is a version of (7) with $\theta = 2/\rho$ and $g = XJ_\rho$. Now since J_ρ is not constant, and M is homogeneous, XJ_ρ cannot be zero for all Killing vectors. So for the first non–trivial eigenvalue of (6), we have $\tau \leq 2/\rho$. Since we know $\rho \geq 2/\tau$, we must have $\rho = 2/\tau$, and the LSI is sharp.

This completes the proof of our theorem. □

It would be quite interesting to get sharp inequalities on the line, other than the usual one.

References

[1] O. S. Rothaus, *Diffusion on Compact Riemannian Manifolds and Logarithmic Sobolev Inequalities*, Journal of Functional Analysis **42, #1** (June 1981), 102–109. MR **83f:**58080a

Department of Mathematics, Cornell University, Ithaca, New York 14853-7901
E-mail address: rothaus@math.cornell.edu