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A RESULT ON THE GELFAND-KIRILLOV DIMENSION OF REPRESENTATIONS OF CLASSICAL GROUPS

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ABSTRACT. Let (G, G') be the reductive dual pair $(O(p,q), Sp(2n,\mathbb{R}))$. We show that if π is a representation of $Sp(2n,\mathbb{R})$ (respectively $O(p,q)$) obtained from duality correspondence with some representation of $O(p,q)$ (respectively $Sp(2n,\mathbb{R})$), then its Gelfand-Kirillov dimension is less than or equal to $(p+q)(2n-\frac{p+q-1}{2})$ (respectively $2n(p+q-\frac{2n+1}{2})$).

0. Gelfand-Kirillov dimension of $\mathcal{U}(\mathfrak{g})$ modules

We shall recall some notions in this section ([V], [KL]).

Let k be a field. A k-algebra is called almost commutative if there exists a filtration $A_0 \subseteq A_1 \subseteq \cdots$ such that

(i) $A_0 = k$.

(ii) A_1 is finite dimensional, and A is generated as an algebra by A_1.

(iii) The associated graded algebra $gr(A) = \sum_{i=0}^{\infty} A_i/A_{i-1}$ is commutative.

Let A be a k-algebra which is almost commutative with respect to the filtration $\mathcal{A} = (A_i)_{i=0}^{\infty}$, and let V be a graded A module with a finite filtration $\mathcal{V} = (V_i)_{i=0}^{\infty}$ such that the associated graded module $gr(V)$ of $gr(A)$ is finitely generated. Then the function $\dim_k V_i$ is a polynomial in i for sufficiently large i, called the Hilbert-Samuel polynomial. The degree of this polynomial is independent of the choice of filtrations of A and V, and is called the Gelfand-Kirillov dimension of V, or GKdim of V for short.

Now let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Let $A = \mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, with the standard filtration given by $A_i = \mathcal{U}(\mathfrak{g})_i$, the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by monomials of the form $X_1 \cdots X_m$, with $m \leq i$ and $X_j \in \mathfrak{g}$, for each j. By the Poincare-Birkhoff-Witt theorem, the associated graded algebra $gr(\mathcal{U}(\mathfrak{g}))$ is canonically isomorphic to $S(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$—a polynomial algebra in $\dim \mathfrak{g}$ variables. Suppose V is a finitely generated $\mathcal{U}(\mathfrak{g})$ module. Choose a finite dimensional generating subspace V_0, and set $V_i = \mathcal{U}(\mathfrak{g})_i V_0$. Then V becomes a graded $\mathcal{U}(\mathfrak{g})$ module, and so the degree of the polynomial $\dim_{\mathbb{C}} V_i$ (for i sufficiently large) gives the Gelfand-Kirillov dimension of V. It is a non-negative integer less than or equal to $\dim \mathfrak{g}$.

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1. Representations from dual pair correspondence

Let $V = \mathbb{R}^{p+q}$ be equipped with the following non-degenerate symmetric form $(,)$ of signature (p,q):

$$(x,y) = x^t I_{p,q} y, \qquad x, y \in \mathbb{R}^{p+q},$$

where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Let $G = O(p,q)$ be its isometry group. Denote $W = V^{2n}$, the direct sum of $2n$ copies of V. We may identify W with the vector space $M_{p+q,2n}(\mathbb{R})$ of $(p+q)\times 2n$ real matrices. It is easy to see that W admits a symplectic structure given by

$$\langle W_1, W_2\rangle = tr(W_1^t I_{p,q} W_2 J_n), \qquad W_1, W_2 \in M_{p+q,2n}(\mathbb{R}),$$

where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, tr denotes the trace function on the space of $2n\times 2n$ real matrices. Let Sp be the group of symplectic transformations of W.

Let $G' = Sp(2n,\mathbb{R}) = \{g \in GL(2n,\mathbb{R}) | gJ_n g^t = J_n\}$. Then G and G' will act on $M_{p+q,2n}(\mathbb{R})$ via matrix multiplication on the left and on the right. Moreover we have a reductive dual pair [H1]

$$(G, G') = (O(p,q), Sp(2n,\mathbb{R})) \subset Sp.$$

Let $\tilde{S}p$ be the metaplectic cover of Sp, and for any subgroup E of Sp, let $\tilde{E}$ be the inverse image of E in $\tilde{S}p$.

Let H be either G or G', and H' the other member.

Theorem 1. *Let π be a representation of $\tilde{H}$ obtained from duality correspondence with $\tilde{H}'$ (*[H3]*).*

a) *Let $H = Sp(2n,\mathbb{R})$; then*

$$\textit{Gelfand-Kirillov dimension of } \pi \le \begin{cases} (p+q)(2n - \frac{p+q-1}{2}), & \text{if } p+q \le 2n, \\ n(2n+1), & \text{if } p+q > 2n. \end{cases}$$

b) *Let $H = O(p,q)$; then*

$$\textit{Gelfand-Kirilliov dimension of } \pi \le \begin{cases} 2n(p+q - \frac{2n+1}{2}), & \text{if } 2n \le p+q, \\ \frac{(p+q)(p+q-1)}{2}, & \text{if } 2n > p+q. \end{cases}$$

Remarks. 1) By a result of Vogan (Proposition 5.7, [V]), the Gelfand-Kirillov dimension of any irreducible admissible $\tilde{H}$ module π is less than or equal to the dimension of a maximal unipotent subgroup. Thus in case a) where $H = Sp(2n,\mathbb{R})$, it implies that the Gelfand-Kirillov dimension of π is at most n^2. So the second part and part of the first part of our inequality are weaker than this general result of Vogan. We thank the referee for pointing out this. However our proof is very elementary, and the main interest here is in the cases where $p+q$ is small relative to n. Similar remarks apply to case b).

2) Analogous results hold for other reductive dual pairs $(G, G') \subset Sp$. The needed adjustment in the argument is very minor, and that is to apply appropriate result from classical invariant theory as in [W]. We shall leave this to the interested reader.

2. Proof of Theorem 1

Recall that H is either $G = O(p,q)$ or $G' = Sp(2n,\mathbb{R})$. Let $\mathfrak{h}$ be the Lie algebra of H, and $\mathfrak{h}_{\mathbb{C}}$ be its complexification. Here and after we use the convention of indicating the appropriate complexification by a subscript $\mathbb{C}$.

Let ω be the oscillator representation of $\tilde{S}p$ ([H2]). Recall that $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ is the universal enveloping algebra of $\mathfrak{h}_{\mathbb{C}}$ endowed with the standard filtration. Then $\omega(\mathcal{U}(\mathfrak{h}_{\mathbb{C}}))$ inherits a corresponding filtration. Let $gr(\omega(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})))$ be the graded algebra associated with this filtration. One version of the First Main Theorem of Classical Invariant Theory [H1] asserts the isomorphism

$$gr(\omega(\mathcal{U}(\mathfrak{h}_{\mathbb{C}}))) \cong P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}},$$

where $P(V_{\mathbb{C}}^{2n})$ denotes the polynomial algebra on $V_{\mathbb{C}}^{2n} = W_{\mathbb{C}}$, and $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$ is the algebra of $H'_{\mathbb{C}}$ invariants in $P(V_{\mathbb{C}}^{2n})$.

Proposition 1. *Let π be a representation of $\tilde{H}$ obtained from duality correspondence with $\tilde{H}'$. Then the Gelfand-Kirillov dimension of π is less than or equal to the Gelfand-Kirillov dimension of $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$.*

Proof. From [H3], we know that π is a finitely generated, admissible quotient of $\omega|_{\tilde{H}'}$. Let $\mathcal{H}_\pi$ be the space of $\tilde{K}$-finite vectors of π, where K is a maximal compact subgroup of H. Then $\mathcal{H}_\pi$ is a module for $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$, and we have a surjective homomorphism

$$\omega(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})) \twoheadrightarrow \pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})).$$

Define a filtration on $\pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}}))$ by

$$\pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}}))_i = \pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})_i).$$

Clearly we have a commutative diagram

$$\begin{array}{ccc} \omega(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})) & \twoheadrightarrow & \pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})) \\ \downarrow & & \downarrow \\ gr\omega(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})) & \longrightarrow & gr\pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})). \end{array}$$

We therefore have a surjective homomorphism

$$P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}} \cong gr\omega(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})) \twoheadrightarrow gr\pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})).$$

This clearly implies that the Gelfand-Kirillov dimension of π is less than or equal to the Gelfand-Kirillov dimension of $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$. $\square$

Proposition 2. a) *Let $H = Sp(2n,\mathbb{R})$; then the Gelfand-Kirillov dimension of $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$ is*

$$\begin{cases} (p+q)(2n - \frac{p+q-1}{2}), & \text{if } p+q \le 2n, \\ n(2n+1), & \text{if } p+q > 2n. \end{cases}$$

b) *Let $H = O(p,q)$; then the Gelfand-Kirillov dimension of $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$ is*

$$\begin{cases} 2n(p+q - \frac{2n+1}{2}), & \text{if } 2n \le p+q, \\ \frac{(p+q)(p+q-1)}{2}, & \text{if } 2n > p+q. \end{cases}$$

Proof. Since $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$ is a commutative algebra, its Gelfand-Kirillov dimension is the same as the Krull dimension ([KL]). In turn it is equal to the transcendence degree of $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$, since it is an integral domain and finitely generated.

First let $H = Sp(2n, \mathbb{R})$ so that $H' = O(p, q)$. We complexify the symmetric bilinear form $(,)$ on V to a symmetric linear form on $V_{\mathbb{C}}$, which we still denote by $(,)$. Let $m = p + q$.

Let $Z = (z_1, ..., z_{2n}) \in V_{\mathbb{C}}^{2n}$ be a typical element. For $1 \leq i, j \leq 2n$, let $r_{i,j}(Z) = (z_i, z_j) = z_i^t I_{p,q} z_j$. Then $r_{i,j} \in P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$. The First Main Theorem of Classical Invariant Theory for orthogonal groups [W] states that $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$ is generated by $r_{i,j}$, where $1 \leq i \leq j \leq 2n$.

Consider the $2n \times 2n$ symmetric matrix

$$R = Z^t I_{p,q} Z = \begin{pmatrix} r_{1,1}(Z) & \cdots & r_{1,2n}(Z) \\ \cdots & \cdots & \cdots \\ r_{2n,1}(Z) & \cdots & r_{2n,2n}(Z) \end{pmatrix}.$$

Observe that for each specialization of Z, the rank of R is less than or equal to m. Thus the determinant of any minor of size $(m+1) \times (m+1)$ of R is equal to zero, namely we have

$$\det \begin{pmatrix} r_{i_1,j_1} & \cdots & r_{i_1,j_{m+1}} \\ \cdots & \cdots & \cdots \\ r_{i_{m+1},j_1} & \cdots & r_{i_{m+1},j_{m+1}} \end{pmatrix} = 0, \tag{1}$$

where $1 \leq i_1 < \cdots < i_{m+1} \leq 2n$, and $1 \leq j_1 < \cdots < j_{m+1} \leq 2n$. In fact the Second Main Theorem of Classical Invariant Theory asserts that these generate all the relations among the generators $r_{i,j} \in P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$, but we shall not use the full force of this result.

Suppose $m \leq 2n$. Let

$$B = \{r_{i,j} | 1 \leq i \leq j \leq 2n, i \leq m\}.$$

In view of the Second Main Theorem of Classical Invariant Theory, we see that B is an algebraically independent set. Moreover by taking $(i_1, ..., i_m, i_{m+1}) = (1, ..., m, m+1)$, $(j_1, ..., j_m, j_{m+1}) = (1, ..., m, m+k)$, where $1 \leq k \leq n - 2m$, the determinant relations in Equation (1) imply that $r_{m+1,m+k}$ is in the field of fractions generated by B. Thus $r_{m+1,j}$ is in this field for any j. Continuing this way, we see that any $r_{i,j} \notin B$ is in the field of fractions of B. Thus the transcendence degree of $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$ is the cardinality of B, which is $m(2n - \frac{m-1}{2})$.

Suppose $m > 2n$. Then the $n(2n+1)$ generators of $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$, namely $\{r_{i,j} | 1 \leq i \leq j \leq 2n\}$, are algebraically independent, and so the transcendence degree of $P(V_{\mathbb{C}}^{2n})^{H'_{\mathbb{C}}}$ is $n(2n+1)$.

Now let $H = O(p, q)$ so that $H' = Sp(2n, \mathbb{R})$. We shall argue similarly as in the above case. Again we identify $V_{\mathbb{C}}^{2n}$ with $M_{m,2n}(\mathbb{C})$. Let $U = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in M_{m,2n}(\mathbb{C})$ be a typical element. Thus $u_1, ..., u_m$ are the row vectors of U. For $1 \leq i, j \leq m$, let $s_{i,j}(U) = u_i J_n u_j^t$. Then $s_{i,j} \in P(M_{m,2n}(\mathbb{C}))^{Sp(2n,\mathbb{C})}$. The First Main Theorem of Classical Invariant Theory for symplectic groups [W] states that $P(M_{m,2n}(\mathbb{C}))^{Sp(2n,\mathbb{C})}$ is generated by $s_{i,j}$, where $1 \leq i < j \leq m$.

Consider the $m \times m$ skewsymmetric matrix

$$S = UJ_nU^t = \begin{pmatrix} s_{1,1}(U) & \cdots & s_{1,m}(U) \\ \cdots & \cdots & \cdots \\ s_{m,1}(U) & \cdots & s_{m,m}(U) \end{pmatrix}.$$

Observe that for each specialization of U, the rank of S is less than or equal to $2n$. Thus the Pfaffian of any principal minor of size $(2n+2) \times (2n+2)$ of S is equal to zero, namely we have

$$Pf \begin{pmatrix} s_{i_1,i_1} & \cdots & s_{i_1,i_{2n+2}} \\ \cdots & \cdots & \cdots \\ s_{i_{2n+2},i_1} & \cdots & s_{i_{2n+2},i_{2n+2}} \end{pmatrix} = 0, \tag{2}$$

where $1 \leq i_1 < \cdots < i_{2n+2} \leq m$.

Suppose $2n \leq m$. Let

$$C = \{s_{i,j} | 1 \leq i < j \leq m, i \leq 2n\}.$$

In view of relations of $s_{i,j}$'s given in the Second Main Theorem of Classical Invariant Theory [W], we see that C is an algebraically independent set. Moreover by taking $(i_1, i_2, ..., i_{2n+1}, i_{2n+2}) = (1, 2, ..., 2n+1, 2n+k)$, where $2 \leq k \leq m-2n$, the Pfaffian relations in Equation (2) imply that $s_{2n+1,2n+k}$ is in the field of fractions generated by C. Thus $s_{2n+1,j}$ is in this field for any j. Continuing this way, we see that any $s_{i,j} \notin C$ is in the field of fractions of C. Thus the transcendence degree of $P(M_{m,2n}(\mathbb{C}))^{Sp(2n,\mathbb{C})}$ is the cardinality of C, which is $2n(m - \frac{2n+1}{2})$.

Suppose $2n > m$. Then the $\frac{m(m-1)}{2}$ generators of $P(M_{m,2n}(\mathbb{C}))^{Sp(2n,\mathbb{C})}$, namely $\{s_{i,j} | 1 \leq i < j \leq m\}$, are algebraically independent, and so the transcendence degree of $P(M_{m,2n}(\mathbb{C}))^{Sp(2n,\mathbb{C})}$ is $\frac{m(m-1)}{2}$.

□

3. Concluding remarks

In the course of works on some representations obtained from duality correspondence, the author and his collaborators have computed the Gelfand-Kirillov dimension of these representations. We list a few results here.

a) Let $q = 0$, and let π be a representation of $\tilde{S}p(2n, \mathbb{R})$ obtained from the duality correspondence of the pair $(O(p), Sp(2n, \mathbb{R}))$. Thus π is a holomorphic representation of $\tilde{S}p(2n, \mathbb{R})$. Then GKdim of $\pi = p(n - \frac{p-1}{2})$ if $p \leq n$, and GKdim of $\pi = \frac{n(n+1)}{2}$ if $p > n$. See [TZ].

b) Let $p + q \leq n$, namely $(O(p, q), Sp(2n, \mathbb{R}))$ is a dual pair in the stable range with $O(p, q)$ the small member ([H5]), and let π be the representation of $\tilde{S}p(2n, \mathbb{R})$ corresponding to the trivial representation of $O(p, q)$. Then GKdim of $\pi = (p + q)(n - \frac{p+q-1}{2})$. See [LZ2].

c) Let $p+q$ is even and $2n \leq \min(p, q)$, namely $(O(p, q), Sp(2n, \mathbb{R}))$ is a dual pair in the stable range with $Sp(2n, \mathbb{R})$ the small member, and let π be the representation of $O(p, q)$ corresponding to the trivial representation of $Sp(2n, \mathbb{R})$. Then GKdim of $\pi = n(p + q - 2n - 1)$. See [ZH].

The above examples suggest that when the size of H' is very small relative to that of H, the Gelfand-Kirillov dimension of a representation π of $\tilde{H}$ obtained from duality correspondence with a representation σ of $\tilde{H}'$ may be independent of σ. They also indicate that our estimate of the Gelfand-Kirillov dimension is not sharp. This is due to the fact that given a finite dimensional generating subspace

V_0 (§0), the map $\pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})_i) \otimes V_0 \mapsto \pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})_i)V_0$ often has a kernel. For example in case a), if we take V_0 to be the lowest $\tilde{U}(n)$-type of the holomorphic representation π of $\tilde{S}p(2n, \mathbb{R})$, then

$$\pi(\mathcal{U}(\mathfrak{h}_{\mathbb{C}})_i)V_0 = \pi(S(\mathfrak{p}^+)_i)V_0, \tag{3}$$

where $S(\mathfrak{p}^+)$ is the symmetric algebra of $\mathfrak{p}^+$, and $\mathfrak{h}_{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is the Harish-Chandra decomposition of $\mathfrak{h}_{\mathbb{C}} = \mathfrak{sp}(2n, \mathbb{C})$.

Finally we remark on the methods the Gelfand-Kirillov dimension is computed in these examples. In case a), we use equation (3) and then apply invariant theoretic considerations, and so the argument is quite similar to the one given in this article. In cases b) and c), our computation of the Gelfand-Kirillov dimension relies on an explicit description of $\tilde{K}$-types of the representations concerned, where K is a maximal compact subgroup. It is worthwhile to mention that they are all $\tilde{K}$ multiplicity-free in cases b) and c).

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