PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 10, October 1998, Pages 2827–2834
S 0002-9939(98)04429-3

VECTOR BUNDLES WITH HOLOMORPHIC CONNECTION OVER A PROJECTIVE MANIFOLD WITH TANGENT BUNDLE OF NONNEGATIVE DEGREE

INDRANIL BISWAS

(Communicated by Ron Donagi)

Abstract. For a projective manifold whose tangent bundle is of nonnegative degree, a vector bundle on it with a holomorphic connection actually admits a compatible flat holomorphic connection, if the manifold satisfies certain conditions. The conditions in question are on the Harder-Narasimhan filtration of the tangent bundle, and on the Neron-Severi group.

1. Introduction

Let E be a holomorphic vector bundle over a connected complex projective manifold M. Assume that E admits a holomorphic connection. Then a natural question to ask is whether E admits a flat holomorphic connection. Since all the rational Chern classes (of degree at least one) of a holomorphic vector bundle with a holomorphic connection vanish, there is no topological obstruction for the existence of a flat connection.

In this paper we consider this question for M satisfying the condition that the degree of the tangent bundle T_M is nonnegative with respect to some polarization on M.

Let

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_l \subset V_{l+1} = T_M$$

be the Harder-Narasimhan filtration of the tangent bundle T_M with respect to a polarization L on M.

In Theorem 2.4 we prove the following (degree of a coherent sheaf on M is computed using L):

Theorem A. *Assume that the degree of the tangent bundle* $\deg T_M \geq 0$*. Let* E *be a holomorphic vector bundle on* M *equipped with a holomorphic connection.*

(1) *If* $\deg (T_M/V_l) \geq 0$ *then the holomorphic vector bundle* E *admits a compatible flat connection. (This inequality condition is satisfied if, for example,* T_M *is semistable, since* $\deg T_M \geq 0$*.)*

Received by the editors February 24, 1997.
1991 *Mathematics Subject Classification.* Primary 14F05, 32L10, 53C07.
Key words and phrases. Holomorphic connection, flat connection, Harder-Narasimhan filtration.

(2) *Consider the case where T_M is not semistable. Assume that the maximal semistable subsheaf of T_M, namely V_1, is locally free. If the rank of the Neron-Severi group, $NS(M)$, of M is 1, i.e.,*

$$H^{1,1}(M) \cap H^2(M, \mathbb{Q}) = \mathbb{Q},$$

then E admits a compatible flat connection.

Under the assumptions either in part (1) or in part (2) of Theorem A, the vector bundle E turns out to be semistable with respect to L [Remark 2.12].

Generalizing the above question one may ask whether a holomorphic fiber bundle admitting a holomorphic connection actually admits a flat holomorphic connection. S. Murakami produced an example of a holomorphic fiber bundle over an abelian variety, with an abelian variety as fiber, such that the fiber bundle admits a holomorphic connection, but it does not admit any flat holomorphic connection [M1], [M2], [M3]. However part (1) of Theorem A implies that any holomorphic vector bundle over a projective manifold with trivial canonical line bundle, which admits a holomorphic connection, actually admits a flat holomorphic connection. Indeed, by a theorem of Yau [Ya] the tangent bundle of such a variety is semistable.

On the other hand, using a method of [Bi2], Theorem A can easily be generalized to principal G-bundles, where the structure group G is a connected affine algebraic reductive group over $\mathbb{C}$. The example of Murakami shows that it is essential for G to be noncompact.

2. Criteria for the existence of a flat connection

Let $M/\mathbb{C}$ be a connected smooth projective variety of complex dimension d. We will denote by T_M (resp. Ω^1_M) the holomorphic tangent bundle (resp. cotangent bundle) of M.

For a holomorphic vector bundle V, the corresponding coherent analytic sheaf given by its local holomorphic sections will also be denoted by V. The basic facts about holomorphic structures used here can be found in [Ko].

A *holomorphic connection* on a holomorphic vector bundle E over M is a first order differential operator

$$D : E \longrightarrow \Omega^1_M \otimes E \tag{2.1}$$

satisfying the following Leibniz condition:

$$D(fs) = fD(s) + df \otimes s \tag{2.2}$$

where f is a local holomorphic function on M and s is a local holomorphic section of E. Extend D as a first order operator

$$D : \Omega^{p,q}_M \otimes E \longrightarrow \Omega^{p+1,q}_M \otimes E$$

using the Leibniz rule. The *curvature* of D is defined to be

$$D^2 := D \circ D$$

which is a holomorphic section of $\Omega^2_M \otimes \operatorname{End} E$ ($\Omega^k_M := \bigwedge^k \Omega^1_M$). The notion of a holomorphic connection was introduced by M. Atiyah [At].

If $\overline{\partial}_E : E \longrightarrow \Omega^{0,1}_M \otimes E$ denotes the first order differential operator defining the holomorphic structure on E, then the operator

$$D + \overline{\partial}_E$$

is a connection on E in the usual sense. Moreover, the curvature of this connection is D^2; in particular, it is a holomorphic section of $\Omega^2_M \otimes \operatorname{End} E$. Conversely, the $(1,0)$ part of a connection on E, such that the $(0,1)$ part of it is $\overline{\partial}_E$ and its curvature is a holomorphic section of $\Omega^2_M \otimes \operatorname{End} E$, is actually a holomorphic connection.

In particular, if ∇ is a flat connection on a C^∞ complex vector bundle M, then the $(0,1)$ part of the connection operator defines a holomorphic structure on E and the $(1,0)$ part defines a holomorphic connection.

Let L be a polarization on M, or equivalently, L is an ample line bundle on M. For a coherent sheaf F on M, the *degree* of F, denoted by $\deg F$, is defined as follows ($d = \dim_{\mathbb{C}} M$):

$$\deg F \quad := \quad \int_M c_1(F) \cup c_1(L)^{d-1}.$$

A torsion-free coherent sheaf F is called *semistable* if for every (nonzero) coherent subsheaf $V \subset F$, the following inequality holds:

$$\frac{\operatorname{rank} V}{\deg V} \quad \leq \quad \frac{\operatorname{rank} F}{\deg F}.$$

Moreover, if the strict inequality holds for every proper coherent subsheaf V with F/V torsion-free, then F is called *stable*.

The quotient $\operatorname{rank} F/\deg F$ is called the *slope* of F and is usually denoted by $\mu(F)$.

Any torsion-free coherent sheaf F admits a unique filtration by coherent subsheaves, known as the *Harder-Narasimhan filtration*, of the following type ([Ko], page 174, Theorem 7.15):

$$0 \,=\, F_0 \,\subset\, F_1 \,\subset\, F_2 \,\subset\, \dots \,\subset\, F_k \,\subset\, F_{k+1} \,=\, F$$

where F_1 is the maximal semistable subsheaf of F. The Harder-Narasimhan filtration is determined by the property that F_{i+1}/F_i is the maximal semistable subsheaf of F/F_i. This implies that $\mu(F_{i+1}/F_i) < \mu(F_i/F_{i-1})$.

Let

$$0 \,=\, V_0 \,\subset\, V_1 \,\subset\, V_2 \,\subset\, \dots \,\subset\, V_l \,\subset\, V_{l+1} \,=\, T_M \tag{2.3}$$

be the Harder-Narasimhan filtration of the tangent bundle T_M.

A flat connection on a holomorphic vector bundle E on M is said to be *compatible* if the $(0,1)$ part of the connection is $\overline{\partial}_E$ (equivalently, (local) flat sections are holomorphic sections). A compatible flat connection is same as a flat holomorphic connection.

Theorem 2.4. *Assume that the degree of the tangent bundle $\deg T_M \geq 0$. Let E be a holomorphic vector bundle on M equipped with a holomorphic connection.*

1. *If $\deg(T_M/V_l) \geq 0$ then the holomorphic vector bundle E admits a compatible flat connection. (This inequality condition is satisfied if, for example, T_M is semistable since $\deg T_M \geq 0$.)*
2. *Consider the case where T_M is not semistable. Assume that the maximal semistable subsheaf of T_M, namely V_1, is locally free. If the rank of the Neron-Severi group, $NS(M)$, of M is 1, i.e.,*

   $$H^{1,1}(M) \cap H^2(M, \mathbb{Q}) \,=\, \mathbb{Q}\,,$$

   then E admits a compatible flat connection.

Proof. Assume that $\deg(T_M/V_l) \geq 0$. Then from Lemma 2.1 of [Bi2] (also Remark 3.7(ii) of [Bi1]) we know that the vector bundle E is semistable. To be self-contained as much as possible we will quickly recall the proof of the semistability of E. Since E admits a holomorphic connection, Theorem 4 (page 192) of [At] says that all the (rational) Chern classes, $c_k(E)$, where $k \geq 1$, of E vanish. In particular $\deg E = 0$. Let W be the maximal semistable subsheaf of E. The key observation is that W is left invariant by the holomorphic connection operator D on E. Indeed, the homomorphism

$$W \longrightarrow \Omega^1_M \otimes \frac{E}{W} \tag{2.5}$$

induced by D is $\mathcal{O}_M$-linear (a simple consequence of the Leibniz identity (2.2)). The Harder-Narasimhan filtration of a tensor product is simply the tensor product of the corresponding Harder-Narasimhan filtrations. Applying this to $\Omega^1_M \otimes (E/W)$, since the degree of any subsheaf of Ω^1_M is nonpositive (this is equivalent to the assertion that the degree of a quotient sheaf of T_M is nonnegative, which, in turn, is warranted by the assumption that $\deg(T_M/V_l) \geq 0$), the slope of the maximal semistable subsheaf of $\Omega^1_M \otimes (E/W)$ is less than or equal to $\mu(E/W)$. Finally from the general properties of Harder-Narasimhan filtrations we have $\mu(W) > \mu(E/W)$. If the image of the homomorphism in (2.5) is nonzero then the slope of the image is simultaneously at least $\mu(W)$ (recall that W is semistable) and as well as it is at most the slope of the maximal semistable subsheaf of $\Omega^1_M \otimes (E/W)$. This contradicts the earlier observation that the slope of the maximal semistable subsheaf of $\Omega^1_M \otimes (E/W)$ is strictly less than $\mu(W)$. Thus the homomorphism in (2.5) must be the zero homomorphism. In other words, W has an induced holomorphic connection. This implies that W is locally free of degree zero. So W cannot be a proper subsheaf of E. In other words, E must be semistable.

Since E is semistable with vanishing first and second Chern classes, the Corollary 3.10 (page 40) of [Si] implies that E admits a flat connection compatible with its holomorphic structure.

To prove part (2) of Theorem 2.4 we assume that T_M is not semistable. The maximal semistable subsheaf of T_M, namely V_1 (in (2.3)), is assumed to be locally free.

Our first step will be to prove that V_1 is closed under the Lie bracket operation on T_M. Towards this goal consider the homomorphism

$$\Gamma : V_1 \otimes V_1 \longrightarrow \frac{T}{V_1} \tag{2.6}$$

defined by composing the Lie bracket operation with the natural projection of T_M onto T_M/V_1. Since the Lie bracket satisfies the Leibniz identity, namely

$$[fv, w] = f[v, w] - \langle df, w\rangle v\,,$$

where $\langle -, - \rangle$ denotes the obvious contraction, the map Γ is actually $\mathcal{O}_M$-linear, i.e., Γ is a homomorphism of vector bundles.

Now we are given that $\mu(V_1) > \mu(T_M) \geq 0$. So

$$\mu(V_1 \otimes V_1) = 2\mu(V_1) > \mu(V_1) > \mu(V_2/V_1)\,, \tag{2.7}$$

the last inequality being a general property of Harder-Narasimhan filtrations. The image of the homomorphism Γ is simultaneously a quotient of $V_1 \otimes V_1$ as well as a subsheaf of T_M/V_1. But V_2/V_1, by definition, is the maximal semistable subsheaf of T_M/V_1. So if $\Gamma \neq 0$ then

$$\mu(V_1 \otimes V_1) \,\leq\, \mu(\text{image}\,\Gamma) \,\leq\, \mu(V_2/V_1).$$

The first inequality is a consequence of the fact that $V_1 \otimes V_1$ is semistable. (A tensor product of semistable vector bundles is again semistable [MR], Remark 6.6 (iii).) This contradicts the inequality (2.7) unless image $\Gamma = 0$. But $\Gamma = 0$ is equivalent to V_1 being closed under the Lie bracket operation. In other words, V_1 is a nonsingular holomorphic foliation on M.

If E is semistable we may complete the proof of Theorem 2.4 by repeating the use of the Corollary 3.10 of [Si] as done in the proof of part (1) of Theorem 2.4. So we may, and we will, assume that E is not semistable. Let

$$0 \,=\, W_0 \,\subset\, W_1 \,\subset\, W_2 \,\subset\, \ldots \,\subset\, W_m \,\subset\, W_{m+1} \,=\, E$$

be the Harder-Narasimhan filtration of E.

Our next step will be to show that the sheaf W_1 has an induced *holomorphic partial connection* along the foliation V_1. In other words, we want to show that the operator D in (2.1) induces an operator

$$D' \,:\, W_1 \,\longrightarrow\, V_1^* \otimes W_1 \tag{2.8}$$

which satisfies the Leibniz condition (2.2); df in (2.2) is realized as a section of V_1^* in (2.8) by using the natural projection of Ω^1_M onto V_1^*. The notion of a partial connection was introduced by R. Bott.

To construct D' first note that, by projecting Ω^1_M onto V_1^*, the operator D in (2.1) induces an operator

$$D_1 \,:\, W_1 \,\longrightarrow\, V_1^* \otimes E. \tag{2.9}$$

Now projecting E onto E/W_1, the operator D_1 in (2.9) induces an operator

$$D_2 \,:\, W_1 \,\longrightarrow\, V_1^* \otimes \frac{E}{W_1}.$$

The Leibniz identity (2.2) implies that D_2 is $\mathcal{O}_M$-linear; i.e., the order of the differential operator D_2 is zero. In other words, D_2 is a homomorphism of vector bundles.

We will show that $D_2 = 0$ by following the steps of the argument for $\Gamma = 0$ (in (2.6)).

If $D_2 \neq 0$ then $\mu(\text{image}\,(D_2)) \geq \mu(W_1)$, since image (D_2) is a quotient of the semistable sheaf W_1. On the other hand, since

$$\text{image}\,(D_2) \,\subseteq\, V_1^* \otimes \frac{E}{W_1},$$

we conclude that the slope of image (D_2) is at most the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$.

Thus if $D_2 \neq 0$, then $\mu(W_1)$ is less than or equal to the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$.

On the other hand, since V_1^* is semistable with strictly negative slope, the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$ is strictly less than the slope of the maximal semistable subsheaf of E/W_1 – which in turn is strictly less than the slope of W_1. Thus the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$ is strictly less than $\mu(W_1)$. This contradicts the inequality obtained in the previous paragraph. So we have $D_2 = 0$.

Since $D_2 = 0$, the differential operator D_1 in (2.9) induces a first order differential operator D' as in (2.8). Clearly D' satisfies the Leibniz identity, as D satisfies it.

The operator D' maps (local) holomorphic sections of W_1 to holomorphic sections of $V_1^* \otimes W_1$. So D' is a partial connection on W_1 along $V_1 \oplus T^{0,1}_M$ in the sense of [BB] (Sections 2 and 3); $T^{0,1}_M$ is the anti-holomorphic tangent bundle.

However, unfortunately, W_1 is not necessarily locally free. (A coherent sheaf equipped with a holomorphic connection must be locally free, but D' is only a partial connection.) To circumvent the problems caused by such a possibility of not being locally free, we will consider the determinant line bundle

$$d(W_1) \quad := \quad \det W_1 \quad = \quad \bigwedge^r W_1$$

where r is the rank of W_1. The details of the construction of the determinant bundle of a torsion-free coherent sheaf can be found in Chapter 5, §6 of [Ko]. We note that the determinant bundle of a torsion-free sheaf is locally free of rank one, i.e., it is a line bundle.

The partial connection D' induces a partial connection on $d(W_1)$, which we will also denote by D'. More precisely, for a local section of $d(W_1)$

$$s \quad := \quad s_1 \wedge s_2 \wedge \ldots \wedge s_r \quad \in \quad \Gamma(U, d(W_1))$$

the action of D' on it is defined as follows:

$$D'(s) \quad := \quad \sum_{j=1}^{r} s_1 \wedge \ldots \wedge D'(s_j) \wedge \ldots \wedge s_r .$$

It is straight-forward to check that the operator D' defined above satisfies the Leibniz identity. Thus D' is a partial holomorphic connection on $d(W_1)$ along V_1.

We may extend the partial connection D' to an actual connection on $d(W_1)$ following [BB]. Fix a Kähler metric, say H, on M. Let ∇' be a hermitian connection on $d(W_1)$; the $(0,1)$ part of ∇' is assumed to be $\overline{\partial}_{d(W_1)}$. For any $v \in T^{1,0}_M$ let $v = v_1 \oplus v_2$ be the decomposition as $T^{1,0}_M = V_1 \oplus V_1^{\perp}$ using the metric H. For $v' \in T^{0,1}_M$ and a smooth section ϕ of $d(W_1)$ define:

$$\nabla_{v \oplus v'} \phi \quad := \quad \langle D'\phi, v_1 \rangle + \nabla'_{v_2} \phi + \langle \overline{\partial}_{d(W_1)} \phi, v' \rangle .$$

Clearly ∇ is a connection in the usual sense whose $(0,1)$ part coincides with $\overline{\partial}_{d(W_1)}$, and it is an extension of the partial connection D'.

Let

$$\mathcal{I} \quad \subseteq \quad \Omega^{1,1}_M \oplus \Omega^{0,2}_M$$

be the degree 2 component of the ideal, in the exterior algebra $\bigwedge \left(\Omega^{1,0}_M \oplus \Omega^{0,1}_M \right)$, generated by the subspace of $\Omega^{0,1}_M$ that annihilates $\overline{V_1}$.

The following simple lemma will be useful:

Lemma 2.10. *The curvature ∇^2, which is a smooth 2-form on M, is actually a section of $\mathcal{I} \oplus \Omega^{2,0}_M$.*

The proof of Lemma 2.10 is a simple computation. It is actually a straight-forward extension of (3.33), page 295 of [BB] to partial holomorphic connections (extension from partial flat connections). All we need to observe is that the curvature of ∇' is of type $(1,1)$ (since ∇' is assumed to be hermitian) and that the curvature of the partial connection D' is a holomorphic section of $\bigwedge^2 V_1^*$. Since the restriction of ∇ to a leaf of the foliation V_1 coincides with $D' + \overline{\partial}_{d(W_1)}$, the restriction of ∇^2 to a leaf is a section of $\bigwedge^2 V_1^*$. It is easy to see that this implies Lemma 2.10. □

Continuing with the proof of Theorem 2.4, our next step will be to establish a lemma on vanishing of characteristic classes of $d(W_1)$, analogous to the Proposition (3.27), page 295, of [BB].

Lemma 2.11. *Let q be an integer with $q > \dim M - \dim V_1$. Then $c_1(d(W_1))^q = 0$.*

Proof of Lemma 2.11. The characteristic class $c_1(d(W_1))^q \in H^{q,q}(M)$, and it is represented by the differential form $(\nabla^2/2\pi\sqrt{-1})^q$. Since the space of forms on M admits Hodge decomposition, to prove Lemma 2.11 it is enough to show that the differential form $(\nabla^2)^q$ is a section of the vector bundle

$$\bigoplus_{j>q} \Omega_M^{j,2q-j}.$$

But Lemma 2.10 implies that $(\nabla^2)^q$ is indeed of the above type. To see this first note that by Lemma 2.10, both the $(1,1)$ and the $(0,2)$ part of ∇^2 is contained in the ideal generated by the subspace of $\Omega_M^{0,1}$ that annihilates $\overline{V_1}$. But the dimension of this annihilator is $\dim M - \dim V_1$. So the component of $(\nabla^2)^q$ in

$$\bigoplus_{j\leq q} \Omega_M^{j,2q-j}$$

vanishes identically. This completes the proof of the lemma. □

To complete the proof of Theorem 2.4 we first note that the given condition that the rank of the Neron-Severi group, $NS(M)$, is 1 implies that if $(\omega)^j = 0$, where $\omega \in NS(M)\otimes_{\mathbb{Z}}\mathbb{Q}$ $(= H^2(M,\mathbb{Q})\cap H^{1,1}(M))$ and $1 \leq j \leq \dim_{\mathbb{C}} M$, then $\omega = 0$. This is simply because ω is a (possibly zero) rational multiple of the hyperplane class, and the j-th power of the hyperplane class is nonzero. Substituting $c_1(d(W_1))$ for ω and using Lemma 2.11 we get that $c_1(d(W_1)) = 0$. Thus we have

$$\deg W_1 \;=\; \deg d(W_1) \;=\; 0.$$

But W_1 is the maximal semistable subsheaf of E and $\deg E = 0$. This contradicts the assumptions that E is not semistable and that W_1 is the maximal semistable subsheaf of E. We already noted that if E is semistable then the Corollary 3.10 (page 40) of [Si] completes the proof of the theorem. This completes the proof of Theorem 2.4. □

Remark 2.12. The proof of Theorem 2.4 shows that under the assumptions in either part 1 or part 2 of the statement of Theorem 2.4, the vector bundle E is actually semistable.

Acknowledgments

The author is thankful to C. Simpson and T. Pantev for explaining some results of [Si] used here.

References

[At] M.F. Atiyah : Complex analytic connections in fibre bundles. Transactions of the Amer. Math. Soc. **85** (1957), 181-207. MR **19:**172c

[BB] P. Baum and R. Bott : Singularities of holomorphic foliations. Jour. Diff. Geom. **7** (1972), 279–342. MR **51:**14092

[Bi1] I. Biswas : On Harder-Narasimhan filtration of the tangent bundle. Comm. Anal. Geom. **3** (1995), 1–10. MR **97a:**32031

[Bi2] I. Biswas : Principal bundles admitting a holomorphic connection. Int. Jour. Math. **7** (1996), 433–440. MR **97f:**32043

[Ko] S. Kobayashi : Differential Geometry of Complex Vector Bundles. Publications of the Math. Society of Japan 15. Iwanami Shoten Publishers and Princeton University Press (1987). MR **89e:**53100

[M1] S. Murakami : Sur certains espaces fibrés principaux admettant des connexions holomorphes. Osaka Math. Jour. **11** (1959), 43–62. MR **22:**958

[M2] S. Murakami : Sur certains espaces fibrés principaux holomorphes dont le groupe est abélien connexe. Osaka Math. Jour. **13** (1961), 143–167. MR **26:**788

[M3] S. Murakami : Harmonic connections and their applications. SEA Bull. Math. Special Issue (1993), 101–103. MR **94k:**53042

[MR] V. Mehta and A. Ramanathan : Semistable sheaves on projective varieties and their restriction to curves. Math. Ann. **258** (1982), 213–224. MR **83f:**14013

[Si] C. Simpson : Higgs bundles and local systems. Pub. Math. IHES **75** (1992), 5–95. MR **94d:**32027

[Ya] S. T. Yau : On Calabi's conjecture and some new results in algebraic geometry. Proc. Nat. Sci. U.S.A. **74** (1977), 1798–1799.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in