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APPLICATIONS OF PSEUDO-MONOTONE OPERATORS WITH SOME KIND OF UPPER SEMICONTINUITY IN GENERALIZED QUASI-VARIATIONAL INEQUALITIES ON NON-COMPACT SETS

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Abstract. Let E be a topological vector space and X be a non-empty subset of E. Let $S : X \to 2^X$ and $T : X \to 2^{E^*}$ be two maps. Then the generalized quasi-variational inequality (GQVI) problem is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$. We shall use Chowdhury and Tan's 1996 generalized version of Ky Fan's minimax inequality as a tool to obtain some general theorems on solutions of the GQVI on a paracompact set X in a Hausdorff locally convex space where the set-valued operator T is either strongly pseudo-monotone or pseudo-monotone and is upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each non-empty finite subset A of X.

1. Introduction

If X is a set, we shall denote by 2^X the family of all non-empty subsets of X and by $\mathcal{F}(X)$ the family of all non-empty finite subsets of X. Let E be a topological vector space. We shall denote by E^* the continuous dual of E, by $\langle w, x\rangle$ the pairing between E^* and E for $w \in E^*$ and $x \in E$ and by $Re\langle w, x\rangle$ the real part of $\langle w, x\rangle$. If $X \subset E$, $S : X \to 2^X$ and $T : X \to E^*$, the quasi-variational inequality problem (QVI) is to find a point $\hat{y} \in S(\hat{y})$ such that $Re\langle T(\hat{y}), \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$. The QVI was first introduced by Bensousson and Lions in 1973 (see, e.g., [2]) in connection with impulse control. Again, if we consider a set-valued map $T : X \to 2^{E^*}$, then the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$. The GQVI was introduced by Chan and Pang [4] in 1982 if $E = \mathbb{R}^n$ and by Shih and Tan [11] in 1985 if E is infinite dimensional.

In this paper, we shall use Chowdhury and Tan's generalized version [5, Theorem 2] of Ky Fan's minimax inequality [8, Theorem 1] as a tool to obtain some

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general theorems on solutions of the GQVI on a paracompact set X in a locally convex Hausdorff topological vector space where the set-valued operator T is strongly pseudo-monotone or pseudo-monotone and is upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$.

We shall use our following set-valued generalization of the classical pseudo-monotone operator. The classical definition of a pseudo-monotone operator was introduced by Brézis, Nirenberg and Stampacchia in [3]. For a slightly general definition of a pseudo-monotone operator we refer to [5, Definition 1].

Definition 1.1. Let E be a topological vector space, X be a non-empty subset of E and $T : X \to 2^{E^*}$. If $h : X \to \mathbb{R}$, then T is said to be (1) *h-pseudo-monotone* if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y with $\limsup_\alpha [\inf_{u\in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \le 0$, we have

$$\liminf_\alpha [\inf_{u\in T(y_\alpha)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)] \ge \inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)$$

for all $x \in X$; (2) *pseudo-monotone* if T is h-pseudo-monotone with $h \equiv 0$.

2. Generalized quasi-variational inequalities for strongly pseudo-monotone operators

In this section we shall introduce the notion of strongly pseudo-monotone operators and obtain some general theorems on solutions of the GQVI on paracompact sets in locally convex Hausdorff topological vector spaces.

We shall begin with the following:

Definition 2.1. Let E be a topological vector space, X be a non-empty subset of E and $T : X \to 2^{E^*}$. If $h : X \to \mathbb{R}$, then T is said to be (1) *strongly h-pseudo-monotone* if for each continuous function $\theta : X \to [0, 1]$, for each $y \in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y with

$$\limsup_\alpha [\theta(y_\alpha)(\inf_{u\in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y))] \le 0$$

we have

$$\begin{aligned}\limsup_\alpha [\theta(y_\alpha)(\inf_{u\in T(y_\alpha)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x))]\\ \ge [\theta(y)(\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x))]\end{aligned}$$

for all $x \in X$; (2) *strongly pseudo-monotone* if T is strongly h-pseudo-monotone with $h \equiv 0$.

Clearly, every strongly pseudo-monotone operator is also a pseudo-monotone operator as defined in [5].

Proposition 2.1. *Let X be a non-empty subset of a topological vector space E. If $T : X \to E^*$ is monotone and continuous from the relative weak topology on X to the weak* topology on E^*, then T is strongly pseudo-monotone.*

Proof. Let us consider any arbitrary continuous function $\theta : X \to [0, 1]$. Suppose $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X and $y \in X$ with $y_\alpha \to y$ (and

$$\limsup_\alpha [\theta(y_\alpha)(Re\langle Ty_\alpha, y_\alpha - y\rangle)] \le 0).$$

Then for any $x \in X$ and $\epsilon > 0$, there are $\beta_1, \beta_2 \in \Gamma$ with $|\theta(y_\alpha)Re\langle Ty, y_\alpha - y\rangle| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_1$ and $|\theta(y_\alpha)Re\langle Ty_\alpha - Ty, y - x\rangle| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_2$. Choose $\beta_0 \in \Gamma$ with $\beta_0 \geq \beta_1, \beta_2$. Thus

$$\begin{aligned}
&\theta(y_\alpha)Re\langle Ty_\alpha, y_\alpha - x\rangle = \theta(y_\alpha)Re\langle Ty_\alpha, y_\alpha - y\rangle + \theta(y_\alpha)Re\langle Ty_\alpha, y - x\rangle \\
&\geq \theta(y_\alpha)Re\langle Ty, y_\alpha - y\rangle + \theta(y_\alpha)Re\langle Ty_\alpha, y - x\rangle \\
&= \theta(y_\alpha)Re\langle Ty, y_\alpha - y\rangle + \theta(y_\alpha)Re\langle Ty_\alpha - Ty, y - x\rangle + \theta(y_\alpha)Re\langle Ty, y - x\rangle \\
&> -\frac{\epsilon}{2} - \frac{\epsilon}{2} + \theta(y_\alpha)Re\langle Ty, y - x\rangle \text{ for all } \alpha \geq \beta_0
\end{aligned}$$

so that $\inf_{\alpha \geq \beta_0} \theta(y_\alpha)Re\langle Ty_\alpha, y_\alpha - x\rangle \geq -\epsilon + \inf_{\alpha \geq \beta_0} \theta(y_\alpha)Re\langle Ty, y - x\rangle$. It follows that $\limsup_\beta \theta(y_\beta)Re\langle Ty_\beta, y_\beta - x\rangle \geq \liminf_\beta \theta(y_\beta)Re\langle Ty_\beta, y_\beta - x\rangle \geq -\epsilon + \theta(y)Re\langle Ty, y - x\rangle$. As $\epsilon > 0$ is arbitrary,

$$\limsup_\beta \theta(y_\beta)Re\langle Ty_\beta, y_\beta - x\rangle \geq \theta(y)Re\langle Ty, y - x\rangle.$$

Hence T is strongly pseudo-monotone. □

We shall now establish the following result:

Theorem 2.1. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex subset of E and $h : E \to \mathbb{R}$ be convex. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be strongly h-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\}$$

is open in X. Suppose further that there exist a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that (i) *$\hat{y} \in S(\hat{y})$ and* (ii) *there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Proof. We divide the proof into two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w \in T(y)} Re\langle w, y - x\rangle + h(y) - h(x) > 0$; that is, $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0$. For each $y \in X$, set $\gamma(y) := \sup_{x \in S(y)}[\inf_{w \in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)]$. Let $V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set $V_p := \{y \in X : Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0\}$.

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 1 in [11] and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X. Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$ (see, e.g., Theorem VIII.4.2 of Dugundji in [7]); that is, for each $p \in E^*$, $\beta_p : X \to [0, 1]$ and $\beta_0 : X \to [0, 1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X \setminus V_0$ and $\{$ support β_0, support $\beta_p : p \in E^*\}$ is locally finite and $\beta_0(y) + \sum_{p \in E^*} \beta_p(y) = 1$ for each $y \in X$. Note that for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see e.g. [10, Corollary 10.1.1, p.83]). Define $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \beta_0(y)[\min_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y-x\rangle$$

for each $x, y \in X$. Then we have the following.

(1) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \min_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)$ is lower semicontinuous on $co(A)$ by Lemma 3 in [5] and the fact that h is continuous on $co(A)$ and therefore the map $y \mapsto \beta_0(y)[\min_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)]$ is lower semicontinuous on $co(A)$ by Lemma 3 in [12]. Also for each fixed $x \in X$, $y \mapsto \sum_{p\in E^*} \beta_p(y) Re\langle p, y-x\rangle$ is continuous on X. Hence, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \longmapsto \phi(x,y)$ is lower semicontinuous on $co(A)$.

(2) For each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x\in A} \phi(x,y) \le 0$. Indeed, if this were false, then for some $A = \{x_1, \cdots, x_n\} \in \mathcal{F}(X)$ and some $y \in co(A)$ (say $y = \sum_{i=1}^n \lambda_i x_i$ where $\lambda_1, \cdots, \lambda_n \ge 0$ with $\sum_{i=1}^n \lambda_i = 1$), we have $\min_{1\le i\le n} \phi(x_i, y) > 0$. Then for each $i = 1, \cdots, n$,

$$\beta_0(y)[\min_{w\in T(y)} Re\langle w, y-x_i\rangle + h(y) - h(x_i)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y-x_i\rangle > 0$$

so that

$$\begin{aligned} 0 = \phi(y,y) &= \beta_0(y)[\min_{w\in T(y)} Re\langle w, y - \sum_{i=1}^n \lambda_i x_i\rangle + h(y) - h(\sum_{i=1}^n \lambda_i x_i)] \\ &\qquad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - \sum_{i=1}^n \lambda_i x_i\rangle \\ &\ge \sum_{i=1}^n \lambda_i (\beta_0(y)[\min_{w\in T(y)} Re\langle w, y-x_i\rangle + h(y) - h(x_i)] \\ &\qquad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y-x_i\rangle) > 0, \end{aligned}$$

which is a contradiction.

(3) Suppose $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y with $\phi(tx + (1-t)y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$ and all $t \in [0,1]$.

Then for $t = 0$ we have $\phi(y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)] + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - y\rangle \le 0$$

for all $\alpha \in \Gamma$. Hence

$$\begin{aligned} &\limsup_\alpha [\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y))] \\ &\qquad + \liminf_\alpha (\sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - y\rangle) \\ &\le \limsup_\alpha [\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)) \\ &\qquad + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - y\rangle] \le 0. \end{aligned}$$

Therefore $\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y))] \le 0$. Since T is strongly h-pseudo-monotone, we have

$$[\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] \\ \ge \beta_0(y)(\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)).$$

Thus

(2.1)

$$\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \\ \ge \beta_0(y)(\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)) + \sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle.$$

For $t = 1$ we have $\phi(x, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)] + \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle \le 0$$

for all $\alpha \in \Gamma$. Therefore

$$\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] \\ + \liminf_\alpha[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle] \\ \le \limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)) \\ + \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle] \le 0.$$

Thus

$$\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] \\ + \sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \le 0. \tag{2.2}$$

Hence by (2.1) and (2.2), we have $\phi(x, y) \le 0$.

(4) By hypothesis, there exist a non-empty compact (and therefore closed) subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0) > 0$ for each $y \in X \setminus K$. Thus for each $y \in X \setminus K$,

$$\beta_0(y)[\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0)] > 0$$

whenever $\beta_0(y) > 0$ and $Re\langle p, y - x_0\rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently,

$$\phi(x_0, y) = \beta_0(y)[\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0)] \\ + \sum_{p\in E^*}\beta_p(y)Re\langle p, y - x_0\rangle > 0$$

for all $y \in X \setminus K$.

Then ϕ satisfies all hypotheses of Theorem 2 in [5]. Hence by Theorem 2 in [5], there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$; i.e.,

$$\beta_0(\hat{y})[\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)] + \sum_{p \in E^*} \beta_p(\hat{y}) Re\langle p, \hat{y} - x\rangle \leq 0 \tag{2.3}$$

for all $x \in X$.

If $\gamma(\hat{y}) = 0$, choose any $\hat{x} \in S(\hat{y})$; if $\gamma(\hat{y}) > 0$, choose any $\hat{x} \in S(\hat{y})$ such that $\inf\limits_{w \in T(\hat{y})} Re\langle w, \hat{y} - \hat{x}\rangle + h(\hat{y}) - h(\hat{x}) \geq \frac{\gamma(\hat{y})}{2} > 0$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$; it follows that

$$\beta_0(\hat{y})[\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - \hat{x}\rangle + h(\hat{y}) - h(\hat{x})] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence $Re\langle p, \hat{y}\rangle > \sup_{x \in S(\hat{y})} Re\langle p, x\rangle \geq Re\langle p, \hat{x}\rangle$ so that $Re\langle p, \hat{y} - \hat{x}\rangle > 0$. Then note that $\beta_p(\hat{y}) Re\langle p, \hat{y} - \hat{x}\rangle > 0$ whenever $\beta_p(\hat{y}) > 0$ for $p \in E^*$.

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x}, \hat{y}) = \beta_0(\hat{y})[\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - \hat{x}\rangle + h(\hat{y}) - h(\hat{x})] + \sum_{p \in E^*} \beta_p(\hat{y}) Re\langle p, \hat{y} - \hat{x}\rangle > 0,$$

which contradicts (2.3). This contradiction proves Step 1.

Step 2. There exists a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$.

Note that for each fixed $x \in S(\hat{y})$, $w \mapsto Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)$ is convex and continuous on $T(\hat{y})$ and for each fixed $w \in T(\hat{y})$, $x \mapsto Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)$ is concave on $S(\hat{y})$. Thus by Kneser's Minimax Theorem in [9] (see also Aubin [1, pp.40-41]), we have

$$\begin{aligned} &\min_{w \in T(\hat{y})} \max_{x \in S(\hat{y})} [Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \\ &\quad = \max_{x \in S(\hat{y})} \min_{w \in T(\hat{y})} [Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)]. \end{aligned}$$

Hence $\min_{w \in T(\hat{y})} \max_{x \in S(\hat{y})} [Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \leq 0$ by Step 1. Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$. □

If X is compact, we obtain the following immediate consequence of Theorem 2.1:

Theorem 2.2. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \to \mathbb{R}$ be convex. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be strongly h-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose the set $\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\}$ is open in X. Then there exists $\hat{y} \in X$ such that* (i) *$\hat{y} \in S(\hat{y})$ and* (ii) *there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Note that if X is also bounded in Theorem 2.1, the map $S : X \to 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semicontinuous at y in X, then the set Σ in Theorem 2.1 is always open in X as can be seen in the proof of the following:

Theorem 2.3. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : E \to \mathbb{R}$ be convex. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be strongly h-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that for each*

$$y \in \Sigma = \{y \in X : \sup_{x\in S(y)} [\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\},$$

T is upper semicontinuous at y from the relative topology on X to the strong topology on E^*. Suppose further that there exist a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and

$$\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0) > 0$$

for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that (i) *$\hat{y} \in S(\hat{y})$ and* (ii) *there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Proof. By virtue of Theorem 2.1, we need only show that the set

$$\Sigma := \{y \in X : \sup_{x\in S(y)} [\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\}$$

is open in X. Indeed, let $y_0 \in \Sigma$; then there exists $x_0 \in S(y_0)$ such that $\alpha := \inf_{w\in T(y_0)} Re\,\langle w, y_0 - x_0\rangle + h(y_0) - h(x_0) > 0$.

Let $W := \{w \in E^* : \sup_{z_1,z_2\in X} |\langle w, z_1 - z_2\rangle| < \frac{\alpha}{6}\}$. Then W is a strongly open neighborhood of 0 in E^* so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in E^*. Since T is upper semicontinuous at y_0 in X, there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Now, the rest of the proof is similar to the proof of Theorem 2.2 in [6]. Hence by the rest of the proof of Theorem 2.2 in [6], Σ is open in X. This proves the theorem. □

If X is compact, we obtain the following immediate consequence of Theorem 2.3:

Theorem 2.4. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \to \mathbb{R}$ be convex. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be strongly h-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x\in S(y)}[\inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^*. Then there exists $\hat{y} \in X$ such that* (i) *$\hat{y} \in S(\hat{y})$ and* (ii) *there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

We remark here that in Theorems 2.1-2.4, the condition "$h : E \to \mathbb{R}$ be convex" can be replaced by the condition "$h : X \to \mathbb{R}$ be convex such that $h|_{co(A)}$ is continuous for each $A \in \mathcal{F}(X)$".

3. Generalized quasi-variational inequalities for pseudo-monotone operators

In this section we shall obtain some existence theorems of generalized quasi-variational inequalities for pseudo-monotone operators (Definition 1.1) on paracompact convex sets.

We shall first establish the following result:

Theorem 3.1. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : E \to \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be h-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that the set $\Sigma = \{y \in X : \sup_{x\in S(y)}[\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\}$ is open in X. Suppose further that there exist a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that* (i) *$\hat{y} \in S(\hat{y})$ and* (ii) *there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Proof. We divide the proof into two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x\in S(\hat{y})}[\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \le 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x) > 0$; that is, $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $Re\langle p, y\rangle - \sup_{x\in S(y)} Re\langle p, x\rangle > 0$. For each $y \in X$, set $\gamma(y) := \sup_{x\in S(y)}[\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)]$. Let $V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set $V_p := \{y \in X : Re\langle p, y\rangle - \sup_{x\in S(y)} Re\langle p, x\rangle > 0\}$.

Then $X = V_0 \cup \bigcup_{p\in E^*} V_p$. Since each V_p is open in X by Lemma 1 in [11] and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X. Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$. Note that for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see e.g. [10, Corollary 10.1.1, p.83]). Define $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \beta_0(y)[\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle$$

for each $x, y \in X$. Then we have the following.

(1) The same argument in proving (1) in the proof of Theorem 2.1 shows that for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \longmapsto \phi(x, y)$ is lower semicontinuous on $co(A)$.

(2) The same argument in proving (2) in the proof of Theorem 2.1 shows that for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x\in A} \phi(x, y) \le 0$.

(3) Suppose $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y with $\phi(tx + (1 - t)y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$.

Case 1. $\beta_0(y) = 0$.

Note that $\beta_0(y_\alpha) \ge 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a bounded net, it follows that

$$\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] = 0. \tag{3.1}$$

Also $\beta_0(y)[\min_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] = 0$. Thus

(3.2)
$$\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle$$
$$= \sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \ \text{ (by (3.1))}$$
$$= \beta_0(y)[\min_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle.$$

For $t = 1$ we have $\phi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

(3.3)
$$\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)] + \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle \leq 0$$

for all $\alpha \in \Gamma$. Therefore

$$\begin{aligned}\limsup_{\alpha}&[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] \\ &+ \liminf_{\alpha}[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle] \\ \leq& \limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)) \\ &+ \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle] \\ \leq& \ 0 \ \text{ (by (3.3))}.\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{\alpha}&[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] \\ &+ \sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \leq 0.\end{aligned} \tag{3.4}$$

Hence by (3.2) and (3.4), we have $\phi(x, y) \leq 0$.

Case 2. $\beta_0(y) > 0$.

Since $\beta_0(y_\alpha) \to \beta_0(y)$, there exists $\lambda \in \Gamma$ such that $\beta_0(y_\alpha) > 0$ for all $\alpha \geq \lambda$. Then for $t = 0$ we have $\phi(y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)] + \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle \leq 0$$

for all $\alpha \in \Gamma$. Thus

$$\begin{aligned}\limsup_{\alpha}&[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)) \\ &+ \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] \leq 0.\end{aligned} \tag{3.5}$$

Hence

$$\begin{aligned}&\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y))]\\ &\qquad + \liminf_{\alpha}[\sum_{p\in E^*} \beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle]\\ &\le \limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y))\\ &\qquad + \sum_{p\in E^*} \beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle]\\ &\le 0 \ \text{(by (3.5))}.\end{aligned}$$

Since $\liminf_\alpha[\sum_{p\in E^*} \beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] = 0$, we have

$$\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y))] \le 0. \tag{3.6}$$

Since $\beta_0(y_\alpha) > 0$ for all $\alpha \ge \lambda$, it follows that

$$\begin{aligned}&\beta_0(y)\limsup_{\alpha}[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y))]\\ &\quad = \limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y))].\end{aligned} \tag{3.7}$$

Since $\beta_0(y) > 0$, by (3.6) and (3.7) we have

$$\limsup_{\alpha}[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \le 0.$$

Since T is h-pseudo-monotone, we have

$$\begin{aligned}&\liminf_{\alpha}[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)]\\ &\qquad \ge \min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x).\end{aligned}$$

Since $\beta_0(y) > 0$, we have

$$\begin{aligned}&\beta_0(y)[\liminf_{\alpha}(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))]\\ &\qquad \ge \beta_0(y)[\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)].\end{aligned}$$

Thus

$$\begin{aligned}&\beta_0(y)[\liminf_{\alpha}(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] + \sum_{p\in E^*} \beta_p(y)Re\langle p, y - x\rangle\\ &\ge \beta_0(y)[\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] + \sum_{p\in E^*} \beta_p(y)Re\langle p, y - x\rangle.\end{aligned} \tag{3.8}$$

For $t = 1$ we also have $\phi(x, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)] + \sum_{p\in E^*} \beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle \le 0$$

for all $\alpha \in \Gamma$. Therefore

$$
\begin{aligned}
0 &\geq \liminf_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)) \\
&\qquad + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle] \\
&\geq \liminf_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] \\
&\qquad + \liminf_{\alpha}[\sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle] \qquad (3.9)\\
&= \beta_0(y)[\liminf_{\alpha}(\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x))] \\
&\qquad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle.
\end{aligned}
$$

Consequently, by (3.8) and (3.9), we have $\phi(x, y) \leq 0$.

Now, the rest of the proof of Step 1 is similar to the proofs in Step 1 of Theorem 2.1 and Theorem 3.1 in [6]. Thus Step 1 is proved.

Step 2. There exists a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$.

Also the same proof of Step 2 of Theorem 2.1 shows that there exists $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$. □

If X is compact, we obtain the following immediate consequence of Theorem 3.1:

Theorem 3.2. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \to \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be h-pseudo-monotone and be upper semicontinuous from co(A) to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that the set $\Sigma = \{y \in X : \sup_{x\in S(y)} [\inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] > 0\}$ is open in X. Then there exists $\hat{y} \in X$ such that* (i) *$\hat{y} \in S(\hat{y})$ and* (ii) *there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Note that if the map $S : X \to 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semicontinuous at y in X, then the set Σ in Theorem 3.1 is always open in X as can be seen in the proof of the following:

Theorem 3.3. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : E \to \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be h-pseudo-monotone and be upper semicontinuous from co(A) to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x\in S(y)}[\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^*. Suppose further that there exist a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that* (i) *$\hat{y} \in S(\hat{y})$ and* (ii) *there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Proof. By virtue of Theorem 3.1, we need only show that the set $\Sigma := \{y \in X : \sup_{x\in S(y)} [\inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] > 0\}$ is open in X.

Now, following the same arguments as in the proofs of Theorem 3.2 in [6] and Theorem 2.3, we can similarly show that the set Σ is open in X. Hence by Theorem 3.1 the conclusion follows. □

If X is compact, we obtain the following immediate consequence of Theorem 3.3:

Theorem 3.4. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \to \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be h-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x\in S(y)}[\inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^*. Then there exists $\hat{y} \in X$ such that* (i) *$\hat{y} \in S(\hat{y})$ and* (ii) *there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

We remark here that in Theorems 3.1-3.4, the condition "$h : E \to \mathbb{R}$ be convex" can be replaced by the condition "$h : X \to \mathbb{R}$ be convex such that $h|_{co(A)}$ is continuous for each $A \in \mathcal{F}(X)$".

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