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DISTRIBUTIONS SUPPORTED IN A HYPERSURFACE AND LOCAL h^p

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ABSTRACT. We give a necessary condition for a distribution with compact support in a hypersurface to be in the local Hardy space $h^p(\mathbf{R}^n)$. We apply this condition to prove a result distinguishing two types of Hardy spaces of distributions on a smooth domain $\Omega \subset \mathbf{R}^n$.

1. INTRODUCTION

The aim of this paper is to solve a problem which arose in the study of various local Hardy spaces of distributions on a smooth bounded domain $\Omega \subset \mathbf{R}^n$. In [CKS], Chang, Krantz and Stein defined the spaces $h^p_r(\Omega)$ and $h^p_z(\Omega)$, $0 < p \leq 1$, (related to the Hardy spaces defined by Miyachi [M] and Jonsson, Sjögren, and Wallin [JSW]) and proceeded to prove regularity results for the Dirichlet and Neumann problems in terms of these spaces. Subsequent work by Chang, Stein and the author ([CDS]) revealed that for certain values of p, namely $p \neq \frac{n}{n+k}$, where $k = 0, 1, 2, \ldots$, these two spaces can be identified. However, it was proved that in the one-dimensional case $n = 1$, the spaces are not equivalent for the values $p = 1, \frac{1}{2}, \frac{1}{3}, \ldots$. Thus it remained to determine whether the two spaces were different for the "critical" values of p for $n > 1$.

The space $h^p_r(\Omega)$ consists of those distributions on Ω which are the restriction to Ω of distributions in $h^p(\mathbf{R}^n)$, the local Hardy spaces defined by Goldberg (see [G].) The space $h^p_z(\Omega)$ is a subspace of $h^p_r(\Omega)$, consisting of those distributions on Ω which are the restriction to Ω of distributions in $h^p(\mathbf{R}^n)$ which also vanish outside $\overline{\Omega}$. Note that two such extensions of the same element of $h^p_z(\Omega)$ differ by a distribution in $h^p(\mathbf{R}^n)$ which is supported on $\partial\Omega$. Thus in order to understand $h^p_z(\Omega)$, it is important to understand the nature of distributions in $h^p(\mathbf{R}^n)$ which are supported in a smooth hypersurface, or (by a choice of local coordinates) in a hyperplane.

Some simple sufficient conditions can be stated for a distribution with compact support in a hyperplane to be in $h^p(\mathbf{R}^n)$ (see for example [S], 5.18, which implies that $f \in h^p(\mathbf{R}^n)$ if its order is strictly less than $\frac{1}{p} - n$). Here we prove a necessary condition, namely that the order of the distribution "in the normal direction" must be strictly less than $n(\frac{1}{p} - 1)$ (Theorem 1). We do this by means of a lower bound on the integral of the maximal function of such a distribution on planes parallel to

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the hyperplane (Lemma 1). It is this lemma that allows us to construct a counterexample showing that for $p = \frac{n}{n+k}$, $k = 0, 1, 2, \dots$, $h^p_r(\Omega) \neq h^p_z(\Omega)$ (Theorem 2).

2. h^p Distributions supported in a hyperplane

For $n \geq 2$, write $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}$, and let $\Pi = \mathbf{R}^{n-1} = \{x_n = 0\}$ be our chosen hyperplane. If f is a distribution in $\mathbf{R}^n$ of order K with compact support in Π, then

$$f(\phi) = \sum_{k=0}^{K} f_k(\phi_k), \tag{1}$$

where each f_k is a distribution of compact support and order $K - k$ in $\mathbf{R}^{n-1}$, and

$$\phi_k(x') = \partial^k_{x_n} \phi(x', x_n)|_{x_n=0}$$

(see [H], Theorem 2.3.5). Let N be the largest integer such that f_N is non-zero. We will call N the *order of f in the normal direction*. Then we have the following:

Theorem 1. *A distribution f in $\mathbf{R}^n$ with compact support in Π is in the local Hardy space $h^p(\mathbf{R}^n)$, $0 < p \leq 1$, only if its order N in the normal direction satisfies*

$$N < n(1/p - 1).$$

Note that for $p = 1$, this agrees with the fact that the only distribution supported in Π which is in h^1 is the zero distribution, since elements of h^1 are functions.

To characterize the spaces $h^p(\mathbf{R}^n)$, we will use a "local grand maximal function" $m(f)$ defined using "normalized bump functions". Such a function φ is smooth ($\mathcal{C}^\infty$), supported in a ball of radius r, and satisfies

$$|\partial^\alpha \varphi| \leq r^{-n-|\alpha|}$$

for all multi-exponents $\alpha = (\alpha_1, \dots, \alpha_n)$ with

$$|\alpha| = \alpha_1 + \ldots + \alpha_n \leq N_p + 1,$$

where N_p is the greatest integer in $n(1/p - 1)$.

For a tempered distribution f, we define

$$m(f)(x) = \sup_{\varphi^x_r} f(\varphi^x_r),$$

where the supremum is taken over all normalized bump functions φ^x_r supported in balls of radii $r \leq 1$ containing x. Then $f \in h^p(\mathbf{R}^n)$ if and only if

$$m(f) \in L^p(\mathbf{R}^n).$$

That this is equivalent to the definition given by Goldberg ([G]) can be seen from the remarks in [S], 5.17.

In view of this characterization, the theorem will follow from the following

Lemma 1. *Let f be a distribution in $\mathbf{R}^n$ with compact support in Π, and let N be the order of f in the normal direction. Then for $0 < p \leq 1$, the local grand maximal function $m(f)$ satisfies*

$$\int_{\mathbf{R}^{n-1}} m(f)^p(x', x_n)dx' \geq C{x_n}^{n-1-(n+N)p}$$

for all sufficiently small $x_n > 0$.

Note that when $N \geq n(1/p-1)$, $n-1-(n+N)p \leq -1$ so the lemma shows that $m(f)^p$ is not locally integrable near $x_n = 0$. This proves the theorem.

Proof of Lemma 1. Write

$$f(\phi) = \sum_{k=0}^{N} f_k(\phi_k),$$

as in equation 1, with f_N nonzero. We may assume the support of f lies in the interior of the unit cube $Q = [0,1]^{n-1}$. Since f_N is nonzero, there is a smooth function ϕ such that $f_N(\phi) = \epsilon > 0$. We may also take the support of ϕ to lie inside Q. By changing ϵ if necessary, we normalize ϕ so that

$$\sup_{|\gamma| \leq N_p+1} |\partial_{x'}^{\gamma} \phi| \leq 1.$$

Consider the dyadic cubes Q_α^j, with $j = 0,1,2,\ldots$ and $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, $0 \leq \alpha_i \leq 2^j - 1$, where the side length of Q_α^j is 2^{-j} and its closest corner to the origin lies at the point $(\alpha_1/2^j, \ldots, \alpha_{n-1}/2^j)$. Thus $Q = Q^0_{(0,\ldots,0)}$.

For each j, we associate with the cubes Q_α^j a partition of unity $\{\eta_\alpha^j\}$, where each η_α^j is a smooth function supported in the double of the cube Q_α^j, i.e. the cube $\widetilde{Q_\alpha^j}$ with the same center but double the side length, and such that

$$\sum_\alpha \eta_\alpha^j = 1$$

on Q. Furthermore, we may assume that, for every j and α,

$$|\partial_{x'}^{\gamma}(\eta_\alpha^j)| \leq 2^{j|\gamma|}$$

for all derivatives of order $|\gamma| \leq N_p + 1$.

Set

$$\phi_\alpha^j = A_{n,p} 2^{j(n-1)} \phi \, \eta_\alpha^j,$$

where the constant $A_{n,p}$ is chosen so that

$$|\partial_{x'}^{\gamma}(\phi_\alpha^j)| \leq 2^{j(n-1+|\gamma|)}$$

for all $|\gamma| \leq N_p + 1$. Consider a smooth one-variable function $\psi(t)$ with

$$\psi(t) = \frac{1}{N!} t^N$$

for $t \in [-1/2, 1/2]$, $\psi(t) = 0$ for $|t| > 1$, and $|d_t^m \psi(t)| \leq 1$ for $m \leq N_p + 1$. Let

$$\psi_j(t) = 2^j \psi(2^j t),$$

and

$$\Phi_\alpha^j(x', x_n) = C_{n,p} \, \phi_\alpha^j(x') \, \psi_j(x_n),$$

where again we chose the constant $C_{n,p}$ so that

$$|\partial^\beta(\Phi_\alpha^j)| \leq (2^{-j}\sqrt{n})^{-(n+|\beta|)}$$

for all $|\beta| \leq N_p + 1$. Since Φ_α^j is supported in the cube $\widetilde{Q_\alpha^j} \times [-2^{-j}, 2^{-j}]$ of side length 2^{-j+1}, hence in a ball of radius $2^{-j}\sqrt{n}$, this makes Φ_α^j a normalized bump function.

Now note that

$$f(\Phi_\alpha^j) = f_N((\Phi_\alpha^j)_N) = C_{n,p} 2^{j(N+1)} f_N(\phi_\alpha^j).$$

Thus for $x \in \widetilde{Q_\alpha^j} \times [-2^{-j}, 2^{-j}]$,

$$m(f) \geq C_{n,p} 2^{j(N+1)} |f_N(\phi_\alpha^j)|.$$

Recall that, for every j,

$$\sum_\alpha f_N(\phi_\alpha^j) = A_{n,p} 2^{j(n-1)} f_N(\phi) = A_{n,p} 2^{j(n-1)} \epsilon.$$

Therefore for $2^{-(j+1)} \leq x_n \leq 2^{-j}$,

$$\begin{aligned}
\int_{\mathbf{R}^{n-1}} m(f)^p(x', x_n) dx' &\geq C_{n,p} \sum_\alpha \left(2^{j(N+1)} |f_N(\phi_\alpha^j)|\right)^p |Q_\alpha^j| \\
&\geq C_{n,p} 2^{j(N+1)p} 2^{-j(n-1)} \left|\sum_\alpha f_N(\phi_\alpha^j)\right|^p \\
&= C'_{n,p} 2^{j(N+1)p} 2^{-j(n-1)} 2^{j(n-1)p} \epsilon^p \\
&= C'_{n,p} \epsilon^p 2^{-j[n-1-(n+N)p]} \\
&\geq C_{n,p,N,\epsilon}\, {x_n}^{n-1-(n+N)p}.
\end{aligned}$$

This proves the lemma. □

We will now give an example to show that Theorem 1 does not hold with N replaced by K, the total order of the distribution.

Example 1. For $0 < p < 1$, there exists a distribution $f \in h^p(\mathbf{R}^n)$ with compact support contained in a hyperplane, such that the order K of f satisfies

$$K \geq N_p,$$

where N_p is the greatest integer in $n(1/p - 1)$.

Take the hyperplane $\Pi = \mathbf{R}^{n-1}$ as above. In $\mathbf{R}^{n-1}$, consider cubes Q_j with centers c_j and side lengths δ_j, such that the double cubes $2Q_j$ are pairwise disjoint and $\sum \delta_j < \infty$. Take functions a_j of the $n-1$ variables $x' = x_1, \ldots, x_{n-1}$ (respectively) supported in Q_j and satisfying

$$\|a_j\|_\infty \leq {\delta_j}^{1-n/p},$$

$$\int a_j(x')(x')^\alpha dx' = 0$$

for all $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ with $|\alpha| \leq N_p$, and

$$\int a_j(x')(x_1 - (c_j)_1)^{N_p+1} dx' = C_n {\delta_j}^{n-n/p+N_p+1}$$

for some $C_n > 0$.

Define linear functionals τ_j on $\mathcal{S}(\mathbf{R}^n)$ by

$$\tau_j(\varphi) = \int_{\mathbf{R}^{n-1}} a_j(x')\varphi(x', 0) dx'$$

for every $\varphi \in \mathcal{S}(\mathbf{R}^n)$. By expanding $\varphi(x', 0)$ in a Taylor expansion around c_j, one can see that

$$|\tau_j(\varphi)| \leq C\|\varphi\|_{\mathcal{C}^{N_p+1}} {\delta_j}^{n-n/p+N_p+1} \leq C_{n,p}\|\varphi\|_{\mathcal{C}^{N_p+1}}$$

since $\delta_j \to 0$ and $n-n/p+N_p+1>0$. This shows the τ_j are continuous on $\mathcal{S}(\mathbf{R}^n)$, and we can define $f \in \mathcal{S}'(\mathbf{R}^n)$ by

$$f = \sum \lambda_j \tau_j,$$

where the sequence λ_j is chosen to satisfy $\sum {\lambda_j}^p < \infty$ and $\lambda_j/\delta_j \to \infty$ as $j \to \infty$ (for example take $\delta_j = 2^{-j}$, $\lambda_j = j^{-1-1/p}$).

To see that the order of f is at least N_p, take a sequence of smooth functions $\varphi_j \in \mathcal{S}(\mathbf{R}^n)$ with $\varphi_j(x',0)$ supported in $2Q_j$,

$$|\partial^\alpha \varphi_j| \leq 1$$

for $|\alpha| \leq N_p - 1$, and

$$\varphi_j(x',0) = \frac{1}{(N_p+1)!}(x_1-(c_j)_1)^{N_p+1}\ {\delta_j}^{-2}$$

on Q_j. Thus

$$f(\varphi_j) = \lambda_j \int_{Q_j} a_j(x')\varphi_j(x',0)dx' = C_{n,p}\lambda_j\ {\delta_j}^{n-n/p+N_p+1-2} \to \infty$$

as $j \to \infty$, since $n - n/p + N_p + 1 - 2 \leq -1$.

Finally, to see that $f \in h^p(\mathbf{R}^n)$, consider the local grand maximal function $m(f)$. It suffices to show $\|m(\tau_j)\|_{L^p(\mathbf{R}^n)} \leq C$ uniformly in j. Take $x \in \mathbf{R}^n$. Note that if φ_t^x is a normalized bump function supported in a ball of radius t containing x, then $\tau_j(\varphi_t^x) \neq 0$ only if $t \geq x_n/2$, and $t\varphi_t^x(y',0)$ is a normalized bump function in $n-1$ variables. Thus

$$m(\tau_j)(x) \leq \frac{2}{x_n} m_{\mathbf{R}^{n-1}}(a_j)(x'),$$

where $m_{\mathbf{R}^{n-1}}$ denotes the local grand maximal function on $\mathbf{R}^{n-1}$. Recalling that $m_{\mathbf{R}^{n-1}}$ is bounded on $L^2(\mathbf{R}^{n-1})$, we get that

$$\begin{aligned}
\int_{2Q_j\times[-\delta_j,\delta_j]} m(\tau_j)^p(x)dx &\leq C\int_{-\delta_j}^{\delta_j} {x_n}^{-p}\int_{2Q_j} m_{\mathbf{R}^{n-1}}(a_j)^p(x')dx' \\
&\leq C{\delta_j}^{1-p}\|m(a_j)\|^p_{L^2(\mathbf{R}^{n-1})}|2Q_j|^{1-p/2} \\
&\leq C{\delta_j}^{1-p}\|a_j\|^p_{L^2(\mathbf{R}^{n-1})}|2Q_j|^{1-p/2} \\
&\leq C{\delta_j}^{1-p}{\delta_j}^{p-n}|Q_j| \\
&= C.
\end{aligned}$$

Note that here we used the fact that $p < 1$.

For $x \notin 2Q_j \times [-\delta_j, \delta_j]$, if φ_t^x is a normalized bump function supported in a ball of radius t containing x, then as above

$$\begin{aligned}
|\tau_j(\varphi_t^x)| &\leq C\|\varphi_t^x\|_{\mathcal{C}^{N_p+1}}{\delta_j}^{n-n/p+N_p+1} \\
&\leq Ct^{-n-(N_p+1)}{\delta_j}^{n-n/p+N_p+1} \\
&\leq C|x-(c_j,0)|^{-n-(N_p+1)}{\delta_j}^{n-n/p+N_p+1},
\end{aligned}$$

so

$$\begin{aligned}
&\int_{x\notin 2Q_j\times[-\delta_j,\delta_j]} m(\tau_j)^p(x)dx \\
&\leq \quad C{\delta_j}^{np-n+(N_p+1)p}\int_{|x-(c_j,0)|\geq\delta_j}|x-(c_j,0)|^{-np-(N_p+1)p}dx \\
&= \quad C.
\end{aligned}$$

This completes the example.

3. The spaces $h_r^p(\Omega)$ and $h_z^p(\Omega)$

Let Ω be a bounded domain in $\mathbf{R}^n$, with smooth boundary. Let us recall the definitions of the spaces $h_r^p(\Omega)$ and $h_z^p(\Omega)$, given in [CKS]:

$$h_r^p(\Omega)=\{f\in\mathcal{D}'(\Omega):\exists F\in h^p(\mathbf{R}^n),F|_\Omega=f\}$$

and

$$h_z^p(\Omega)=\{f\in\mathcal{D}'(\Omega):\exists F\in h^p(\mathbf{R}^n),F|_\Omega=f,F|_{\mathbf{R}^n\setminus\overline{\Omega}}=0\}.$$

Clearly $h_z^p(\Omega)$ is a subspace of $h_r^p(\Omega)$. It was shown in [CDS] that in fact $h_r^p(\Omega)=h_z^p(\Omega)$ for all values of p with $\frac{n}{n+j+1}<p<\frac{n}{n+j}$, $j=0,1,\ldots.$ We will now show that this is not true when $n(1/p-1)$ is an integer.

Theorem 2. *For* $p=1,\frac{n}{n+1},\frac{n}{n+2},\ldots,$

$$h_r^p(\Omega)\neq h_z^p(\Omega).$$

Proof. Assume $0\in\partial\Omega$, and let $U\subset V$ be open neighborhoods of 0. We may take these so that there is a smooth choice of coordinates in V in which $V=B(0,2)$, $U=B(0,1)$ and $\Omega\cap V=\mathbf{R}_+^n\cap B(0,2)$. Here $\mathbf{R}_+^n$ denotes the open upper half-space $\{x:x_n>0\}$.

We will construct a distribution $f\in h_r^p(\Omega)$, supported in U, such that there is no distribution $G\in h^p(\mathbf{R}^n)$ satisfying

$$G|_\Omega=f$$

and

$$G|_{\mathbf{R}^n\setminus\overline{\Omega}}=0.$$

Note that the space of distributions in $h_r^p(\Omega)$ which are supported in U does not depend on the choice of coordinates.

Let us construct f as follows. For $j=1,2,\ldots$, consider the cylinder

$$S_j=B^{n-1}(0,2^{-(j+1)})\times[2^{-(j+1)},2^{-j}],$$

where $B^{n-1}(0,r)$ denotes the $(n-1)$-dimensional ball of radius r in the variables $x'=x_1,\ldots,x_{n-1}$. Note that S_j is contained in the (much larger) cube

$$Q_j=[-2^{-(j+1)},2^{-(j+1)}]^{n-1}\times[2^{-(j+1)},3\cdot 2^{-(j+1)}]$$

of volume $|Q_j|=2^{-jn}$. Set

$$a_j=|Q_j|^{-1/p}\chi_{S_j}.$$

If $\lambda_j=j^{-(1+1/p)}$, then $\sum{\lambda_j}^p<\infty$ so by the atomic decomposition for h_r^p (see [CKS], Theorem 2.7,) the distribution $f\in\mathcal{D}'(\Omega)$ defined by

$$f=\sum_{j=1}^\infty\lambda_j a_j$$

is an element of $h^p_r(\Omega)$. Furthermore, f is supported in $\Gamma \cap B(0,1)$, where Γ is the cone

$$\{x \in \mathbf{R}^n : |x'| \le x_n < 1\}.$$

In order to extend f to a distribution $F \in \mathcal{S}'(\mathbf{R}^n)$, we define distributions τ_j by

$$\tau_j(\varphi) = \int a_j(x) \left[\varphi(x) - \sum_{|\alpha| \le N_p - 1} \frac{1}{\alpha!} \partial^\alpha \varphi(0) x^\alpha \right] dx$$

for every $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Since

$$\begin{aligned} |\tau_j(\varphi)| &= \left| \int a_j(x) \left[\varphi(x) - \sum_{|\alpha| \le N_p - 1} \frac{1}{\alpha!} \partial^\alpha \varphi(0) x^\alpha \right] dx \right| \\ &\le C 2^{jn/p} \|\varphi\|_{\mathcal{C}^{N_p}} \int_{Q_j} |x|^{N_p} dx \\ &\le C 2^{j(n/p - N_p - n)} \|\varphi\|_{\mathcal{C}^{N_p}} \\ &= C \|\varphi\|_{\mathcal{C}^{N_p}}, \end{aligned}$$

and $\sum \lambda_j < \infty$, we can define a distribution $F \in \mathcal{S}'(\mathbf{R}^n)$ by

$$F = \sum_{j=1}^{\infty} \lambda_j \tau_j.$$

This distribution is supported in $B(0,1) \cap \overline{\Omega}$ and satisfies

$$F|_\Omega = \sum_{j=1}^{\infty} \lambda_j \tau_j|_\Omega = \sum_{j=1}^{\infty} \lambda_j a_j = f.$$

Now suppose there was a distribution $G \in \mathcal{S}'(\mathbf{R}^n)$ with G supported on $\overline{\Omega}$ and $G|_\Omega = f$. We want to show that G cannot be an element of $h^p(\mathbf{R}^n)$. Since $h^p(\mathbf{R}^n)$ is closed under multiplication by smooth functions of compact support, we may assume G is supported in $V = B(0,2)$.

Consider the distribution $G - F \in \mathcal{S}'(\mathbf{R}^n)$. Since both G and F are supported on $\overline{\Omega}$, and $G|_\Omega = f = F|_\Omega$, we must have that $G - F$ is supported on $\partial\Omega$, and in particular on $B^{n-1}(0,2) = B(0,2) \cap \partial \mathbf{R}^n_+$. Thus $G - F$ is of the form of equation (1).

Let N denote the order of $G - F$ in the normal direction.

Claim 1. If $N < N_p = n(1/p - 1)$, then the distribution G does not belong to $h^p(\mathbf{R}^n)$, since its local grand maximal function satisfies

$$m(G)(x) \ge C|x|^{-n/p} \left(\log \frac{1}{|x|} \right)^{-1/p}$$

for $x \in \Gamma$, $x \ne 0$.

To get the lower bound for $m(G)$, we will construct a special test function ϕ. First take a smooth function ψ of one variable, supported in $(-1,2)$, with $\psi(y) \ge 0$ for $y \ge 0$,

$$\psi(y) = \frac{y^{N_p}}{N_p!} \quad \text{for } y \in [0,1]$$

and

$$\left| \frac{d^k}{dy^k} (\psi) \right| \le 1$$

for $k \leq N_p + 1$. Now take φ to be a smooth function of $n-1$ variables, supported in $B^{n-1}(0,2)$, with $\varphi \geq 0$, $\varphi = 1$ on $B^{n-1}(0,1)$ and

$$|\partial^\alpha(\varphi)| \leq 1$$

for $|\alpha| \leq N_p + 1$. Defining ϕ by

$$\phi(x) = \varphi(x')\psi(x_n),$$

we see that ϕ is supported in $B(0,3)$,

$$\phi(x) = \frac{{x_n}^{N_p}}{N_p!}$$

for $x \in B^{n-1}(0,1) \times [0,1]$, and

$$|\partial^\alpha(\phi)| \leq 1$$

for $|\alpha| \leq N_p + 1$.

Suppose $x \in \Gamma$, $x \neq 0$. Let $x^* = (x', 0)$, and define

$$\phi_x(y) = 3^{-n-N_p}|x|^{-n}\phi\left(\frac{y - x^*}{|x|}\right).$$

Then ϕ_x is supported in the ball $B(x^*, 3|x|)$, with $\phi_x \geq 0$ on $\mathbf{R}^n_+$ and

$$\phi_x(y) = \frac{{y_n}^{N_p}}{N_p!(3|x|)^{n+N_p}}$$

for $y \in S_x = B^{n-1}(x', |x|) \times [0, |x|]$. Furthermore

$$|\partial^\alpha(\phi_x)| \leq (3|x|)^{-n-|\alpha|}$$

for $|\alpha| \leq N_p + 1$. Thus ϕ_x is a normalized bump function.

Note that, for $k < N_p$,

$$\frac{\partial^k}{\partial {y_n}^k}\phi_x(y', y_n)|_{y_n=0} = 3^{-n-N_p}|x|^{-n-k}\varphi\left(\frac{y' - x'}{|x|}\right)\frac{\partial^k \psi}{\partial {y_n}^k}(0) = 0,$$

so by equation (1), if $N < N_p$,

$$(G - F)(\phi_x) = 0.$$

Moreover, $\tau_j(\phi_x) = \int a_j \phi_x$ for all j. Thus

$$\begin{aligned} G(\phi_x) &= F(\phi_x) \\ &\geq \sum_{S_j \subset S_x} \lambda_j \int_{S_j} a_j(y)\phi_x(y)dy \\ &= \frac{1}{N_p!(3|x|)^{n+N_p}} \sum_{S_j \subset S_x} \lambda_j \int_{S_j} a_j(y){y_n}^{N_p}dy \\ &= C_{N_p}(3|x|)^{-n-N_p} \sum_{S_j \subset S_x} \lambda_j 2^{j(n/p - N_p - n)} \\ &\geq C'_{N_p}|x|^{-n-N_p} \sum_{j \geq -\log_2 x_n + 2} j^{-1/p-1} \\ &\approx |x|^{-n/p}\left(\log\frac{1}{|x|}\right)^{-1/p}. \end{aligned}$$

Here we have used the fact that for $x \in \Gamma$, if $2^{-j} \leq (\sqrt{2}-1)x_n$, then

$$S_j = B^{n-1}(0, 2^{-(j+1)}) \times [2^{-(j+1)}, 2^{-j}] \subset S_x = B^{n-1}(x', |x|) \times [0, |x|].$$

Claim 2. If $N \geq N_p$, then the distribution G does not belong to $h^p(\mathbf{R}^n)$, since its local grand maximal function satisfies

$$\int_{\mathbf{R}^{n-1}} m(G)^p(x', x_n)dx' \geq C{x_n}^{-1}$$

for all sufficiently small $x_n > 0$.

By Lemma 1, with $N \geq N_p$, the distribution $G - F$ satisfies

$$\int_{\mathbf{R}^{n-1}} m(G-F)^p(x', x_n)dx' \geq C{x_n}^{n-1-(n+N_p)p} = C{x_n}^{-1}$$

for all sufficiently small $x_n > 0$. Since

$$\int_{\mathbf{R}^{n-1}} m(G)^p(x', x_n)dx' \geq \int_{\mathbf{R}^{n-1}} m(G-F)^p(x', x_n)dx' - \int_{\mathbf{R}^{n-1}} m(F)^p(x', x_n)dx',$$

it suffices to prove that

$$\int_{\mathbf{R}^{n-1}} m(F)^p(x', x_n)dx' = o({x_n}^{-1})$$

as $x_n \to 0$. In fact, we will show that

$$\int_{\mathbf{R}^{n-1}} m(F)^p(x', x_n)dx' \leq C(x_n \log(1/x_n))^{-1}$$

for sufficiently small $x_n > 0$.

To prove the upper bound on the maximal function of F, take $x \in \mathbf{R}^n_+$, and let φ^x_t be a bump function supported in a ball of radius $t \leq 1$ containing x. Since F is supported in $B(0,1)$, we can assume $|x| < 3$.

If $t \geq |x|/4$, write

$$F(\varphi^x_t) = \sum_{j=1}^{\infty} \lambda_j \tau_j(\varphi^x_t).$$

Note that if $j < -\log_2(t) - 4$, then $2^{-(j+1)} > 8t > 2t + x_n$, so $\varphi^x_t = 0$ on S_j and

$$\begin{aligned}
\sum_{j<-\log_2(t)-4} |\lambda_j \tau_j(\varphi^x_t)| &= \sum_{j<-\log_2(t)-4} \left| \lambda_j 2^{jn/p} \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} \partial^\alpha \varphi^x_t(0) \int_{S_j} x^\alpha dx \right| \\
&\leq C \sum_{j<-\log_2(t)-4} \lambda_j \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} t^{-n-|\alpha|} 2^{j(n/p-n-|\alpha|)} \\
&\leq C \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} t^{-n-|\alpha|} \sum_{j<-\log_2(t)-4} \frac{2^{j(n/p-n-|\alpha|)}}{j^{1+1/p}} \\
&\leq C \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} t^{-n-|\alpha|} \frac{t^{-(n/p-n-|\alpha|)}}{\log(1/t)^{1/p}} \\
&\leq C t^{-n/p} (\log(1/t))^{-1/p}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{j\geq -\log_2(t)-4} |\lambda_j \tau_j(\varphi_t^x)| &\leq C \sum_{j\geq -\log_2(t)-4} \frac{1}{j^{1+1/p}} \|\varphi_t^x\|_{\mathcal{C}^{N_p}} \\ &\leq Ct^{-n-N_p} \sum_{j\geq -\log_2(t)-4} \frac{1}{j^{1+1/p}} \\ &\leq Ct^{-n/p}(\log(1/t))^{-1/p}.\end{aligned}$$

Thus

$$\begin{aligned}|F(\varphi_t^x)| &\leq Ct^{-n/p}(\log(1/t))^{-1/p} \\ &\leq C|x|^{-n/p}(\log(1/|x|))^{-1/p},\end{aligned}$$

since $(|x|/t)^{n/p}(\log|x|/\log t)^{1/p}$ remains bounded when $|x| << t$.

If $t < |x|/4$, then φ_t^x vanishes in a neighborhood of 0, and $S_j \cap \operatorname{supp}(\varphi_t^x) \neq \emptyset$ only if $j \leq 3/2 - \log_2|x|$, so

$$\begin{aligned}|F(\varphi_t^x)| &\leq C \sum_{j\leq -C\log|x|} \lambda_j \left| \int a_j \varphi_t^x \right| \\ &\leq C \sum_{j\leq -C\log|x|} \frac{1}{j^{1+1/p}} 2^{jn/p} \|\varphi_t^x\|_{L^1} \\ &\leq C|x|^{-n/p}(\log(1/|x|))^{-1/p}.\end{aligned}$$

In both cases we are assuming of course that $|x|$ is sufficiently small, i.e. $|x| \leq a << 1$, so that $\log(1/|x|)$ is bounded below. Otherwise we just have

$$|F(\varphi_t^x)| \leq C|x|^{-n/p} \sum \lambda_j \leq C_a.$$

Now integrating in the first $n-1$ variables, we have

$$\begin{aligned}&\int_{\mathbf{R}^{n-1}} m(F)^p(x', x_n)dx' = \\ &\qquad \int_{|x'|\leq x_n} m(F)^p(x', x_n)dx' + \int_{x_n\leq|x'|\leq a} m(F)^p(x', x_n)dx' \\ &\qquad + \int_{a\leq|x'|\leq 3} m(F)^p(x', x_n)dx' \\ &\leq C \int_{|x'|\leq x_n} {x_n}^{-n}(\log(1/x_n))^{-1}dx' \\ &\qquad + C\int_{x_n\leq|x'|\leq a} |x'|^{-n}(\log(1/|x'|))^{-1}dx' + C \\ &\leq C{x_n}^{-1}(\log(1/x_n))^{-1} + C\int_{x_n}^{a} \frac{dr}{r^2(-\log r)} + C.\end{aligned}$$

But

$$\begin{aligned}\int_{x_n}^{a} \frac{dr}{r^2(-\log r)} &= \int_{x_n}^{a} \frac{dr}{r^2\log^2 r} + {x_n}^{-1}\left(\log\frac{1}{x_n}\right)^{-1} + C \\ &\leq (-\log a)^{-1} \int_{x_n}^{a} \frac{dr}{r^2(-\log r)} + {x_n}^{-1}(\log(1/x_n))^{-1} + C\end{aligned}$$

so

$$\int_{x_n}^{a} \frac{dr}{r^2(-\log r)} \leq C{x_n}^{-1}(\log(1/x_n))^{-1} + C$$

for a sufficiently small.

This concludes the proof of Theorem 2. □

References

[CDS] D.C. Chang, G. Dafni, and E. M. Stein, *Hardy spaces, BMO, and boundary value problems for the Laplacian on a smooth domain in* $\mathbf{R}^n$, Trans. Amer. Math. Soc., to appear. CMP 97:15

[CKS] D.C. Chang, S. G. Krantz, and E. M. Stein, *H^p Theory on a smooth domain in $\mathbf{R}^N$ and elliptic boundary value problems*, J. Funct. Anal. **114**, No. 2 (1993), 286-347. MR **94j:**46032

[G] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), 27-42. MR **80h:**46052

[H] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Springer-Verlag, Berlin 1983. MR **85g:**35002a

[JSW] A. Jonsson, P. Sjögren, and H. Wallin, *Hardy and Lipschitz spaces on subsets of* $\mathbf{R}^n$, Studia Math. **80**, No. 2 (1984), 141-166. MR **87b:**46022

[M] A. Miyachi, *H^p spaces over open subsets of* $\mathbf{R}^n$, Studia Math. **95**, No. 3 (1990), 205-228. MR **91m:**42022

[S] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, Princeton, New Jersey, 1993. MR **95c:**42002

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