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A NOTE ON HARMONIC FORMS ON COMPLETE MANIFOLDS

LUEN-FAI TAM

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ABSTRACT. In this note, we will prove that under certain conditions, the space of polynomial growth harmonic functions and harmonic forms with a fixed growth rate on manifolds which are asymptotically nonnegatively curved is finite dimensional. This is a partial generalization of the works of Li and Colding-Minicozzi. We will also give an explicit estimate for the dimension in case the manifold is a complete surface of finite total curvature. This is a generalization to harmonic forms of the work of Li and the author.

0. INTRODUCTION

After the work [Y], it was conjectured by Yau that the space of harmonic functions of polynomial growth of degree at most d on a complete noncompact manifold with nonnegative Ricci curvature is finite dimensional. This conjecture has stimulated many interesting works; see [L 2] for details. Finally, the conjecture was proved to be true in [L-T 1] for the case of linear growth, and by Colding and Minicozzi for the general case in a series of papers [C-M 1]–[C-M 6]. In fact, they prove that the the conjecture is also true if we only assume the manifold satisfy volume doubling and that Poincaré inequality holds on the manifold. It is well-known that these two properties are satisfied by manifolds with nonnegative Ricci curvature. In [L 1], Li gives a very simple proof of a more general result. He proves that on a complete manifold M satisfying volume doubling and on which mean value inequality for positive subharmonic functions holds, then the space of harmonic functions of polynomial growth of degree at most d is finite dimensional. Note that if a manifold satisfies volume doubling and that Poincaré inequality holds on the manifold then mean value inequality will hold on such a manifold [G, SC], but the converse is not true [L 1]. In [C-M 7], simple proofs of the main results in their previous works [C-M 1]–[C-M 6] are also given. Sharp estimates on the dimension are obtained under various assumptions; see [C-M 6], [C-M 7] and in particular [L 1]. Using similar methods, results on harmonic sections can also be obtained. For example, in [L 1], Li proves that on a complete noncompact manifold with nonnegative curvature operator, then the space of harmonic forms with polynomial growth of order at most d is finite dimensional, and sharp bound on the dimension is also obtained; see also [C-M 7].

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It is proved in [C-M 7] that if a manifold has Ricci curvature decay at least quadratically to zero near infinity, and satisfies a volume comparison property introduced in [L-T 5] on the whole manifold, then it will satisfy volume doubling and the mean value inequality for positive subharmonic function also holds. One can apply the main theorem in [L 1] to such manifolds. However, if we assume only that the Ricci curvature of M is nonnegative outside a compact set, then M may not satisfy the above mentioned properties. In this direction, it is proved by Wang [W] that if M has nonnegative Ricci curvature outside a compact set with finite first Betti number, then the space of linear growth harmonic function is of finite dimension, and the dimension can be estimated in terms of geometric data. By introducing a similar inner product as in [W], in this note, we will prove that if M is a complete noncompact manifold with nonnegative Ricci curvature outside a compact set, and if each unbounded component of $M \setminus D$, where D is a bounded domain, satisfies a certain kind of volume comparison property, then the space of polynomial growth harmonic functions of degree at most d is again finite dimensional. A sharp estimate can also be obtained. This class of manifolds include manifolds with nonnegative sectional curvature outside a compact set; see [L-T 5]. If we further assume that the curvature operator is nonnegative outside a compact set, then similar results are true for harmonic forms. In fact, the curvature assumptions can be relaxed further as in [W]. We should emphasis that the arguments are more or less the same as in [L 1] together with some observations.

In [L-T 2], spaces of polynomial growth harmonic functions on a complete noncompact surface with finite total curvature have been studied thoroughly. Lower and upper bounds for the dimensions of those spaces are given explicitly. Hence we have the same bounds for spaces of polynomial growth harmonic two forms. In the second part of this note, we will fill the gap and give an explicit upper bound for the polynomial growth harmonic one forms on a complete surface with finite total curvature.

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1. Dimension estimates of harmonic forms

Let M be a complete noncompact manifold of dimension n, and $p \in M$ be a fixed point. Let D be a compact set, and E be an unbounded component of $M \setminus D$. We call E an end of M with respect to D. We say that E satisfies (**VC**) if there exists a constant $\zeta > 0$, such that

$$V_E(r) \leq \zeta V_x(\frac{r}{2}) \tag{\textbf{VC}}$$

for all $x \in \partial B_E(r)$ and for r large enough. Here we use $B_E(r)$ to denote $B_p(r) \cap E$, $\partial B_E(r) = \partial B(r) \cap E$, $V_E(r)$ is the volume of $B_E(r)$ and $A_E(r)$ is the area of $\partial B_E(r)$.

Let $\mathfrak{V}$ be a rank m vector bundle over M with a metric. Let D be a compact set of M, and let $E_1, \ldots, E_L$ be the ends of M with respect to D. For $r > r_0 > 0$, with $B_p(r_0) \supset D$, we define a positive semidefinite symmetric bilinear form S_{r,r_0} on the space of sections $\Gamma(\mathfrak{V})$ of $\mathfrak{V}$ by

$$S_{r,r_0}(u,v) = V_p^{-1}(2r_0)\int_{B_p(2r_0)}\langle u,v\rangle + \sum_{l=1}^{L} V_{E_l}^{-1}(r)\int_{B_{E_l}(r)}\langle u,v\rangle \tag{1.1}$$

for $u,\ v \in \Gamma(\mathfrak{V})$. In practice, S_{r,r_0} is always positive definite, and $(\mathfrak{V}, S_{r,r_0})$ is an inner product space.

Lemma 1.1. *Let M^n be a complete manifold such that $Ric(x) \geq -K(1+r(x))^{-2}$, for some constant K, where $r(x)$ is the distance of x from p. Let $E_1, \ldots, E_L$ be the ends of M with respect to $B_p(1)$. Suppose each E_l satisfies* (**VC**) *. Let $\mathcal{K}$ be a k-dimensional subspace of $\Gamma(\mathfrak{V})$ with the property that*

$$\Delta |u|^2 \geq 0$$

outside $B_p(1)$ and

$$\Delta |u|^2 \geq -a|u|^2$$

in $B_p(1)$ for some $a \geq 0$ for all $u \in \mathcal{K}$. Then there is a constant C depending only on $n, K, \zeta,\ a,\ L,\ m$ such that if $0 < \epsilon < \frac{1}{4}$, and $\delta > 0$, and if $\{u_1, \ldots, u_k\}$ is an orthonormal basis for $\mathcal{K}$ with respect to $S_{(1+\epsilon)r,1}$, then

$$\sum_{i=1}^{k} S_{r,1}(u_i, u_i) \leq C\epsilon^{-n-\delta+1}.$$

for all r large enough.

Proof. Let $E = E_l$ be one of the ends of M with respect to $B_p(1)$. Since E satisfies (VC), there is a constant ζ such that for all $r > 2$ and for all $x \in \partial B_E(r)$,

$$V_E(r) \leq \zeta V_x(\frac{r}{2}). \tag{1.2}$$

By the assumption on M, and volume comparison, see [B-C], there is a constant C_1 and $\nu > 0$ depending only on n and K, such that for all $r' > r > 2$,

$$V_E(r') \leq \left(\frac{r'}{r}\right)^{\nu} V_E(r), \tag{1.3}$$

and

$$rA_E(r) \leq C_1 V_E(r). \tag{1.4}$$

Moreover, given any $\delta > 0$, there is $r_1 > 2$ such that for all $x \in \partial B_E(R)$, with $R > r_1$, and for all $\frac{2}{3}R > r' > r > 0$,

$$V_x(r') \leq \left(\frac{r'}{r}\right)^{n+\delta} V_x(r). \tag{1.5}$$

Let $r > \frac{4}{3}r_1$, and let $\{u_1, \ldots, u_k\}$ be an orthonormal basis for $(\mathcal{K}, S_{(1+\epsilon)r,1})$. Following [L 1], let $F(x) = \sum_{i=1}^{k} |u_i|^2(x)$. Then F is subharmonic in E. For any $0 < \epsilon < \frac{1}{4}$, and for $\frac{3}{4}r < \rho < r$, let $x_0 \in \partial B_E(\rho)$ be such that

$$F(x_0) = \sup_{\partial B_E(\rho)} F.$$

As in [L 1], one can find an orthogonal transformation A of $(\mathcal{K}, S_{(1+\epsilon)r,1})$, such that if $v_i = Au_i$, then $v_i(x_0) = 0$ for $i \geq m+1$. Note that $v_i = \sum_j a_{ji}u_j$ for some $k \times k$ orthogonal matrix (a_{ij}). Hence $F(x) = \sum_{i=1}^{k} |v_i|^2(x)$, and $F(x) \leq \sum_{i=1}^{m} |v_i|^2(x_0)$ for $x \in \partial B_E(\rho)$. Observe that since $\rho > \frac{3}{4}r$ and $\epsilon < \frac{1}{4}$, we have $(1+\epsilon)r - \rho < \frac{1}{2}r < \frac{2}{3}\rho$. Hence by the mean value inequality of [L-S] and the assumption on the Ricci curvature of M, there is a constant $C_2 > 0$ depending only on $n,\ K$ such that

$$
\begin{aligned}
\sup_{\partial B_E(\rho)} F &= F(x_0)\\
&= \sum_{i=1}^m |v_i|^2(x_0)\\
&\le C_2 V_{x_0}^{-1}((1+\epsilon)r-\rho)\sum_{i=1}^m \int_{B_{x_0}((1+\epsilon)r-\rho)} |v_i|^2 \qquad (1.6)\\
&\le C_2 V_{x_0}^{-1}((1+\epsilon)r-\rho)\sum_{i=1}^m \int_{B_E((1+\epsilon)r)} |v_i|^2\\
&\le mC_2 V_{x_0}^{-1}((1+\epsilon)r-\rho)V_E((1+\epsilon)r)
\end{aligned}
$$

where we have used the fact that $\{v_1,\ldots,v_k\}$ is orthonormal with respect to $S_{(1+\epsilon)r,r_0}$, so that

$$
\begin{aligned}
\int_{B_E((1+\epsilon)r)} |v_i|^2 &\le V_E((1+\epsilon)r)S_{(1+\epsilon)r,1}(v_i,v_i)\\
&= V_E((1+\epsilon)r).
\end{aligned}
$$

By (1.5)

$$
V_{x_0}^{-1}((1+\epsilon)r-\rho) \le \left(\frac{\frac{1}{2}r}{(1+\epsilon)r-\rho)}\right)^{n+\delta} V_{x_0}^{-1}(\frac{1}{2}r).
$$

By (1.2), we have

$$
V_{x_0}^{-1}((1+\epsilon)r-\rho) \le \zeta\cdot\left(\frac{\frac{1}{2}r}{(1+\epsilon)r-\rho}\right)^{n+\delta}(V_E(r))^{-1}.
$$

Hence using (1.3), (1.6) implies that

$$
\begin{aligned}
\sup_{\partial B_E(\rho)} F &\le 2^{-n-\delta} mC_2\zeta V_E((1+\epsilon)r)V_E^{-1}(r)(1+\epsilon-\frac{\rho}{r})^{-n-\delta} \qquad (1.7)\\
&\le 2^{-n-\delta}mC_2\zeta(1+\epsilon)^\nu(1+\epsilon-\frac{\rho}{r})^{-n-\delta}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{B_E(r)\setminus B_E(\frac{3}{4}r)} F &= \int_{\frac{3}{4}r}^r \int_{\partial B_E(\rho)} F d\rho\\
&\le \int_{\frac{3}{4}r}^r A_E(\rho)(\sup_{\partial B_E(\rho)} F)d\rho\\
&\le 2^{-n-\delta}mC_2\zeta(1+\epsilon)^\nu \int_{\frac{3}{4}r}^r A_E(\rho)f(\rho)d\rho
\end{aligned}
$$

where $f(\rho) = (1+\epsilon-\rho/r)^{-n-\delta}$. Using (1.4), we have

$$\begin{aligned}\int_{\frac{3}{4}r}^{r} A_E(\rho) f(\rho) d\rho &\le C_1 \int_{\frac{3}{4}r}^{r} \frac{V_E(\rho)}{\rho} f(\rho) d\rho \\ &\le \frac{4C_1}{3}\frac{V_E(r)}{r}\int_{\frac{3}{4}r}^{r} f(\rho)d\rho \\ &\le \frac{4C_1}{3(n+\delta-1)} V_E(r)\epsilon^{-n-\delta+1}.\end{aligned}$$

Hence

$$\int_{B_E(r)\setminus B_E(\frac{3}{4}r)} F \le C_3\epsilon^{-n-\delta+1}V_E(r) \tag{1.8}$$

for some constant C_3 depending only on $n,\ K,\ \zeta,\ m$. Since F is subharmonic in $M\setminus B_p(1)$,

$$\sup_{B_E(\frac{3}{4}r)} F \le \max\{\sup_{\partial B_E(\frac{3}{4}r)} F, \sup_{B_p(1)} F\}. \tag{1.9}$$

By (1.7), take $\rho = \frac{3}{4}r$, we have

$$\sup_{\partial B_E(\frac{3}{4}r)} F \le C_5 \tag{1.10}$$

for some constant C_5 depending only on $n,\ K,\ \zeta,\ m$. Since $\Delta F \ge -aF$ in $B_p(1)$, using the mean value inequality in [L-T 3] and arguing as before, we have

$$\sup_{B_p(1)} F \le C_6, \tag{1.11}$$

where C_6 depends only on $a,\ n,\ K,\ m$. Combining this with (1.9) and (1.10), we have

$$\sup_{B_E(\frac{3}{4}r)} F \le C_7$$

where $C_7 = \max\{C_5, C_6\}$. Hence

$$\int_{B_E(\frac{3}{4}r)} F \le C_7 V_E(r). \tag{1.12}$$

By (1.8) and (1.12), using the fact the $0<\epsilon<\frac{1}{4}$, we have

$$\int_{B_E(r)} F \le C_8\epsilon^{-n-\delta+1}V_E(r). \tag{1.13}$$

The inequality is true for each end E of M with respect to $B_p(1)$. Hence

$$\sum_{l=1}^{L}\sum_{i=1}^{k} V_{E_l}^{-1}(r)\int_{B_{E_l}(r)} |u_i|^2 \le LC_8\epsilon^{-n-\delta+1}. \tag{1.14}$$

On the other hand, if r is large enough so that all bounded components of $M\setminus B_p(1)$ are contained in $B_p(\frac{3}{4}r)$, we have

$$\begin{aligned}\sup_{B_p(2)} F &= \max\{\sup_{B_p(2)\setminus B_p(1)} F, \sup_{B_p(1)} F\} \\ &\le \max\{\sup_{B_p(\frac{3}{4}r)\setminus B_p(1)} F, \sup_{B_p(1)} F\} \\ &\le C_7.\end{aligned}$$

Hence

$$V_p^{-1}(2)\sum_{i=1}^k\int_{B_p(2)}|u_i|^2\le C_7.$$

Combining this with (1.14), the result follows. □

Suppose that the Ricci curvature of M^n satisfies $Ric(x)\ge-\lambda(r(x))$ where $\lambda:[0,\infty)\to[0,\infty)$ is a nonincreasing function such that $\int_0^\infty r^{n-1}\lambda(r)dr<\infty$. In this case M has finitely many ends [L-T 4]; that is, the number of unbounded components of $M\setminus D$ is bounded from above uniformly for all compact sets D. Assume each end of M with respect to $B_p(1)$ satisfies (**VC**) . This implies that each end of M with respect to $B_p(r)$ also satisfies (**VC**) with the same constant ζ for all $r>1$. By modifying the proof of Lemma 1.1, we have the following generalization:

Lemma 1.2. *Let M^n be as above with $n\ge 3$. Let $\mathfrak{V}$ be a rank m vector bundle over M with a metric. Let $\mathcal{K}$ be a k-dimensional subspace of $\Gamma(\mathfrak{V})$, with the property that*

$$\Delta|u|^2(x)\ge-a\lambda(r(x))|u|^2(x)$$

for all $u\in\mathcal{K}$. Then there is an $r_0>0$ depending only on n, λ, a, M, and a constant C depending also on the number L of unbounded components of $M\setminus B_p(r_0)$, such that if $0<\epsilon<\frac14$, and $\delta>0$, and if $\{u_1,\dots,u_k\}$ is an orthonormal basis of $(\mathcal{K},S_{(1+\epsilon)r,r_0})$, then

$$\sum_{i=1}^k S_{r,r_0}(u_i,u_i)\le C\epsilon^{-n-\delta+1}$$

for r large enough. Here for any r, S_{r,r_0} is defined as in (1.1) with $D=B_p(r_0)$.

Proof. By the assumption on λ we have $\lambda(r)=o(r^{-n+1})$. As in [L-T 4, p.378], there is a function $g>0$ defined on M such that

$$\Delta g(x)\ge a\lambda(r(x)),\tag{1.15}$$

in the sense of distribution on $M\setminus\{p\}$, and $g(x)\to 0$ as $x\to\infty$. Choose $r_0>$ large enough so that $g\le\frac12$ on $M\setminus B_p(r_0)$. Let $r>2r_0$, and let $\{u_1,\dots,u_k\}$ be an orthonormal basis for $(\mathcal{K},S_{(1+\epsilon)r,r_0})$. Let $F=\sum_{i=1}^k|u_i|^2$ as before. Let E be an end of M with respect to $B_p(r_0)$ and let $b=\sup_{B_E(\frac34 r)}F$. Then in $B_E(\frac34 r)$

$$\begin{aligned}\Delta(b^{-1}F)&\ge-a\lambda\cdot(b^{-1}F)\\&\ge-a\lambda.\end{aligned}\tag{1.16}$$

By (1.15), we have

$$\Delta(g+b^{-1}F)\ge 0$$

in $B_E(\frac34 r)$. Hence

$$\sup_{B_E(\frac34 r)}(g+b^{-1}F)\le\max\{\sup_{\partial B_p(r_0)}(g+b^{-1}F),\sup_{\partial B_p(\frac34 r)\cap E}(g+b^{-1}F)\}.$$

However, $g\le\frac12$ outside $B_p(r_0)$, hence

$$1\le\frac12+\max\{\sup_{\partial B_p(r_0)}b^{-1}F,\sup_{\partial B_p(\frac34 r)\cap E}b^{-1}F\},$$

and so

$$\sup_{B_E(\frac{3}{4}r)} F \le 2\max\{\sup_{\partial B_p(r_0)} F, \sup_{\partial B_p(\frac{3}{4}r)\cap E} F\}. \tag{1.17}$$

Using (1.17), one can proceed as in the proof of Lemma 1.1 to get the result. □

Lemma 1.3. *Let M be a complete noncompact manifold and let $p \in M$ be a fixed point. Let $\mathfrak{V}$ be a vector bundle of rank m over M with a metric. Given $r_0 > 0$, let $E_1, \dots, E_L$ be the unbounded components of $M \setminus B_p(r_0)$. Assume each S_{r,r_0} is an inner product for $r > r_0$, where S_{r,r_0} is defined as in (1.1) with $D = B_p(r_0)$. Let $\mathcal{K}$ be a k-dimensional subspace of $\mathfrak{V}$ with polynomial growth of degree at most d. Then for all $\beta > 1$ and $r_1 > r_0$ there is $r > r_1$ such that if $\{u_1, \dots, u_k\}$ is an orthonormal basis for $(\mathcal{K}, S_{\beta r, r_0})$ then*

$$\sum_{i=1}^{k} S_{r,r_0}(u_i, u_i) \ge k\beta^{-(2d+1)}.$$

Proof. Following exactly as in [L 1], let us denote S_{r,r_0} by S_r, the trace of S_r with respect to $S_{r'}$ by $\mathrm{tr}_{r'} S_r$, and the determinant of S_r with respect to $S_{r'}$ by $\det_{r'} S_r$. Suppose the theorem is not true; then for some $\beta > 1$, and $r_1 > r_0$ such that for all $r > r_1$,

$$\mathrm{tr}_{\beta r} S_r < k\beta^{-(2d+1)}.$$

Since

$$(\det_{\beta r} S_r)^{\frac{1}{k}} \le k^{-1}\mathrm{tr}_{\beta r} S_r,$$

we have

$$\det_{\beta r} S_r \le \beta^{-k(2d+1)}.$$

Let $r = r_1 + 1$ and iterating we have for all integers $t \ge 1$,

$$\det_{\beta^t r} S_r \le \beta^{-tk(2d+1)}.$$

Now choose a fixed orthonormal basis $\{u_1, \dots, u_k\}$ of $\mathcal{K}$, with respect to S_r; then

$$\begin{aligned} S_{\beta^t r}(u_i, u_j) &= V_p^{-1}(2r_0)\int_{B_p(2r_0)} \langle u_i, u_j\rangle + \sum_{l=1}^{L} V_{E_l}^{-1}(\beta^t r)\int_{B_{E_l}(\beta^t r)} \langle u_i, u_j\rangle \\ &\le C_1(1+\beta^t r)^{2d} \end{aligned}$$

for some constant C_1 independent of t. Hence

$$\det_r S_{\beta^t r} \le (C_1)^k k!(1+\beta^t r)^{2dk}.$$

Hence

$$(C_1)^{-k}(k!)^{-1}(1+\beta^t r)^{-2dk} \le \beta^{-tk(2d+1)}.$$

This is impossible if $t \to \infty$. □

Let M^n be a complete noncompact manifold, for $0 \le q \le n$ and for $d > 0$, let

$$\mathcal{H}_d^q(M) = \{u \mid u \text{ is a harmonic } q \text{ form, and } |u|(x) \le C(1+r(x))^d, \text{ for some } C\}.$$

In case $q = 0$, that is, in the case of harmonic functions, we simply write $\mathcal{H}_d(M)$ instead of $\mathcal{H}_d^0(M)$. Using Lemmas 1.1 and 1.3, as in [L 1], we have

Theorem 1.4. *Let M^n be a complete manifold such that $Ric(x) \geq -K(1+r(x))^{-2}$, for some constant K, where $r(x)$ is the distance of x from p. Suppose that each end of M with respect with $B_p(1)$ satisfies* (**VC**) *. Then there is a constant C independent of d such that*

$$\dim \mathcal{H}_d(M) \leq Cd^{n-1}.$$

Proof. Note that if $u \in \mathcal{H}_d(M)$, then $\Delta u^2 \geq 0$. Let $\mathcal{K}$ be a k-dimensional subspace of $\mathcal{H}_d(M)$. We may assume that $d \geq 1$. Let $1 > \delta > 0$, and let $\epsilon = \frac{1}{5d}$. By Lemma 1.1, there is a constant C_1 independent of d and δ, and there is $r_0 > 0$ which may depend on δ such that if $r > r_0$, and if $\{u_1, \ldots, u_k\}$ is an orthonormal basis for $(\mathcal{K}, S_{(1+\epsilon)r,1})$, then

$$\sum_{i=1}^{k} S_{r,1}(u_i, u_i) \leq C_1 \epsilon^{-n-\delta+1}.$$

On the other hand by Lemma 1.3, we can find $r > r_0$ so that if $\{u_1, \ldots, u_k\}$ is an orthonormal basis for $(\mathcal{K}, S_{(1+\epsilon)r,1})$, then

$$\sum_{i=1}^{k} S_{r,1}(u_i, u_i) \geq k(1+\epsilon)^{-(2d+1)}.$$

Hence

$$\begin{aligned} k &\leq C_1(1+\epsilon)^{(2d+1)}\epsilon^{-n-\delta+1} \\ &= C_1(1+\frac{1}{5d})^{2d+1}(5d)^{n+\delta-1} \\ &\leq C_2 d^{n+\delta-1} \end{aligned}$$

for some constant C_2 independent of d and δ. Hence

$$\dim \mathcal{H}_d(M) \leq C_2 d^{n+\delta-1}.$$

Let $\delta \to 0$; the result follows. $\square$

Similarly, we can prove

Theorem 1.5. *Suppose that the Ricci curvature of M^n, $n \geq 3$, satisfies $Ric(x) \geq -\lambda(r(x))$ where $\lambda : [0, \infty) \to [0, \infty)$ is a nonincreasing function such that*

$$\int_0^\infty r^{n-1}\lambda(r)dr < \infty.$$

Suppose also that each end of M with respect to $B_p(1)$ satisfies (**VC**) *. Then*

$$\dim \mathcal{H}_d^1(M) \leq Cd^{n-1}.$$

If we assume that the curvature operator at a point x is no less than $-\lambda(r(x))$, then for all q, we also have

$$\dim \mathcal{H}_d^q(M) \leq Cd^{n-1}.$$

Here C is a constant independent of d.

Proof. By assumptions on the Ricci curvature, we have

$$\Delta|u|^2 \geq -C_1\lambda|u|^2,$$

for all harmonic one forms. If in addition, the assumption on the curvature operator is also satisfied, then the above inequality is also true for any harmonic forms. Using

Lemma 1.2 instead of Lemma 1.1, the proof is similar to the proof of Theorem 1.4. □

Remark 1.6. The following are some examples of manifolds M satisfying the assumptions of Theorem 1.4:

1. M has nonnegative sectional curvature outside a compact set, or more generally, the sectional curvature satisfies $K_M(x) \geq -\lambda(r(x))$, where λ is a nonnegative, nonincreasing function on $[0,\infty)$ such that $\int_0^\infty r\lambda(r)dr < \infty$ [L-T 5].
2. M^n has nonnegative Ricci curvature outside $B_p(1)$, $Ric \geq -\epsilon(n)$, where $\epsilon(n)$ is a small number depending only on n, and if M has two ends, [L-T 5].
3. M had nonnegative Ricci curvature outside a compact set, with finite first Betti number. In this case, even though it is not exactly true that each end of M satisfies (**VC**) , however, it is almost true so that by modifying some arguments of Lemma 1.1, Theorem 1.4 still holds for such manifolds; see [L-T 5, Corollary 6.2] in particular.

2. Harmonic forms on complete surfaces

In Theorem 1.5 we assume that the dimension of the manifold is at least 3. In case M is a surface, and if M satisfies the curvature assumption in the theorem, then M will have finite total curvature; see [Hu]. Hence it would be interesting to understand the dimension of the space of polynomial growth harmonic forms on such a surface. The case of harmonic functions (0-forms), and hence harmonic two forms, have been studied in [L-T 2]. Only the case for harmonic 1-forms remains to be studied. Let M be a complete noncompact surface with finite total curvature, and let $p \in M$ be a fixed point. Then M is conformally equivalent with a compact Riemann surface of genus g with finitely many points deleted [Hu]. Let g be the genus of the surface, and l be the number of points deleted. Let E_i, $1 \leq i \leq l$, be the ends of M, and let

$$\alpha_i = \lim_{r\to\infty} \frac{A(B_p(r)\cap E_i)}{\pi r^2}$$

which exists by [Ha]. Denote the dimension of the space of harmonic polynomials of degree less than or equal to k with zero constant term in $\mathbb{R}^2$ by N_k. If $k < 0$, then $N_k = 0$ by convention.

Lemma 2.1. *Let M be a complete surface of finite total curvature. Let $\mathcal{H}_d^{cl}(M)$ be the space of closed harmonic 1-forms with polynomial growth of degree at most d. Then*

$$\dim \mathcal{H}_d^{cl}(M) \leq \sum_{i=1}^{l} N_{\alpha_i(d+1)} + 2(g+l) - 1.$$

Proof. Since the singular homology class $H_1(M;\mathbb{R})$ is a vector space of dimension $2g+l-1$,

$$\dim \mathcal{H}_d^{cl}(M) \leq 2g + l - 1 + \dim\left(\mathcal{H}_d^{cl}(M) \cap \{\text{exact 1-forms}\}\right). \tag{2.1}$$

Let $u \in \mathcal{H}_d^{cl}(M) \cap \{\text{exact 1-forms}\}$. Then $du = 0$. Since u is harmonic, $d\delta u = 0$, and δu is a constant function. Hence we can define a linear map

$$\Phi : \mathcal{H}_d^{cl}(M) \cap \{\text{exact 1-forms}\} \to \mathbb{R},$$

such that u is mapped into δu. The kernel of this map consists of harmonic 1-forms u on M which is both closed and co-closed with polynomial growth of degree at most d. In fact, u is exact. Hence $u = df$ for some function f. We normalize f so that $f(p) = 0$. In this case, f is uniquely determined. Since $\delta u = 0$, f is harmonic. Now $u \in \mathcal{H}_d^{cl}(M)$ implies

$$|u|(x) \le C(1 + r(x))^d$$

for some C. So

$$|f|(x) \le C(1 + r(x))^{d+1}.$$

By [L-T 2], we have

$$\dim(\ker(\Phi)) \le \sum_{i=1}^{l} N_{\alpha_i(d+1)} + l - 1.$$

So

$$\dim\left(\mathcal{H}_d^{cl}(M) \cap \{\text{exact 1-forms}\}\right) \le \sum_{i=1}^{l} N_{\alpha_i(d+1)} + l. \tag{2.2}$$

From (2.1) and (2.2), the lemma follows. □

Lemma 2.2. *With the same assumptions and notations in Lemma 2.1, we have*

$$\dim\{du|\ u \in \mathcal{H}_d^1(M)\} \le \sum_{i=1}^{l} N_{\alpha_i(d-1)} + l.$$

Proof. Note that

$$\dim\{du|\ u \in \mathcal{H}_d^1(M)\} = \dim\{*du|\ u \in \mathcal{H}_d^1(M)\}.$$

Let $u \in \mathcal{H}_d^1(M)$; then $*du$ is a harmonic function, where $*$ is the usual star operator. For any $R > 0$, let $\phi \ge 0$ be a smooth function with compact support in $B_p(4R)$, such that $\phi = 1$ in $B_p(2R)$, and $|d\phi| \le \frac{2}{R}$. Then

$$\begin{aligned} 0 &= \int_M \langle (d\delta + \delta d)u, \phi^2 u\rangle \\ &= \int_M \left(\langle \delta u, \delta(\phi^2 u)\rangle + \langle du, d(\phi^2 u)\rangle\right) \\ &= \int_M \phi^2\left(|\delta u|^2 + |du|^2\right) - \int_M \delta u \langle d(\phi^2), u\rangle + \int_M \langle du, d(\phi^2)u\rangle, \end{aligned}$$

where δ is the formal adjoint of d. By Schwartz inequality, we have

$$\int_M \phi^2\left(|\delta u|^2 + |du|^2\right) \le 2\int_M |d\phi|^2 |u|^2.$$

Hence

$$\int_{B_p(2R)} |du|^2 \le \frac{4}{R^2}\int_{B_p(4R)} |u|^2.$$

Let $f = *du$; then

$$\int_{B_p(2R)} f^2 \le \frac{4}{R^2}\int_{B_p(4R)} |u|^2.$$

By the assumption on u, and the fact that the volume growth of M is at most quadratic, see [Ha], there is a constant C_1, such that

$$\int_{B_p(2R)} f^2 \leq C_1(1 + R^{2d}). \tag{2.3}$$

Let E_i be an end. We may assume that E_i is conformally equivalent to the exterior of the unit disk in the complex plane. We may assume that the metric is extended conformally and smoothly into the unit disk. Let o be the origin; by (2.3), we conclude that

$$\int_{B_o(2R)} f^2 \leq C_1(1 + R^{2d}).$$

Since f is harmonic outside the Euclidean unit disk, using cut off function, it is not hard to prove

$$\int_{B_o(R)} |\nabla f|^2 \leq C_2(1 + R^{2d-2}) \tag{2.4}$$

for some constant C_2 which is independent of R. Let R_0 be the largest ρ such that the Euclidean disk with center at o and radius ρ lies inside $B_o(R)$. In this case, there is a point $x \in \partial B_o(R)$, such that the Euclidean distance of x from o is R_0. By [L-T 2], for all $\epsilon > 0$,

$$R \leq (R_0)^{\alpha_i + \epsilon}$$

if R is large enough. Hence by (2.4),

$$\int_{\tilde{B}_o(R_0)} |\nabla_0 f|^2 \leq C_4 \left(1 + R_0^{2(d-1)(\alpha_i+\epsilon)}\right) \tag{2.5}$$

for some constant C_4, where $\tilde{B}_o(r)$ is the Euclidean disk of radius r with center at o, and ∇_0 is the Euclidean gradient of f. By the mean value inequality in $\mathbb{R}^2$, if $r(x)$ is large then

$$|\nabla_0 f(x)| \leq C_5 \left(r_0^{-1}(x) + r_0(x)^{(d-1)(\alpha_i+\epsilon)-1}\right)$$

where $r_0(x)$ is the Euclidean distance of x from o. By the method in [L-T 2], we conclude that

$$f(x) = f_i(x) + f_i^*(x) + \beta_i \log r_0(x)$$

near infinity of E_i, where f_i is a harmonic polynomial on $\mathbb{R}^2$ with degree not greater than $(d-1)(\alpha_i + \epsilon)$ with zero constant term, β_i is a constant, and f_i^* is a bounded harmonic functions. As in [L-T 2], we can conclude that

$$\dim\{du |\ u \in \mathcal{H}^1_d(M)\} \leq \sum_{i=1}^{l} N_{\alpha_i(d-1)} + l.$$

□

Combining Lemmas 2.1 and 2.2, we have:

Theorem 2.3. *Let M be a complete surface with finite total curvature. Using the notations as in Lemma 2.1, we have*

$$\dim \mathcal{H}_d^1(M) \le \sum_{i=1}^{l} \left(N_{\alpha_i(d+1)} + N_{\alpha_i(d-1)}\right) + 2g + 3l - 1.$$

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Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: `lftam@math.cuhk.edu.hk`