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A THEOREM ON ITERATIONS OF POLYNOMIAL MAPS IN SEVERAL VARIABLES

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ABSTRACT. We establish a polynomial version of a theorem obtained by Enflo, Gurarii, Lomonosov and Lyubich for linear operators. As a consequence, we also derive a polynomial version of a result due to Pták.

1. INTRODUCTION

Throughout this paper $\mathbf{K}$ denotes either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers, $\mathbf{N}$ denotes the set of non-negative integers and $\mathbf{N}^* = \mathbf{N} - \{0\}$. We shall always assume $\mathbf{K}^n$ ($n \in \mathbf{N}^*$) endowed with its euclidean norm, and if $P : \mathbf{K}^n \to \mathbf{K}^n$ is a homogeneous polynomial map we consider

$$\|P\| = \sup_{\|x\| \leq 1} \|Px\|.$$

If X and Y are sets, then the relation "$X \supset Y$" means that X contains Y *properly* and the relation "$X \supseteq Y$" means that X contains or is equal to Y. Moreover, we denote by $X - Y$ the difference set $\{x \in X; x \notin Y\}$.

Pták [4] established the following result:

If $T : \mathbf{K}^n \to \mathbf{K}^n$ is a linear operator which satisfies $1 = \|T\| = ... = \|T^n\|$, then $\|T^j\| = 1$ for all $j \geq 1$.

Later, Enflo, Gurarii, Lomonosov and Lyubich [1] obtained the following pointwise version of Pták's result :

There is a constant $C = C_{\mathbf{K}}(n) \in \mathbf{N}^*$ such that for any linear operator $T : \mathbf{K}^n \to \mathbf{K}^n$ and any $x \in \mathbf{K}^n$, the relation $1 = \|x\| = \|Tx\| = ... = \|T^C x\|$ implies $\|T^j x\| = 1$ for all $j \geq 1$.

The goal of the present paper is to obtain polynomial versions of the above mentioned theorems. Our main result is the following:

Theorem 1. *Let n and d be positive integers. Then there is a constant $M = M_{\mathbf{K}}(n, d) \in \mathbf{N}^*$ such that for any homogeneous polynomial map $P : \mathbf{K}^n \to \mathbf{K}^n$ of degree d and any $x \in \mathbf{K}^n$, the relation*

$$1 = \|x\| = \|Px\| = ... = \|P^M x\| \tag{1}$$

implies

$$\|P^j x\| = 1 \text{ for all } j \geq 1. \tag{2}$$

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As an immediate consequence of this theorem, we have the following

Theorem 2. *Let n and d be positive integers. Then there is a constant $N = N_{\mathbf{K}}(n,d) \in \mathbf{N}^*$ such that for any homogeneous polynomial map $P : \mathbf{K}^n \to \mathbf{K}^n$ of degree d, the relation $1 = \|P\| = ... = \|P^N\|$ implies $\|P^j\| = 1$ for all $j \geq 1$.*

In the next section we prove theorem 1 modulo two lemmas. These lemmas will be proved in §3. Finally, in §4 we show that the argument used to prove theorem 1 can also be applied to establish other results of similar type.

2. Proof of theorem 1

Without loss of generality, we may assume $\mathbf{K} = \mathbf{R}$. Put

$$\alpha_j = 2^j d^{1+2+...+j} \qquad (j \geq 1).$$

We consider a set B of the form

$$B = \{b_{i_0,...,i_r}; r \in \mathbf{N} \text{ and } i_0, ..., i_r \in \mathbf{N}^*\},$$

where the $b_{i_0,...,i_r}$'s are distinct objects. The elements of B will be called *pieces*. We define a relation "$>$" on B by

$$b_{i_0,...,i_r} > b_{j_0,...,j_s} \quad \text{if and only if} \quad s = r+1 \text{ and } j_0 = i_0, ..., j_r = i_r.$$

If $b, b' \in B$ and $b > b'$, we say that b' is a *subpiece* of b. We also consider the collection Ω of all finite sequences

$$S = (B_1, ..., B_r)$$

of subsets of B with the following properties :
(a) $B_1 \subset \{b_i; i \in \mathbf{N}^*\}$;
(b) Each B_j has at most α_j pieces;
(c) For each $j \in \{2, ..., r\}$ and for each piece x of B_j, either x is a piece of B_{j-1} or x is a subpiece of some piece y of B_{j-1}; moreover, in the latter case, we cannot have $y \in B_j$;
(d) $B_1 \neq B_2 \neq ... \neq B_r$;
(e) There cannot exist integers $1 \leq i_0 < ... < i_n \leq r$ and pieces $x_{i_0} \in B_{i_0}, ..., x_{i_n} \in B_{i_n}$ so that $x_{i_0} > x_{i_1} > ... > x_{i_n}$.

We claim that the following holds :

(3) There is a constant $M = M(n,d) \in \mathbf{N}^*$ such that every sequence in Ω has length $\leq M$, where the length of a sequence $S = (B_1, ..., B_r) \in \Omega$ is defined to be the number r.

Assume (3) for the moment, and let us see how it can be used to prove theorem 1. For each $j \in \mathbf{N}^*$, let q_j be the real homogeneous polynomial given by

$$q_j(y) = \langle P^j y, P^j y\rangle - \langle y, y\rangle^{d^j} \qquad (y \in \mathbf{R}^n),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbf{R}^n$. Define

$$A_j = \{y \in \mathbf{R}^n; q_k(y) = 0 \text{ for } k = 1, ..., j\}$$
$$= \{y \in \mathbf{R}^n; \|P^k y\| = \|y\|^{d^k} \text{ for } k = 1, ..., j\},$$

for $j \geq 1$. Obviously,

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots.$$

For each $j \in \mathbf{N}^*$, consider the algebraic subset

$$Y_j = Z(\tilde{q}_1) \cap \ldots \cap Z(\tilde{q}_j)$$

of the projective $(n-1)$-space $\mathbf{P}^{n-1}$ over $\mathbf{C}$ (where $\tilde{q}_j$ denotes the polynomial q_j regarded as a complex polynomial in n complex variables and $Z(\tilde{q}_j)$ is the zero set of $\tilde{q}_j$ in $\mathbf{P}^{n-1}$; see [3]). Obviously, $Y_1 \supseteq Y_2 \supseteq \cdots$. Since $\mathbf{P}^{n-1}$ is a Noetherian topological space, there is a greatest integer γ such that

$$Y_1 \supset Y_2 \supset \ldots \supset Y_\gamma.$$

Now we need the following:

Lemma 3. *There is a sequence in Ω with length γ.*

The proofs of this and the next lemma will be left to the next section.

By (3) and lemma 3, $\gamma \leq M$. Our definition of γ implies that $Y_\gamma = Y_{\gamma+1}$. Hence, if $y \in A_\gamma - \{0\}$ then

$$\phi(y) \in Z(\tilde{q}_1) \cap \ldots \cap Z(\tilde{q}_\gamma) = Y_\gamma = Y_{\gamma+1} \subseteq Z(\tilde{q}_{\gamma+1})$$

(where $\phi(y)$ denotes the equivalence class of y in $\mathbf{P}^{n-1}$), which implies $y \in A_{\gamma+1}$. Thus, $A_\gamma = A_{\gamma+1}$. Now, if $y \in A_{\gamma+1}$ then $Py \in A_\gamma$ and so $Py \in A_{\gamma+1}$, which implies $y \in A_{\gamma+2}$. Hence, $A_{\gamma+1} = A_{\gamma+2}$. By induction, we see that $A_\gamma = A_{\gamma+1} = A_{\gamma+2} = \cdots$. In particular,

$$A_M = A_{M+1} = A_{M+2} = \cdots.$$

If x satisfies (1), then $x \in A_M$ and so $x \in A_j$ for all $j \geq M$, which gives (2). This proves theorem 1.

We shall establish (3) by presenting an algorithm that constructs a longest possible sequence in Ω. The main motivation for the algorithm is based on the following observation: By (c), we can think of B_j as obtained from B_{j-1} by repeating or deleting or replacing by subpieces each of the pieces of B_{j-1}.

First, let us make some definitions. Let $S = (B_1, \ldots, B_r) \in \Omega$. If $x \in B_j$ for some $j \in \{1, \ldots, r\}$, say $x = b_{j_0,\ldots,j_m}$, we define the depth of x (denoted $depth(x)$) as being the number m. Note that this is the largest integer with the property that we can find integers $1 \leq i_0 < \ldots < i_m = j$ and pieces $x_{i_0} \in B_{i_0}, \ldots, x_{i_m} \in B_{i_m}$ so that

$$x_{i_0} > x_{i_1} > \ldots > x_{i_m} = x.$$

With this definition we see that condition (e) merely says that each piece in each B_j must have depth $\leq n-1$. We also define the numbers

$$N_i(S, j) = card\{x \in B_j; depth(x) = i\} \quad (i = 0, \ldots, n-1; j = 1, \ldots, r)$$

and put

$$N(S, j) = (N_0(S, j), \ldots, N_{n-1}(S, j)) \quad (j = 1, \ldots, r).$$

Note that $N(S, j) \in \mathbf{N}^n$. In the sequel we consider $\mathbf{N}^n$ endowed with its lexicographic order relation, which we denote by "$\leq$".

Algorithm. 1. *Put $\tilde{B}_1 = \{b_1, \ldots, b_{\alpha_1}\}$.*

2. *While $\tilde{B}_j \neq \emptyset$ do:*
   - *if $\tilde{B}_j$ has a piece of depth $n-1$, choose one such piece x and define $\tilde{B}_{j+1} = \tilde{B}_j - \{x\}$;*

- *if $\tilde{B}_j$ has no piece of depth $n-1$, choose a piece x of $\tilde{B}_j$ with maximum depth. Then choose $\alpha_{j+1} - card(\tilde{B}_j) + 1$ subpieces $x_1, \dots, x_{\alpha_{j+1} - card(\tilde{B}_j)+1}$ of x and define $\tilde{B}_{j+1} = (\tilde{B}_j - \{x\}) \cup \{x_1, \dots, x_{\alpha_{j+1}-card(\tilde{B}_j)+1}\}$.*

3. *If $\tilde{B}_j = \emptyset$, stop.*

Let $\tilde{S} = (\tilde{B}_1, \tilde{B}_2, \dots)$ be the sequence generated by our algorithm. Clearly our algorithm never generates pieces of depth $\geq n$, and in each step it decreases the depth numbers in lexicographic order: $N(\tilde{S}, j+1) < N(\tilde{S}, j)$. Since the lexicographic order relation on $\mathbf{N}^n$ is a well-order, it follows that our algorithm must end in a finite number of steps, so that $\tilde{S}$ is a finite sequence; say

$$\tilde{S} = (\tilde{B}_1, \dots, \tilde{B}_M).$$

Moreover, it is immediate to check that $\tilde{S} \in \Omega$. Now, we will need the fact that our sequence $\tilde{S}$ has "maximal depth numbers":

Lemma 4. *If $S = (B_1, \dots, B_r)$ is any element of Ω, then*

$$N(\tilde{S}, j) \geq N(S, j) \quad \textit{for all } j = 1, \dots, min\{M, r\}.$$

Let us finally prove (3). We claim that for any sequence $S = (B_1, \dots, B_r) \in \Omega$, we have $r \leq M$. Suppose that this is not the case, and let $S \in \Omega$ be a sequence with length $r > M$. By lemma 4, $N(S, M) \leq N(\tilde{S}, M)$. Since $\tilde{B}_M = \emptyset$, we have $N(\tilde{S}, M) = (0, \dots, 0)$. Hence, $N(S, M) = (0, \dots, 0)$, which implies $B_M = \emptyset$. It now follows from (c) that $B_{M+1} = \emptyset$. Hence, $B_M = B_{M+1}$, which contradicts (d).

3. Proofs of the lemmas

Proof of lemma 3. For each $j \in \{1, \dots, \gamma\}$, let C_j be the set of all irreducible components of Y_j. Since each $\tilde{q}_j$ has degree $\leq 2d^j$, it follows that

$$card(C_j) \leq (2d)(2d^2)\dots(2d^j) = \alpha_j \quad \text{for } j = 1, \dots, \gamma$$

([2], example 8.4.6). For each $j \in \{1, \dots, \gamma\}$ and each $Z \in C_j$, we define the depth of Z (denoted $depth(Z)$) as the greatest integer $m \in \mathbf{N}$ such that we can find integers $1 \leq i_0 < \dots < i_m = j$ and elements $Z_{i_0} \in C_{i_0}, \dots, Z_{i_m} \in C_{i_m}$ so that

$$Z_{i_0} \supset Z_{i_1} \supset \dots \supset Z_{i_m} = Z.$$

Note that since $\mathbf{P}^{n-1}$ has dimension $n-1$ [3], each element of each C_j must have depth $\leq n-1$. We shall construct, inductively, subsets $B_1, \dots, B_\gamma$ of B and bijections $\phi_1 : C_1 \to B_1, \dots, \phi_\gamma : C_\gamma \to B_\gamma$, so that

$$(B_1, ..., B_j) \in \Omega \tag{4}$$

and

$$depth(\phi_j(Z)) \leq depth(Z) \text{ for all } Z \in C_j, \tag{5}$$

for $j = 1, \dots, \gamma$. This will prove the lemma.

We begin by putting $B_1 = \{b_1, \dots, b_{card(C_1)}\}$ and by choosing an arbitrary bijection $\phi_1 : C_1 \to B_1$. It is clear that (4) and (5) hold for $j = 1$. Suppose that for some $t \in \{1, \dots, \gamma - 1\}$ we have already constructed $B_1, \dots, B_t \subset B$ and bijections $\phi_1 : C_1 \to B_1, \dots, \phi_t : C_t \to B_t$ so that (4) and (5) hold for $j = 1, \dots, t$. Since $Y_t \supset Y_{t+1}$, each element of C_{t+1} is either an element of C_t or a proper subset of some element of C_t. Let $Z_1, \dots, Z_\beta$ be the elements of $C_{t+1} \cap C_t$. We define

$$\phi_{t+1}(Z_i) = \phi_t(Z_i) \quad \text{for } i = 1, \dots, \beta.$$

Let $W_1, \dots, W_\alpha$ be an enumeration of the elements of $C_t - C_{t+1}$. Let $W_{1,1}, \dots, W_{1,\beta_1}$ be the elements of C_{t+1} that are proper subsets of W_1. By definition, $\phi_t(W_1) \in B$, say $\phi_t(W_1) = b_{i_1,\dots,i_s}$. We then define

$$\phi_{t+1}(W_{1,i}) = b_{i_1,\dots,i_s,i} \quad \text{for } i = 1, \dots, \beta_1.$$

Now, let $W_{2,1}, \dots, W_{2,\beta_2}$ be the elements of $C_{t+1} - \{W_{1,1}, \dots, W_{1,\beta_1}\}$ that are proper subsets of W_2. If $\phi_t(W_2) = b_{j_1,\dots,j_w}$, we then define

$$\phi_{t+1}(W_{2,i}) = b_{j_1,\dots,j_w,i} \quad \text{for } i = 1, \dots, \beta_2.$$

By continuing this process, we obtain a function $\phi_{t+1} : C_{t+1} \to B$. Put

$$B_{t+1} = \phi_{t+1}(C_{t+1}).$$

The main ingredient for checking that B_{t+1} and ϕ_{t+1} have the desired properties is the fact that for any $S' = (B'_1, ..., B'_r) \in \Omega$ no two pieces of any B'_j can be comparable under "$>$" (which can be easily proved by induction on j). We leave the details to the reader.

Proof of lemma 4. We proceed by induction on j. If $j = 1$ then

$$N(\tilde{S}, 1) = (\alpha_1, 0, 0, \dots, 0) \geq N(S, 1).$$

Suppose that for a certain $j \in \{1, \dots, min\{M, r\} - 1\}$ we have $N(\tilde{S}, j) \geq N(S, j)$. We have to show that

$$N(\tilde{S}, j+1) \geq N(S, j+1). \tag{6}$$

We have two cases:

Case 1. $N(\tilde{S}, j) = N(S, j)$: This means that

$$N_i(\tilde{S}, j) = N_i(S, j) \quad \text{for } i = 0, \dots, n-1.$$

Let β be the maximum depth of the pieces of $\tilde{B}_j$, so that

$$N_{\beta+1}(\tilde{S}, j) = \ldots = N_{n-1}(\tilde{S}, j) = 0.$$

By our construction of $\tilde{B}_{j+1}$,

$$N_i(\tilde{S}, j+1) = N_i(\tilde{S}, j) = N_i(S, j) \quad \text{for } i = 0, \dots, \beta - 1. \tag{7}$$

Suppose that for some $i \in \{1, \dots, \beta\}$ we have $N_i(S, j+1) > N_i(S, j)$. Then, in the construction of B_{j+1}, at least one piece of B_j with depth $i-1$ must have been replaced by subpieces. Let $i_0 \in \{0, \dots, \beta - 1\}$ be the smallest integer such that some piece of B_j with depth i_0 was replaced by subpieces. Then, $N_i(\tilde{S}, j+1) = N_i(S, j) \geq N_i(S, j+1)$ for $i = 0, \dots, i_0 - 1$, and $N_{i_0}(\tilde{S}, j+1) = N_{i_0}(S, j) > N_{i_0}(S, j+1)$ (by (7)), and so (6) holds. Thus, assume that

$$N_i(S, j+1) \leq N_i(S, j) \quad \text{for } i = 1, \dots, \beta.$$

If for some $i \in \{0, \dots, \beta - 1\}$ we have $N_i(S, j+1) < N_i(S, j)$, then (6) holds (by (7)). Hence, assume

$$N_i(S, j+1) = N_i(S, j) \quad \text{for } i = 0, \dots, \beta - 1. \tag{8}$$

By (7) and (8), we obtain

$$N_i(\tilde{S}, j+1) = N_i(S, j+1) \quad \text{for } i = 0, \dots, \beta - 1. \tag{9}$$

By (8), in the construction of B_{j+1}, no piece of B_j with depth $\leq \beta - 1$ was removed nor replaced by subpieces. Since $N_i(S, j) = N_i(\tilde{S}, j) = 0$ for $j = \beta + 1, \dots, n-1$, some piece of B_j with depth β must have been removed or replaced by subpieces, so that $N_\beta(S,j+1) \leq N_\beta(S,j)-1 = N_\beta(\tilde{S},j)-1 = N_\beta(\tilde{S},j+1)$. If $N_\beta(S,j+1) < N_\beta(\tilde{S},j+1)$, then (6) follows from (9). Thus, assume

$$N_\beta(\tilde{S},j+1) = N_\beta(S,j+1). \tag{10}$$

If $\beta = n-1$, (9) and (10) give equality in (6). Suppose $\beta < n-1$. By our construction of $\tilde{B}_{j+1}$, we have

$$N_{\beta+1}(\tilde{S},j+1) = \alpha_{j+1} - (N_0(\tilde{S},j+1) + \ldots + N_\beta(\tilde{S},j+1)) \tag{11}$$

and

$$N_{\beta+2}(\tilde{S},j+1) = \ldots = N_{n-1}(\tilde{S},j+1) = 0. \tag{12}$$

Since $card(B_{j+1}) \leq \alpha_{j+1}$,

$$N_{\beta+1}(S,j+1) \leq \alpha_{j+1} - (N_0(S,j+1) + \ldots + N_\beta(S,j+1))$$
$$= \alpha_{j+1} - (N_0(\tilde{S},j+1) + \ldots + N_\beta(\tilde{S},j+1)) = N_{\beta+1}(\tilde{S},j+1)$$

(by (11)). Again, if $N_{\beta+1}(S,j+1) < N_{\beta+1}(\tilde{S},j+1)$, then (6) follows from (9) and (10). So, suppose

$$N_{\beta+1}(S,j+1) = N_{\beta+1}(\tilde{S},j+1).$$

Then $N_0(S,j+1)+\ldots+N_{\beta+1}(S,j+1) = N_0(\tilde{S},j+1)+\ldots+N_{\beta+1}(\tilde{S},j+1) = \alpha_{j+1}$, and therefore

$$N_{\beta+2}(S,j+1) = \ldots = N_{n-1}(S,j+1) = 0.$$

In view of (12), we conclude that we have equality in (6).

Case 2. $N(\tilde{S},j) > N(S,j)$: In this case we shall show that

$$N(\tilde{S},j+1) > N(S,j+1). \tag{13}$$

Our hypothesis implies that there is an $\alpha \in \{0,\ldots,n-1\}$ such that

$$N_0(\tilde{S},j) = N_0(S,j), \ldots, N_{\alpha-1}(\tilde{S},j) = N_{\alpha-1}(S,j) \quad \text{and} \quad N_\alpha(\tilde{S},j) > N_\alpha(S,j).$$

In particular, $N_\alpha(\tilde{S},j) \neq 0$, and therefore

$$N_i(\tilde{S},j+1) = N_i(\tilde{S},j) = N_i(S,j) \quad \text{for } i = 0,\ldots,\alpha-1. \tag{14}$$

Suppose that for some $i \in \{1,\ldots,\alpha\}$ we have $N_i(S,j+1) > N_i(S,j)$. Then, in the construction of B_{j+1}, at least one piece of B_j with depth $i-1$ must have been replaced by subpieces. Let $i_0 \in \{0,\ldots,\alpha-1\}$ be the smallest integer such that some piece of B_j with depth i_0 was replaced by subpieces. Then, $N_i(\tilde{S},j+1) = N_i(S,j) \geq N_i(S,j+1)$ for $i = 0,\ldots,i_0-1$, and $N_{i_0}(\tilde{S},j+1) = N_{i_0}(S,j) > N_{i_0}(S,j+1)$ (by (14)), and so (13) holds. Thus, assume that

$$N_i(S,j+1) \leq N_i(S,j) \quad \text{for } i = 1,\ldots,\alpha. \tag{15}$$

If for some $i \in \{0,\ldots,\alpha-1\}$ we have $N_i(S,j+1) < N_i(S,j)$, then (13) holds (by (14) and (15)). Hence, assume

$$N_i(S,j+1) = N_i(S,j) \quad \text{for } i = 0,\ldots,\alpha-1. \tag{16}$$

By (14) and (16),

$$N_i(\tilde{S},j+1) = N_i(S,j+1) \quad \text{for } i = 0,\ldots,\alpha-1. \tag{17}$$

Now, if some piece of $\tilde{B}_j$ has depth $> \alpha$, then

$$N_\alpha(\tilde{S}, j+1) = N_\alpha(\tilde{S}, j) > N_\alpha(S, j) \geq N_\alpha(S, j+1)$$

(by (15)), and so (13) follows from (17). Suppose that the maximum depth of the pieces in $\tilde{B}_j$ is exactly equal to α. Then

$$N_\alpha(\tilde{S}, j+1) = N_\alpha(\tilde{S}, j) - 1 \geq N_\alpha(S, j) \geq N_\alpha(S, j+1).$$

If $N_\alpha(\tilde{S}, j+1) > N_\alpha(S, j+1)$, then (13) follows from (17). So, assume

$$N_\alpha(\tilde{S}, j+1) = N_\alpha(S, j+1). \tag{18}$$

Hence, $N_\alpha(S, j) = N_\alpha(S, j+1)$, and so we conclude from (16) that $\alpha < n-1$ and that

$$N_{\alpha+1}(S, j+1) \leq N_{\alpha+1}(S, j).$$

Now, the fact that $\tilde{B}_j$ has no piece with depth $> \alpha$ implies that

$$N_{\alpha+1}(\tilde{S}, j+1) = \alpha_{j+1} - (N_0(\tilde{S}, j+1) + \ldots + N_\alpha(\tilde{S}, j+1)).$$

Consequently,

$$\begin{aligned} N_{\alpha+1}(S, j+1) &\leq N_{\alpha+1}(S, j) \leq \alpha_j - (N_0(S, j) + \ldots + N_\alpha(S, j)) \\ &< \alpha_{j+1} - (N_0(\tilde{S}, j+1) + \ldots + N_\alpha(\tilde{S}, j+1)) = N_{\alpha+1}(\tilde{S}, j+1). \end{aligned}$$

In view of (17) and (18), we conclude that (13) holds. This completes the proof of lemma 4.

Remark 5. The above proof shows that if $S = (B_1, \ldots, B_r) \in \Omega$ and if $N(\tilde{S}, j) > N(S, j)$ for some $j \in \{0, \ldots, min\{M, r\}\}$, then

$$N(\tilde{S}, t) > N(S, t) \quad \text{for all} \quad t \in \{j, \ldots, min\{M, r\}\}.$$

In particular, $r < M$. This shows that the sequences $S \in \Omega$ that have length M are exactly those that satisfy the condition

$$N(S, j) = N(\tilde{S}, j) \quad \text{for all} \quad j \in \{0, \ldots, M\}.$$

4. Further results

We now remark that the argument used to prove theorem 1 can also be applied to establish other results of similar type. For instance, we have the following

Theorem 6. *Let n and d be positive integers. Then there is a constant $L = L_{\mathbf{K}}(n, d) \in \mathbf{N}^*$ such that for any homogeneous polynomial maps $P, Q : \mathbf{K}^n \to \mathbf{K}^n$ of degree d and any $x, y \in \mathbf{K}^n$, the relation*

$$\|P^j x\| = \|Q^j y\| \text{ for } j = 1, \ldots, L$$

implies

$$\|P^j x\| = \|Q^j y\| \text{ for all } j \geq 1.$$

Proof. We may assume $\mathbf{K} = \mathbf{R}$. Now, we have to consider the homogeneous polynomial $q_j : \mathbf{R}^{2n} \to \mathbf{R}$ given by

$$q_j(z, w) = \langle P^j z, P^j z\rangle - \langle Q^j w, Q^j w\rangle \quad (z, w \in \mathbf{R}^n).$$

As before, we define

$$\begin{aligned} A_j &= \{(z, w) \in \mathbf{R}^{2n}; q_k(z, w) = 0 \text{ for } k = 1, \ldots, j\} \\ &= \{(z, w) \in \mathbf{R}^{2n}; \|P^k z\| = \|Q^k w\| \text{ for } k = 1, \ldots, j\} \end{aligned}$$

and consider the algebraic subset

$$Y_j = Z(\tilde{q}_1) \cap \ldots \cap Z(\tilde{q}_j)$$

of $\mathbf{P}^{2n-1}$. The same argument used to prove theorem 1 then applies (the α_j's are the same, but we have to consider $2n$ in place of n).

Similarly, we obtain

Theorem 7. *Let n and d be positive integers. Then there is a constant $R = R_{\mathbf{K}}(n,d) \in \mathbf{N}^*$ such that for any homogeneous polynomial maps $P, Q : \mathbf{K}^n \to \mathbf{K}^n$ of degree d and any $x, y \in \mathbf{K}^n$, the relation*

$$1 = \|x - y\| = \|Px - Qy\| = \ldots = \|P^R x - Q^R y\|$$

implies

$$\|P^j x - Q^j y\| = 1 \text{ for all } j \geq 1.$$

Remark 8. Note that theorem 7 generalizes theorem 1 (just set $Q = P$ and $y = 0$).

We close the paper by proposing the following question:

Open problem. What are the best values of the constants $M_{\mathbf{R}}(n,d)$, $M_{\mathbf{C}}(n,d)$, $N_{\mathbf{R}}(n,d)$ and $N_{\mathbf{C}}(n,d)$?

Only the following are known: $N_{\mathbf{R}}(n,1) = N_{\mathbf{C}}(n,1) = n$ (see [4]), $M_{\mathbf{R}}(1,1) = M_{\mathbf{C}}(1,1) = 1$, $M_{\mathbf{R}}(2,1) = 3$, $M_{\mathbf{C}}(2,1) = 4$ and $M_{\mathbf{C}}(3,1) = 8$ (see [1]). Moreover, some estimates for $M_{\mathbf{R}}(n,1)$ and $M_{\mathbf{C}}(n,1)$ were obtained in [1].

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