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UNIQUE DECOMPOSITION OF RIEMANNIAN MANIFOLDS

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Abstract. We prove an extension of de Rham's decomposition theorem to the non-simply connected case.

1. Introduction

A connected Riemannian manifold may allow more than one decomposition into a product of indecomposable factors: Euclidean space of dimension ≥ 2 splits orthogonally into a product of one-dimensional subspaces in many different ways. But by the classical theorem of de Rham ([dR], cf. also [KN], [M], [P]), this is essentially the only *simply* connected example with that property. The purpose of our note is to generalize this result to the non-simply connected case as well.

Theorem. *Any complete connected Riemannian manifold M decomposes into a Riemannian product*

$$M = M_0 \times M_1 \times ... \times M_p \tag{1}$$

where M_0 is a maximal factor isometric to euclidean space and each M_i, $i > 0$, is indecomposable. This decomposition is unique up to the order of $M_1, ..., M_p$.

We call a Riemannian manifold *indecomposable* if it is not isometric to a Riemannian product of lower dimensional manifolds. Any (holonomy) *irreducible* manifold is indecomposable and by de Rham's theorem also the converse is true for simply connected manifolds. But in general the two notions differ: A non-rectangular flat 2-torus is indecomposable but not irreducible. By decomposing a manifold further and further it is clear that any Riemannian manifold admits a decomposition into a product of indecomposable ones. Therefore, the only question is about uniqueness. We say that a product decomposition is *unique* if the corresponding foliations are uniquely determined.

If M is compact, there is no euclidean factor. Hence we get the following

Corollary 1. *Let M be a compact Riemannian manifold. Then M decomposes uniquely into a Riemannian product of indecomposable factors. Any isometry of M must preserve or interchange these factors. In particular, for any Riemannian product decomposition $M = M_1 \times M_2$, the identity component of the isometry group splits as $I_0(M) = I_0(M_1) \times I_0(M_2)$.*

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Another immediate consequence is a theorem due to Uesu [U] generalizing a previous result of Takagi [T]:

Corallary 2. *Let M, N and B be complete connected Riemannian manifolds. If $M \times B$ is isometric to $N \times B$ then M is isometric to N.*

The main idea of the proof of our Theorem is to use a special (so-called "short") set of generators of the fundamental group which is compatible to *any* Riemannian decomposition of M. The same generating set had been used by Gromov in order to estimate the number of generators of the fundamental group (cf. [G]).

2. Proof of the Theorem

By the remark above, we only have to show uniqueness. This will follow from a series of lemmas. We always denote by $\tilde{M}$ the universal cover of M and by Γ its group of deck transformations. A decompositon of $\tilde{M}$ into a product $\tilde{M} = X_1 \times ... \times X_k$ determines k foliations on $\tilde{M}$ whose leaves through a point $p \in \tilde{M}$ will be denoted by $X_i(p)$. We say that an isometry ϕ of $\tilde{M}$ acts *only on* X_i (or *trivially on all* $X_j,\ j \neq i$) if

$$\phi(x_1, ..., x_i, ..., x_k) = (x_1, ..., \phi_i x_i, ..., x_k)$$

for some isometry ϕ_i of X_i. In the language of foliations this means that each leaf $X_i(p)$ is ϕ-invariant, i.e. $\phi(p) \in X_i(p)$ for all $p \in \tilde{M}$.

Lemma 1. *The maximal euclidean factor M_0 of M is uniquely determined.*

Proof. By de Rham's theorem $\tilde{M}$ splits uniquely into $E \times N$ where E is euclidean and N has no euclidean factor. Furthermore Γ preserves this splitting, i.e. each $\gamma \in \Gamma$ is of the form (γ_E, γ_N) where γ_E and γ_N are isometries of E and N, respectively. Now any euclidean factor of M corresponds to a factor E_1 of E on which Γ acts trivially. If $E = E_1 \times E_2$ as a Riemannian manifold, then $E_i(x) = \vec{E}_i + x$ where $\vec{E}_1 \oplus \vec{E}_2$ is an orthogonal splitting of the euclidean vector space $\vec{E}$ acting simply transitively on the affine space E by translations. By the remark before Lemma 1, $\gamma \in \Gamma$ acts trivially on E_1 if

$$\gamma_E(x) \in E_2(x) = \vec{E}_2 + x$$

for all $x \in E$ which in turn is equivalent to $(\gamma_E x - x) \perp \vec{E}_1$. Thus E_1 is maximal if

$$\vec{E}_1 = \{\gamma_E x - x;\ x \in E,\ \gamma \in \Gamma\}^{\perp} \subset \vec{E}_1$$

but this is uniquely determined. □

Lemma 2. *Let $\tilde{M} = \tilde{M}_1 \times ... \times \tilde{M}_p = \tilde{M}'_1 \times ... \times \tilde{M}'_q$ be two decompositions of $\tilde{M}$. Then there exists a decomposition $\tilde{M} = \prod_{i,j} \tilde{M}_{ij} \times F$ of $\tilde{M}$ where F is a euclidean factor and $\tilde{M}_{ij}(p) = \tilde{M}_i(p) \cap \tilde{M}'_j(p)$.*

Proof. $\tilde{M}_i(p)$ and $\tilde{M}'_j(p)$ are totally convex in the sense that any minimal geodesic of $\tilde{M}$ joining two points in $\tilde{M}_i(p)$ or $\tilde{M}'_j(p)$ lies completely in $\tilde{M}_i(p)$ or $\tilde{M}'_j(p)$, respectively. Therefore, $\tilde{M}_{ij}(p)$ is a totally geodesic connected submanifold of $\tilde{M}$. The tangent spaces $\mathcal{D}_{ij}(p) = T_p(\tilde{M}_{ij}(p)) = T_p(\tilde{M}_i) \cap T_p(\tilde{M}'_j)$ form a distribution $\mathcal{D}_{ij}$ which is invariant under parallel translations. Therefore we get from de Rham's theorem $\tilde{M} = \prod_{i,j} \tilde{M}_{ij} \times F$ where F is some complementary factor. Since each irreducible de Rham factor is contained in some $\tilde{M}_i$ and in some $\tilde{M}'_j$, it is also contained in some $\tilde{M}_{ij}$. Thus the complementary factor F must be euclidean. □

Recall that a splitting $\tilde{M} = \tilde{M}_1 \times ... \times \tilde{M}_p$ of the universal cover is induced by a splitting $M = M_1 \times ... \times M_p$ of the manifold M itself if and only if the group Γ of deck transormations splits accordingly. This means that Γ has a set of generators each of which acts only on one of the factors $\tilde{M}_i$. We now show that there is even a set of generators which has this property for all splittings of M at the same time:

Lemma 3. *There exists a generating set Σ of Γ such that for any decomposition $M = M_1 \times ... \times M_p$ of M, each $\sigma \in \Sigma$ acts only on one factor of the corresponding decompositon $\tilde{M} = \tilde{M}_1 \times ... \times \tilde{M}_p$.*

Proof. Choose $o \in \tilde{M}$ and let $|\gamma| := dist(o, \gamma o)$ for each $\gamma \in \Gamma$. Let $\Sigma = \{\sigma_1, \sigma_2, ...\}$ be a *short generating set* in the sense of Gromov [G], i.e. σ_1 is chosen with $|\sigma_1| = \min\{|\gamma|;\ \gamma \in \Gamma \setminus \{1\}\}$ and σ_k inductively with $|\sigma_k| = \min\{|\gamma|;\ \gamma \in \Gamma \setminus \Gamma_{k-1}\}$ where Γ_{k-1} denotes the subgroup generated by $\sigma_1, ..., \sigma_{k-1}$. Each $\sigma_k \in \Sigma$ (in fact each $\sigma \in \Gamma$) can be written as $\sigma_k = \gamma_1\gamma_2...\gamma_p$ where γ_i acts only on $\tilde{M}_i$. Hence

$$|\sigma_k|^2 = \sum_{i=1}^{p} |\gamma_i|^2 \geq |\gamma_j|^2$$

for all j. In case of a strict inequality we have $\gamma_j \in \Gamma_{k-1}$ by the choice of σ_k. But this cannot happen for all j since $\sigma_k \notin \Gamma_{k-1}$. Thus there exists $i \in \{1, ..., p\}$ with $|\sigma_k| = |\gamma_i|$ and $|\gamma_j| = 0$ for all $j \neq i$ which means $\sigma_k = \gamma_i$. □

Proof of the Theorem. By Lemma 1 we may assume that M contains no euclidean factor. Let $M = M_1 \times ... \times M_p = M'_1 \times ... \times M'_q$ be two decompositions of M into indecomposable factors. According to Lemma 2 we get a decomposition $\tilde{M} = \prod_{i,j} \tilde{M}_{ij} \times F$. Now, if σ is any element of the special generating set of Lemma 3, there exist $i \in \{1, ..., p\}$ and $j \in \{1, ..., q\}$ such that the leaves $\tilde{M}_i(p)$ and $\tilde{M}'_j(p)$ and hence $\tilde{M}_{ij}(p)$ are σ-invariant for all $p \in \tilde{M}$. In particular, σ and hence Γ act trivially on F. Since M has no euclidean factor, F must be trivial, i.e. $\tilde{M} = \prod_{i,j} \tilde{M}_{ij}$. Furthermore, Γ is generated by elements σ which act only on one of the factors $\tilde{M}_{ij}$. Thus, by the remark before Lemma 3 we get a corresponding decomposition $M = \prod_{i,j} M_{ij}$ of M with $M_{ij}(m) = M_i(m) \cap M'_j(m)$ for all $m \in M$. Since the M_i and M'_j are indecomposable, the theorem follows. □

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