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# OPTIMAL CONTROL OF A FUNCTIONAL EQUATION ASSOCIATED WITH CLOSED RANGE SELFADJOINT OPERATORS

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Abstract. Necessary and sufficient conditions for the optimality of a pair $(y^*, u^*)$ subject to $Ay^* = Bu^* + f$ are given. Here $A$ is a selfadjoint operator with closed range on a Hilbert space $\mathcal{H}$ and $B \in L(\mathcal{H})$. The case $B$– unbounded is also discussed, which leads to some open problems. This general functional scheme includes most of the previous results on the optimal control of the $T$–periodic wave equation for all $T$ in a dense subset of $\mathbb{R}$. It also includes optimal control problems for some elliptic equations.

## 1. Introduction

This paper is concerned with the following optimal control problem

$$\text{(P)} \qquad \text{Minimize} : L(y, u) = G(y) + H(u)$$

over all $(y, u)$ subject to

$$Ay = Bu + f. \tag{1.1}$$

Here $A$ is a selfadjoint operator $A\colon D(A) \subset \mathcal{H} \to R(A) \subset \mathcal{H}$ with closed range, where $\mathcal{H}$ is a real Hilbert space. $G$ and $H$ are lower-semicontinuous convex functionals from $\mathcal{H}$ into $(-\infty, +\infty]$ and satisfying Hypothesis (H1) in Section 2.

$B$ is supposed to be linear bounded one-to-one and onto from $\mathcal{H}$ into itself.

However, the case when $B$ is unbounded will be also discussed, as it is more important in applications and leads to delicate open problems.

The motivation of the problem (P) consists in the fact that it includes the optimal control problems for the time-periodic wave equation. It is interesting in itself, too.

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As an example of $A$, consider the problem

$$\begin{aligned} & y_{tt} - \Delta y = f(t,x), \;\; x \in \Omega,\, t \in (0,T), \\ & y(t,x) = 0 \;\text{ on }\; (0,T) \times \partial\Omega, \\ & y(0,x) = y(T,x);\;\; y_t(0,x) = y_t(T,x), \;\; x \in \Omega, \end{aligned} \tag{1.2}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $\Delta$ is the Laplace operator in the sense of distributions, $f \in L^2(Q)$.

Set $Q = (0,T) \times \Omega$ and

$$\begin{aligned} C_T^2(\overline{Q}) = \{\, \phi \in C^2(\overline{Q});\;\; & \phi(t,x) = 0 \;\text{ on } (0,T) \times \partial\Omega, \\ & \phi \text{ satisfies the } T\text{-periodic conditions in (1.2)} \,\}, \end{aligned} \tag{1.3}$$

$$\mathcal{B}\phi = \phi_{tt} - \Delta\phi, \;\; \phi \in C_T^2(\overline{Q}).$$

**Definition 1.** $y \in L^2(Q)$ is said to be a weak ($T$-periodic) solution to the problem (1.2) if

$$\int_Q y\mathcal{B}\overline{\phi}\, dt = \int_Q f\overline{\phi}\, dx, \;\; \forall \phi \in C_T^2(\overline{Q}). \tag{1.4}$$

Note that (1.4) can formally be obtained by multiplying by $\phi$, integrating by parts with respect to $t$ and using Green's formula

$$\int_\Omega (\phi\Delta y - y\Delta\phi)\, dx = \int_{\partial\Omega} \left( \phi\frac{\partial y}{\partial\eta} - y\frac{\partial\phi}{\partial\eta} \right) d\sigma \tag{1.5}$$

where $\frac{\partial\phi}{\partial\eta}$ is the outward derivative of $\phi$ on $\partial\Omega$.

Finally, set

$$D(A) = \{\, y \in L^2(Q);\; \exists f \in L^2(Q) \;\text{ such that (1.4) holds} \,\} \tag{1.6}$$

and

$$Ay = f, \;\; y \in D(A). \tag{1.7}$$

In other words, $Ay = f$ if and only if (1.4) holds. It was recently proved (cf. [6], [7]) that for $\Omega = (0,\pi) \times (0,\pi)$ and for every $T$ with $T^2 =$ a rational multiple of $\pi^2$, the "weak solution operator" $A$ defined by (1.6)+(1.7) satisfies the above conditions, i.e. it is selfadjoint in $\mathcal{H} = L^2$, with $R(A)$ closed. Moreover, dim $N(A) = +\infty$.

Other examples of such operators can be found in ([1]–[2]). This example will be revisited at the end of the paper, in order to give an application of our main results. Another example of such $A$ is the elliptic operator $Ay = \Delta y - ay$ with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. Indeed, this operator is selfadjoint in $L^2(\Omega)$ with $R(A) = L^2(\Omega)$, for all nonnegative numbers $a$.

## 2. Main results

Let $\mathcal{H}$ be a real Hilbert space, $A\colon D(A) \subset \mathcal{H} \to \mathcal{H}$ a selfadjoint operator with closed range $R(A)$ and $B$ a continuous linear operator on $\mathcal{H}$. We formulate our problem as follows: For a given $f \in \mathcal{H}$, find a solution of

$$\text{(P)} \qquad \operatorname{Min}\{\, L(y,u) \mid (y,u) \in M_f \,\}$$

where

$$M_f = \{\, (y,u) \in \mathcal{H} \times \mathcal{H} \mid Ay = Bu + f, \;\; y \in D(A), u \in \mathcal{H} \,\} \tag{2.1}$$

and $G$ and $H$ satisfy:

(H1): $G$ and $H$ are lower semicontinuous convex functionals on $\mathcal{H}$ with

$$G(y) \geq a\|y\|^2 + b, \text{ for every } y \in \mathcal{H}, \tag{2.2}$$

$$H(u) \geq a_1\|u\|^2 + b_1, \text{ for every } u \in \mathcal{H}, \tag{2.3}$$

where $a, a_1 > 0, b, b_1 \in \mathbb{R}$.

*Remark* 1. $M_f$ is weakly closed, i.e. if $(y_n, u_n) \in M_f, y_n \rightharpoonup y, u_n \rightharpoonup u$, then $(y, u) \in M_f$. This is because $B$ is bounded and $A$ is weakly closed in the following sense: $y_n \in D(A), y_n \rightharpoonup y$ and $Ay_n$ -bounded imply $y \in D(A)$ and $Ay_n \rightharpoonup Ay$.

**Definition 2.** We say that $(y, u) \in \mathcal{H} \times \mathcal{H}$ is an admissible pair if $(y, u) \in M_f$.

**Definition 3.** We say that $(y^*, u^*) \in \mathcal{H} \times \mathcal{H}$ is an optimal pair for the problem (P) if and only if it is an admissible pair and it satisfies

$$L(y^*, u^*) = \mathrm{Min}\{\, L(y, u) \mid (y, u) \in M_f \,\}. \tag{2.4}$$

In this case, we also say that the problem (P) admits an optimal pair.

We now state our main result.

**Theorem 1.** *Suppose $A : D(A) \subset \mathcal{H} \to R(A) \subset \mathcal{H}$, with $R(A)$ closed, is self-adjoint; $B\colon \mathcal{H} \to \mathcal{H}$ is a linear continuous operator which is also one-to-one and onto. In addition,* (H1) *holds and $\partial G$ and $\partial H$ are locally bounded at $(y^*, u^*)$ respectively. Then $(y^*, u^*)$ is an optimal pair of* (P) *if and only if there exists $p^* \in D(A)$ such that*

$$Ap^* \in -\partial G(y^*), \;\; B^* p^* \in \partial H(u^*) \tag{2.5}$$

*where $\partial$ stands for the subdifferential operator.*

**Lemma 1.** *Suppose the conditions of Theorem 1 hold. Then problem* (P) *admits an optimal pair.*

*Proof.* $B$ is onto, so $M_f$ in not empty. Denote

$$d = \mathrm{Inf}\{\, L(y, u) \mid (y, u) \in M_f \,\}. \tag{2.6}$$

Condition (H1) ensures that $d > -\infty$. Then there exists

$$\{\, (y_n, u_n) \,\}_{n=1}^{\infty}, \;\; (y_n, u_n) \in M_f \tag{2.7}$$

for each $n \in \mathbb{N}$, such that

(2.8)
$$d + 1 \geq L(y_n, u_n) \geq a\|y_n\|^2 + a_1\|u_n\|^2 + e \text{ for all } n \geq N \text{ for some } N \in \mathbb{N}.$$

(2.8) implies that both $\{\, y_n \,\}_1^{\infty}$ and $\{\, u_n \,\}_1^{\infty}$ are bounded and hence weakly convergent, say, $y_n \rightharpoonup y^*$ and $u_n \rightharpoonup u^*$ weakly in $\mathcal{H}$, as $n \to \infty$ (relabeling if necessary). Therefore

$$Ay_n = Bu_n + f \rightharpoonup Bu^* + f \text{ weakly in } \mathcal{H}, \text{ as } n \to \infty. \tag{2.9}$$

Note that the graph of $A$ is weakly closed in $\mathcal{H} \times \mathcal{H}$, so (2.9), in conjunction with $y_n \rightharpoonup y^*$ weakly in $\mathcal{H}$, shows that $(y^*, u^*) \in M_f$, i.e.

$$Ay^* = Bu^* + f. \tag{2.10}$$

Therefore $(y^*, u^*)$ is an admissible pair. The lower semicontinuity of $L$ gives the following:

$$L(y^*, u^*) \leq \underline{\lim}_{n\to\infty} L(y_n, u_n) = d. \tag{2.11}$$

On the other hand, $L(y^*,u^*) \geq d$. This, together with (2.11), shows that

$$L(y^*,u^*) = \mathrm{Min}\{\, L(y,u) \mid (y,u) \in M_f \,\}, \tag{2.12}$$

i.e., the problem (P) admits an optimal pair. □

**Lemma 2.** *Suppose that the conditions on $A$ and $B$ in Theorem 1 hold. In addition, suppose that $G$ and $H$ are Fréchet differentiable. Then the necessary condition for an admissible pair $(y^*,u^*)$ to be an optimal pair is the existence of an element $p^* \in D(A)$ such that*

$$Ap^* = -\dot{G}(y^*), \;\; B^*p^* = \dot{H}(u^*) \tag{2.13}$$

*where $\dot{G}$ is the Fréchet derivative of $G$.*

*Proof.* One can show that the tangent space $TM_f(y^*,u^*)$ to $M_f$, $(y^*,u^*) \in M_f$, or simply $TM_f$, is given by

$$TM_f = \{\, (z,w) \in \mathcal{H} \times \mathcal{H} \mid Az = Bw, z \in D(A), w \in \mathcal{H} \,\} \tag{2.14}$$

independently of the point $(y^*,u^*) \in M_f$.

Now, if $(y^*,u^*)$ is optimal, we have

$$(\dot{L}(y^*,u^*))(z,w) = 0, \;\; \text{for every } \; (z,w) \in TM_f, \tag{2.15}$$

i.e.,

$$\langle\, \dot{G}(y^*), z \,\rangle + \langle\, \dot{H}(u^*), w \,\rangle = 0, \;\; \text{for every } \; (z,w) \in TM_f \tag{2.16}$$

where $\dot{G}(y^*)$ and $\dot{H}(u^*)$ are the Fréchet derivatives of the functionals $G$ and $H$ at $y^*$ and $u^*$, respectively. For $w = 0$, i.e., for any $z \in N(A)$, (2.16) yields

$$\langle\, \dot{G}(y^*), z \,\rangle = 0, \;\; \text{for every } z \in N(A). \tag{2.17}$$

This implies that $\dot{G}(y^*) \in R(A)$ since $\mathcal{H}$ admits the decomposition

$$\mathcal{H} = R(A) \oplus N(A) \tag{2.18}$$

under the conditions on $A$. Therefore there exists $\tilde{p}^* \in D(A)$ such that

$$A\tilde{p}^* = \dot{G}(y^*). \tag{2.19}$$

Thus (2.16) becomes

$$\langle\, A\tilde{p}^*, z \,\rangle + \langle\, \dot{H}(u^*), w \,\rangle = 0, \;\; \text{for every } (z,w) \in TM_f \tag{2.20}$$

or equivalently (as $B^{-1}$ exists)

$$\langle\, \tilde{p}^*, Az \,\rangle + \langle\, (B^*)^{-1}\dot{H}(u^*), Bw \,\rangle = 0, \;\; \text{for every } (z,w) \in TM_f, \tag{2.21}$$

i.e.,

$$\langle\, \tilde{p}^* + (B^*)^{-1}\dot{H}(u^*), Az \,\rangle = 0, \;\; \text{for every } z \in D(A) \tag{2.22}$$

as $R(B) = \mathcal{H}$. Indeed, for every $z \in D(A)$, there is $w$ such that $Az = Bw$, as $B$ is onto. Consequently, one has

$$\tilde{p}^* + (B^*)^{-1}\dot{H}(u^*) \in N(A) \tag{2.23}$$

again because of the decomposition (2.18) of $\mathcal{H}$. Denote

$$\tilde{p}^* + (B^*)^{-1}\dot{H}(u^*) = \eta \in N(A), \;\; \text{and } p^* = \eta - \tilde{p}^*. \tag{2.24}$$

Clearly (2.24) implies (2.13), which completes the proof. Note that Conditions (2.13) may not be sufficient, due to the fact that in Lemma 2, $G$ and $H$ are not supposed to be convex so they may not be subdifferentiable.When they are, (2.13) are sufficient, too(see (2.38)). ☐

*Proof of Theorem 1.* **Necessity**.

**Step 1.** If $G$ and $H$ are Fréchet differentiable, then by Lemma 2, there is $p^* \in D(A)$ such that equalities in (2.13) hold. In this case (2.13) and (2.5) are equivalent as $\partial G = \dot{G}$ and $\partial H = \dot{H}$.

**Step 2.** Suppose $G$ and $H$ are merely lower semicontinuous and $\partial G$ and $\partial H$ are locally bounded at $y^*$ and $u^*$, respectively. We can use the convex approximation scheme $L_\lambda$ of $L$, i.e., consider the following optimal problem of Barbu's type:

$$(P_\lambda) \qquad \text{Minimize} : L_\lambda(y,u) = G_\lambda(y) + H_\lambda(u) + \frac{1}{2}\|y-y^*\|^2 + \frac{1}{2}\|u-u^*\|^2 \tag{2.25}$$

over $M_f$, where $G_\lambda$ and $H_\lambda$ are the convex regularizations of $G$ and $H$, respectively, i.e.,

$$G_\lambda(y) = \text{Min}\{\, G(x) + \frac{1}{2\lambda}\|y-x\|^2 \mid \}, \quad \lambda > 0, y \in D(G). \tag{2.26}$$

Here $G_\lambda$ and $H_\lambda$ are Fréchet differentiable, and $\dot{G}_\lambda(y) = (\partial G)_\lambda(y) \in \partial G(J_\lambda^G y)$, $J_\lambda^G = (I - \lambda\partial G)^{-1}$ (see e.g. [8], Appendix, on these Yosida operators $J_\lambda$ and $A_\lambda$, and convex regularizations). On the basis of Step 1, there exists an optimal pair $(y_\lambda^*, u_\lambda^*)$ for $(P_\lambda)$ and, correspondingly, $p_\lambda^* \in D(A)$ such that

$$Ap_\lambda^* = -\dot{G}_\lambda(y_\lambda^*) - (y_\lambda^* - y^*), B^*p_\lambda^* = \dot{H}_\lambda(u_\lambda^*) + (u_\lambda^* - u^*). \tag{2.27}$$

Equalities

$$G_\lambda(y_\lambda^*) = G(J_\lambda y_\lambda^*) + \frac{\lambda}{2}\|\dot{G}_\lambda(y_\lambda^*)\|^2, \;\; J_\lambda = J_\lambda^G \tag{2.28}$$

and

$$J_\lambda(y_\lambda^*) + \lambda\dot{G}_\lambda(y_\lambda^*) = y_\lambda^* \tag{2.29}$$

together with the inequalities

$$G_\lambda(y_\lambda^*) \geq a\|J_\lambda y_\lambda^*\|^2 + \frac{\lambda}{2}\|\dot{G}_\lambda(y_\lambda^*)\|^2 + b \tag{2.30}$$

and

$$\begin{aligned} G(J_\lambda^G y_\lambda^*) + H(J_\lambda^H u_\lambda^*) &\leq G_\lambda(y_\lambda^*) + H_\lambda(u_\lambda^*) \\ &\leq L_\lambda(y_\lambda^*, u_\lambda^*) \leq L_\lambda(y^*, u^*) \\ &= G_\lambda(y^*) + H_\lambda(u^*) \leq L(y^*, u^*) \end{aligned} \tag{2.31}$$

imply that $\sqrt{\lambda}\,\|\dot{G}_\lambda(y_\lambda^*)\|$ and $\|J_\lambda(y_\lambda^*)\|$ are bounded, so by (2.29) $y_\lambda^*$ is bounded, too. Say $y_\lambda^* \rightharpoonup \tilde{y}^*$ weakly in $\mathcal{H}$ as $\lambda \to 0$ for some $\tilde{y}^* \in \mathcal{H}$. Then by (2.29), $J_\lambda(y_\lambda^*) \rightharpoonup \tilde{y}^*$. Similarly, $u_\lambda \rightharpoonup \tilde{u}^*$ weakly in $\mathcal{H}$ as $\lambda \to 0$ for some $\tilde{u}^* \in \mathcal{H}$. The same argument used in the proof of Lemma 1 can be used here to show that $(\tilde{y}^*, \tilde{u}^*) \in M_f$.

The above inequality, together with the semicontinuity of $G$ and $H$, implies that

$$
\begin{aligned}
L(y^*, u^*) &\leq L(\tilde{y}^*, \tilde{u}^*) \\
&\leq \lim_{\lambda\to 0}(G_\lambda(y^*) + H_\lambda(u^*)) \\
&\leq \lim_{\lambda\to 0} L_\lambda(y^*_\lambda, u^*_\lambda) \\
&\leq L(y^*, u^*);
\end{aligned} \tag{2.32}
$$

therefore

$$
\lim_{\lambda\to 0} L_\lambda(y^*_\lambda, u^*_\lambda) = \lim_{\lambda\to 0}(G_\lambda(y^*_\lambda) + H_\lambda(u^*_\lambda)) = L(y^*, u^*). \tag{2.33}
$$

This yields $\lim_{\lambda\to 0}\|y^*_\lambda - y^*\| = \lim_{\lambda\to 0}\|u^*_\lambda - u^*\| = 0$ and consequently, $\tilde{y}^* = y^*$ and $\tilde{u}^* = u^*$.

We now rely on the facts:

$$
\dot{G}_\lambda(y^*_\lambda) \in \partial G(J^G_\lambda y^*_\lambda),\;\; \dot{H}_\lambda(u^*_\lambda) \in \partial H(J^H_\lambda u^*_\lambda), \tag{2.34}
$$

$$
J^G_\lambda(y^*_\lambda) \to y^*,\;\; J^H_\lambda(u^*_\lambda) \to u^* \text{ strongly.} \tag{2.35}
$$

Note that (2.35) follows from (2.29) in conjunction with $y^*_\lambda \to y^*$, due to

$$
\|\, J^G_\lambda(y^*_\lambda) - y^*_\lambda \,\| \leq \sqrt{\lambda}\; (\sqrt{\lambda}\; \|\dot{G}_\lambda(y^*_\lambda)\|\,). \tag{2.36}
$$

As $\partial G$ and $\partial H$ are locally bounded at $y^*$ and $u^*$, respectively, it follows that $\dot{G}_\lambda(y^*_\lambda)$ and $\dot{H}_\lambda(u^*_\lambda)$ are bounded, so we may assume $\dot{G}_\lambda(y^*_\lambda) \rightharpoonup \alpha$ and $\dot{H}_\lambda(u^*_\lambda) \rightharpoonup \beta$ weakly.

As $\partial G$ and $\partial H$ are maximal monotone, we can pass to limit in (2.34) for $\lambda \to 0$, to get $\alpha \in \partial G(y^*), \beta \in \partial H(u^*)$.

From (2.27) it follows that $p^*_\lambda$ is bounded. Say $p^*_\lambda \rightharpoonup p^*$. Letting $\lambda \to 0$ in (2.27) we get

$$
Ap^* = -\alpha \in -\partial G(y^*),\;\; B^*p^* = \beta \in \partial H(u^*) \tag{2.37}
$$

which completes the proof.

**Sufficiency.** For any $(y, u) \in M_f$, we have

$$
\begin{aligned}
L(y, u) - L(y^*, u^*) &= G(y) - G(y^*) + H(u) - H(u^*) \\
&\geq \langle -Ap^*, y - y^* \rangle + \langle B^*p^*, u - u^* \rangle \\
&= -\langle p^*, A(y - y^*) \rangle + \langle p^*, B(u - u^*) \rangle \\
&= 0
\end{aligned} \tag{2.38}
$$

which completes the proof. □

In the case in which $B$ is not one-to-one, the following result holds:

**Theorem 2.** *Let $A$ and $B$ be selfadjoint operators in $\mathcal{H}$ with $R(A)$ and $R(B)$ closed in $\mathcal{H}$ and $R(A) \subset R(B)$. Let $G$ and $H$ be Frechét differentiable, satisfying* (H1). *Then a necessary and sufficient condition for an admissible pair $(y^*, u^*)$ to be optimal is the existence of an element $p^* \in D(A)$ such that*

$$
Ap^* = -\dot{G}(y^*);\;\; Bp^* = \dot{H}(u^*). \tag{2.39}
$$

*Proof.* In this case $TM_f(y^*, u^*) \supset S$ with

$$
S = \{\, (z, w);\;\; z \in D(A), w \in D(B), Az = Bw \,\}. \tag{2.40}
$$

Indeed, $(y^*, u^*) + \lambda(z, w) \in M_f$, $\forall \lambda \in \mathbb{R}$. Arguing as in the proof of Lemma 2, we have

$$\langle \dot{G}(y^*), z \rangle + \langle \dot{H}(u^*), w \rangle = 0, \ \forall (z, w) \in S \tag{2.41}$$

which yields (2.19).

For $z = 0$, (2.41) implies $\langle \dot{H}(u^*), w \rangle = 0 \ \forall Bw = 0$ so due to (2.18) we have

$$\dot{H}(u^*) \in R(B). \tag{2.42}$$

But $B$ is invertible on $R(B)$ so in view of (2.41), (2.21) makes sense for $(z, w) \in S$. Therefore, according to (2.19)–(2.24) the conclusion of Theorem 2 holds. □

We now can briefly sketch an application of our general results.

Let $g \colon \mathbb{R} \to \mathbb{R}$ and $h \colon \mathbb{R} \to \mathbb{R}$ be lower semicontinuous convex functions such that their $L^2$ convex integrands $G$ and $H$ satisfy the Hypothesis (H1). Recall that $G$ is precisely defined by $Gy = \int_Q g(y(t,x))\, dt\, dx$.

Let $\phi$ be such that the operator $B, (Bu)(t,x) = \phi(x)u(t,x)$ is one-to-one and onto in $L^2(Q)$. Then the problem

$$\text{Minimize:} \int_Q (g(y(t,x)) + h(u(t,x)))\, dt\, dx \tag{2.43}$$

subject to

$$\begin{aligned} & y_{tt} - \Delta_x y = \phi(x)u(t,x) + f(t,x), \ \ x \in \Omega, \ t \in (0,T), \ \ \Omega = (0,\pi) \times (0,\pi), \\ & y(t,x) = 0 \ \text{ on } \partial((0,\pi) \times (0,\pi)), \ \ t \in (0,T), \\ & y(0,x) = y(T,x); \ \ y_t(0,x) = y_t(T,x), \end{aligned} \tag{2.44}$$

where $T^2$ = a rational multiple of $\pi^2$ as indicated in Section 1, has a solution (optimal pair) $(y^*, u^*)$. Moreover, a pair $(y^*, u^*)$ is optimal iff there is $p^*$ such that

$$\begin{aligned} p^*_{tt} - \Delta p^* &= -w(t,x), \\ p^*(t,x) &= 0 \ \text{ on } \ \partial((0,\pi) \times (0,\pi)), \\ p^*(0,x) &= p^*(T,x); \ \ p^*_t(0,x) = p^*_t(T,x), \\ w(t,x) &\in \partial g(y^*(t,x)), \\ \phi(x)p^*(t,x) &\in \partial h(u^*(t,x)) \text{ a.e. in } Q, \end{aligned} \tag{2.45}$$

or equivalently

$$u^*(t,x) \in \partial h^*(\phi(x)p^*(t,x)) \text{ a.e. in } Q \tag{2.46}$$

where $h^*$ is the conjugate of $h$, i.e.

$$h^*(r) = \sup\{\, rv - h(v); \ \ v \in \mathbb{R} \,\}, \ \ r \in \mathbb{R}. \tag{2.47}$$

If $\phi \colon \Omega \to \mathbb{R}$ is a measurable function satisfying

$$0 < a \le \phi(x) \le b < \infty, \ \text{ a.e. in } \Omega, \tag{2.48}$$

then the operator $B$ given by

$$(Bu)(t,x) = \phi(x)u(t,x), \ \text{ a.e. in } Q \tag{2.49}$$

is one-to-one and onto in $L^2(Q)$ so the above results hold.

If instead of (2.48) $\phi$ satisfies only

$$0 \leq \phi(x), \text{ a.e. in } \Omega, \tag{2.50}$$

then $B$ is (unbounded) selfadjoint operator in $L^2(Q)$.

Indeed, $B$ is monotone (i.e. $\langle Bu, u \rangle \geq 0, \ \forall u \in D(B)), R(I+B) = L^2(Q)$, so $B$ is maximal monotone. On the other hand $B$ is also symmetric, or a symmetric maximal monotone operator in a Hilbert space is selfadjoint, as claimed.

It follows that if

$$0 < a \leq \phi(x), \tag{2.51}$$

then the results above on the optimal pair hold.

Indeed, (2.51) implies that $N(B) = 0$, so $R(B) = L^2(Q) \supset R(A)$ and $B$ is selfadjoint with closed range.

Details and other examples, as well as the case $B$-nonlinear will be studied elsewhere. The existence of optimal pairs in the case in which $B$ is unbounded, remains an open problem.

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