A NEW CHARACTERISTIC OF MÖBIUS TRANSFORMATIONS BY USE OF APollONIUS QUADRILATERALS

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Abstract. The purpose of this paper is to give a new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping. To this end a new concept of “Apollonius quadrilaterals” is used.

1. Introduction and statement of results

Throughout the paper, unless otherwise stated, let \( w = f(z) \) be a nonconstant meromorphic function of a complex variable \( z \) in \( |z| < +\infty \). We consider the following Property A:

Property A. The function \( w = f(z) \) transforms circles on the \( z \)-plane onto circles on the \( w \)-plane, including straight lines among circles.

The well-known principle of circle-transformation (see [1], [4], [5], [7, p. 160]) reads:

Theorem A. \( w = f(z) \) satisfies Property A iff \( w = f(z) \) is a Möbius transformation.

Before stating Property B we shall give the definition of Apollonius quadrilaterals on the complex plane.

Definition. Let \( ABCD \) be an arbitrary quadrilateral (not necessarily simple) on the complex plane. If \( AB \cdot CD = BC \cdot DA \) holds, then \( ABCD \) is said to be an Apollonius quadrilateral. (\( AB = |z_1 - z_2| \) where \( z_1, z_2 \) are the complex numbers corresponding to the points \( A, B \) respectively.)

Example 1. Each of a square, a rhombus and a kite is an Apollonius quadrilateral.

Example 2. If from a point outside a circle a secant and two tangents are drawn, then the four points on the circle form an Apollonius quadrilateral.

Proof. The proof follows from the following theorem (see [3, p. 7]): If a chord of a circle is drawn from the point of contact of a tangent, the angle made by the chord with the tangent is equal to the angle subtended by the chord at a point on that part of the circumference which lies on the far side of the chord.
We may now state Property B. In section 3 we shall prove that Property A implies the following Property B.

**Property B.** Suppose that \( w = f(z) \) is analytic and univalent in a non-empty domain \( R \) on the \( z \)-plane. Let \( ABCD \) be an arbitrary Apollonius quadrilateral contained in \( R \). If we set \( A' = f(A), B' = f(B), C' = f(C), D' = f(D) \), then \( A'B'C'D' \) is also an Apollonius quadrilateral on the \( w \)-plane.

The purpose of this paper is to give a new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping and to give a new proof of the only if part of Theorem A. The results in this paper are contained in the following theorem and its corollary.

**Main Theorem.** The function \( w = f(z) \) satisfies Property B iff \( w = f(z) \) is a Möbius transformation.

**Corollary.** This theorem gives a new proof of the only if part of Theorem A.

2. **Lemmas**

In sections 3 and 4 we shall apply the following three lemmas:

**Lemma 1.** If the function \( w = f(z) \) satisfies Property A, then \( w = f(z) \) is univalent in \( |z| < +\infty \).

Proof. See [4].

**Lemma 2** (The Theorem of Apollonius). We include straight lines among circles.

(i) The locus of a point, which moves so that the ratio of its distances from two fixed points is constant, is a circle with respect to which those points are inverse.

(ii) If \( A, B \) are two fixed points inverse with respect to a given circle, then the ratio \( \frac{PA}{PB} \) is constant for all positions of \( P \) on that circle.

Proof. (i) See [6, p. 129].

(ii) See [6, pp. 129–130].

**Lemma 3.** If the function \( w = f(z) \) is analytic and univalent in a non-empty domain \( R \), then \( f'(z) \neq 0 \) in \( R \).

Proof. See [9, p. 302].

3. **Proof that Property A implies Property B**

Suppose that \( w = f(z) \) is analytic in a non-empty domain \( R \) on the \( z \)-plane. Since, by hypothesis, \( w = f(z) \) satisfies Property A, by Lemma 1 \( w = f(z) \) is univalent in \( |z| < +\infty \). Thus \( w = f(z) \) is univalent in \( R \). Let \( ABCD \) be an arbitrary Apollonius quadrilateral contained in \( R \). Then, by definition we obtain

\[
AB \cdot CD = BC \cdot DA,
\]

and so

\[
\frac{AB}{CB} = \frac{AD}{CD}.
\]

We denote (1) by \( N \). By Lemma 2 (i) the locus of a point, which moves so that the ratio of its distances from two fixed points \( A, C \) is \( N (= \text{const}) \), is a circle with
respect to which \(A, C\) are inverse. We denote the above circle of Apollonius by \(K\) and let \(K' = f(K)\). Since, by hypothesis, \(w = f(z)\) satisfies Property A, \(K'\) is also a circle on the \(w\)-plane. If we set \(A' = f(A), B' = f(B), C' = f(C), D' = f(D)\), then by the fact that \(B, D\) are on \(K\) and by \(K' = f(K)\) \(B', D'\) are on the circle \(K'\). Since \(A, C\) are inverse with respect to the circle \(K\), by the Reflection Principle [10, p. 155] of analytic functions it follows that \(A' = f(A), C' = f(C)\) are also inverse with respect to the circle \(K' = f(K)\). Consequently, by Lemma 2 (ii) we obtain

\[
\frac{AB'}{CB'} = \frac{AD'}{CD'}
\]

which implies

\[
AB' \cdot CD' = BC' \cdot DA'.
\]

Therefore, the quadrilateral \(A'B'C'D'\) on the \(w\)-plane is also an Apollonius quadrilateral, and so we get the desired result.

4. Proof of the main theorem

If \(w = f(z)\) is a Möbius transformation, then, by the if part of Theorem A, \(w = f(z)\) satisfies Property A. Thus, by the result in section 3, \(w = f(z)\) satisfies Property B. Then, we shall prove the only if part of the main theorem, i.e., if \(w = f(z)\) satisfies Property B, then \(w = f(z)\) is a Möbius transformation. Since \(w = f(z)\) is analytic and univalent in the domain \(R\), by Lemma 3 it follows that

\[
f'(z) \neq 0.
\]

If \(x\) is an arbitrarily fixed point of \(R\), then, by (2) we obtain

\[
f'(x) \neq 0.
\]

Let \(E\) be the point represented by \(x\). Since \(E \in R\), there exists a positive real number \(r\) such that the \(r\) closed neighbourhood of \(E\) is contained in \(R\). We denote this closed neighbourhood by \(V\). Throughout the rest of the proof let \(ABCD\) denote an arbitrary square which is contained in \(V\) and whose centre is at \(E\). Here the sense of \(A, B, C, D\) is counterclockwise. Since \(ABCD\) is a square contained in \(V\), we can represent \(A, B, C, D\) by complex numbers

\[
x + y, x + iy, x - y, x - iy \quad (|y| \leq r),
\]

respectively. Since \(w = f(z)\) is univalent in \(R, A' = f(A), B' = f(B), C' = f(C), D' = f(D)\) are different points. Since, by hypothesis, \(A'B'C'D'\) is an Apollonius quadrilateral, by definition we obtain

\[
AB' \cdot CD' = BC' \cdot DA'.
\]

Since \(A', B', C', D'\) are represented by

\[
f(x + y), f(x + iy), f(x - y), f(x - iy),
\]

respectively, we obtain

\[
\frac{AB'}{CB'} = |f(x + y) - f(x + iy)|,
\]

\[
\frac{B'C'}{CD'} = |f(x + iy) - f(x - y)|,
\]

\[
\frac{C'D'}{AD'} = |f(x - y) - f(x - iy)|,
\]
Since the numerator and the denominator of \( g(11) \) are analytic for all \( y \) satisfying \( 0 < |y| \leq r \), we have

\[
\lim_{y \to 0} \frac{|(f(x+y) - f(x+iy))(f(x-y) - f(x-iy))|}{|(f(x+iy) - f(x-y))(f(x-iy) - f(x+y))|} = 1.
\]

If we set

\[
g(y) = \frac{(f(x+y) - f(x+iy))(f(x-y) - f(x-iy))}{(f(x+iy) - f(x-y))(f(x-iy) - f(x+y))},
\]

then, by (9) we have

\[
|g(y)| = 1.
\]

Since the numerator and the denominator of \( g(y) \) in (10) are analytic for all \( y \) satisfying \( 0 < |y| \leq r \) and since, by the fact that \( w = f(z) \) is univalent in \( R \),

the denominator of \( g(y) \) in (10) never vanishes in \( 0 < |y| \leq r \), \( g(y) \) is analytic in \( 0 < |y| \leq r \). Next we will prove that \( g(y) \) is also analytic at \( y = 0 \). As \( y \to 0 \), by L'Hopital's Rule (see [2]) and by (3) we have

\[
f(x+y) - f(x+iy) \xrightarrow{y \to 0} f'(x) - i f'(x) = \frac{1-i}{1+i}.
\]

and

\[
f(x-y) - f(x-iy) \xrightarrow{y \to 0} - f'(x) + i f'(x) = \frac{1-i}{1+i}.
\]

Hence, by (10), (12), (13), as \( y \to 0 \),

\[
g(y) \to \left( \frac{1-i}{1+i} \right)^2 = -1.
\]

If we define

\[
g(0) = -1
\]

by (14), by Riemann's Theorem on removable singularities, the function \( g(y) \) is analytic at \( y = 0 \). Furthermore, by (15) the equality (11) still holds at \( y = 0 \).

Thus \( g(y) \) is analytic in \( |y| \leq r \) and \( |g(y)| = 1 \) holds in \( |y| \leq r \). Therefore, by the Maximum Modulus Principle [8, p. 141] of analytic functions we obtain

\[
g(y) = K
\]

in \( |y| \leq r \), where \( K \) is a complex constant.

Setting \( y = 0 \) in (16) and using (15) it follows

\[
K = -1.
\]

By (10), (16), (17) we obtain

\[
(f(x+y) - f(x+iy))(f(x-y) - f(x-iy)) + (f(x+iy) - f(x-y))(f(x-iy) - f(x+y)) = 0
\]

for all \( y \) satisfying \( |y| \leq r \).
Using Leibnitz’s Rule for differentiation, differentiating both sides of (18) four times with respect to \( y \), setting \( y = 0 \) and simplifying the resulting equality yields

\[
f''''(x)f'(x) - \frac{3}{2}f''(x)^2 = 0.
\]

(19)

Since \( x \in R \) (the domain) was arbitrarily fixed, we can replace \( x \) by a variable \( z \) and thus by (19) we have

\[
f''''(z)f'(z) - \frac{3}{2}f''(z)^2 = 0
\]

in \( R \). By the Identity Theorem (see [7, p. 106]) the above equality holds in \( |z| < +\infty \). Hence

\[
\frac{f''''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 = 0
\]

holds for all \( z \) satisfying \( f'(z) \neq 0 \).

Thus, the Schwarzian derivative of \( f \) vanishes for all \( z \) satisfying \( f'(z) \neq 0 \). Therefore, by a well-known fact \( f(z) \) is a Möbius transformation of \( z \).

**Proof of the Corollary.** By hypothesis \( w = f(z) \) satisfies Property A. Hence, by the result of section 3, \( w = f(z) \) satisfies Property B. Consequently, by the main theorem \( w = f(z) \) is a Möbius transformation.

**References**


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