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A NEW CHARACTERISTIC OF MÖBIUS TRANSFORMATIONS BY USE OF APOLLONIUS QUADRILATERALS

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ABSTRACT. The purpose of this paper is to give a new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping. To this end a new concept of "Apollonius quadrilaterals" is used.

1. Introduction and statement of results

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function of a complex variable z in $|z| < +\infty$. We consider the following Property A:

Property A. The function $w = f(z)$ transforms circles on the z-plane onto circles on the w-plane, including straight lines among circles.

The well-known principle of circle-transformation (see [1], [4], [5], [7, p. 160]) reads:

Theorem A. *$w = f(z)$ satisfies Property A iff $w = f(z)$ is a Möbius transformation.*

Before stating Property B we shall give the definition of Apollonius quadrilaterals on the complex plane.

Definition. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on the complex plane. If $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$ holds, then $ABCD$ is said to be an *Apollonius quadrilateral*. ($\overline{AB} = |z_1 - z_2|$ where z_1, z_2 are the complex numbers corresponding to the points A, B respectively.)

Example 1. Each of a square, a rhombus and a kite is an Apollonius quadrilateral.

Example 2. If from a point outside a circle a secant and two tangents are drawn, then the four points on the circle form an Apollonius quadrilateral.

Proof. The proof follows from the following theorem (see [3, p. 7]): If a chord of a circle is drawn from the point of contact of a tangent, the angle made by the chord with the tangent is equal to the angle subtended by the chord at a point on that part of the circumference which lies on the far side of the chord.

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We may now state Property B. In section 3 we shall prove that Property A implies the following Property B.

Property B. Suppose that $w = f(z)$ is analytic and univalent in a non-empty domain R on the z-plane. Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also an Apollonius quadrilateral on the w-plane.

The purpose of this paper is to give a new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping and to give a new proof of the only if part of Theorem A. The results in this paper are contained in the following theorem and its corollary.

Main Theorem. *The function $w = f(z)$ satisfies Property B iff $w = f(z)$ is a Möbius transformation.*

Corollary. *This theorem gives a new proof of the only if part of Theorem A.*

2. Lemmas

In sections 3 and 4 we shall apply the following three lemmas:

Lemma 1. *If the function $w = f(z)$ satisfies Property A, then $w = f(z)$ is univalent in $|z| < +\infty$.*

Proof. See [4].

Lemma 2 (The Theorem of Apollonius)**.** *We include straight lines among circles.*

(i) *The locus of a point, which moves so that the ratio of its distances from two fixed points is constant, is a circle with respect to which those points are inverse.*

(ii) *If A, B are two fixed points inverse with respect to a given circle, then the ratio $\frac{\overline{PA}}{\overline{PB}}$ is constant for all positions of P on that circle.*

Proof. (i) See [6, p. 129].
(ii) See [6, pp. 129–130].

Lemma 3. *If the function $w = f(z)$ is analytic and univalent in a non-empty domain R, then $f'(z) \neq 0$ in R.*

Proof. See [9, p. 302].

3. Proof that Property A implies Property B

Suppose that $w = f(z)$ is analytic in a non-empty domain R on the z-plane. Since, by hypothesis, $w = f(z)$ satisfies Property A, by Lemma 1 $w = f(z)$ is univalent in $|z| < +\infty$. Thus $w = f(z)$ is univalent in R. Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R. Then, by definition we obtain

$$\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA},$$

and so

$$\frac{\overline{AB}}{\overline{CB}} = \frac{\overline{AD}}{\overline{CD}}. \qquad (1)$$

We denote (1) by N. By Lemma 2 (i) the locus of a point, which moves so that the ratio of its distances from two fixed points A, C is N (= const), is a circle with respect to which A, C are inverse. We denote the above circle of Apollonius by K and let $K' = f(K)$. Since, by hypothesis, $w = f(z)$ satisfies Property A, K' is also a circle on the w-plane. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then by the fact that B, D are on K and by $K' = f(K)$ B', D' are on the circle K'. Since A, C are inverse with respect to the circle K, by the Reflection Principle [10, p. 155] of analytic functions it follows that $A' = f(A)$, $C' = f(C)$ are also inverse with respect to the circle $K' = f(K)$. Consequently, by Lemma 2 (ii) we obtain

$$\frac{\overline{A'B'}}{\overline{C'B'}} = \frac{\overline{A'D'}}{\overline{C'D'}}$$

which implies

$$\overline{A'B'} \cdot \overline{C'D'} = \overline{B'C'} \cdot \overline{D'A'}.$$

Therefore, the quadrilateral $A'B'C'D'$ on the w-plane is also an Apollonius quadrilateral, and so we get the desired result.

4. Proof of the main theorem

If $w = f(z)$ is a Möbius transformation, then, by the if part of Theorem A, $w = f(z)$ satisfies Property A. Thus, by the result in section 3, $w = f(z)$ satisfies Property B. Then, we shall prove the only if part of the main theorem, i.e., if $w = f(z)$ satisfies Property B, then $w = f(z)$ is a Möbius transformation. Since $w = f(z)$ is analytic and univalent in the domain R, by Lemma 3 it follows that

$$f'(z) \neq 0. \tag{2}$$

If x is an arbitrarily fixed point of R, then, by (2) we obtain

$$f'(x) \neq 0. \tag{3}$$

Let E be the point represented by x. Since $E \in R$, there exists a positive real number r such that the r closed neighbourhood of E is contained in R. We denote this closed neighbourhood by V. Throughout the rest of the proof let $ABCD$ denote an arbitrary square which is contained in V and whose centre is at E. Here the sense of A, B, C, D is counterclockwise. Since $ABCD$ is a square contained in V, we can represent A, B, C, D by complex numbers

$$x + y, x + iy, x - y, x - iy \qquad (|y| \leq r),$$

respectively. Since $w = f(z)$ is univalent in R, A' $(= f(A))$, B' $(= f(B))$, C' $(= f(C))$, D' $(= f(D))$ are different points. Since, by hypothesis, $A'B'C'D'$ is an Apollonius quadrilateral, by definition we obtain

$$\overline{A'B'} \cdot \overline{C'D'} = \overline{B'C'} \cdot \overline{D'A'}. \tag{4}$$

Since A', B', C', D' are represented by

$$f(x + y), f(x + iy), f(x - y), f(x - iy),$$

respectively, we obtain

$$\overline{A'B'} = |f(x + y) - f(x + iy)|, \tag{5}$$

$$\overline{B'C'} = |f(x + iy) - f(x - y)|, \tag{6}$$

$$\overline{C'D'} = |f(x - y) - f(x - iy)|, \tag{7}$$

$$\overline{D'A'} = |f(x-iy) - f(x+y)|. \tag{8}$$

Substituting (5), (6), (7), (8) into (4) yields

$$\begin{aligned}|(f(x+y) - f(x+iy))(f(x-y) - f(x-iy))|\\ = |(f(x+iy) - f(x-y))(f(x-iy) - f(x+y))|\end{aligned}$$

and therefore

$$\frac{|(f(x+y) - f(x+iy))(f(x-y) - f(x-iy))|}{|(f(x+iy) - f(x-y))(f(x-iy) - f(x+y))|} = 1. \tag{9}$$

If we set

$$g(y) = \frac{(f(x+y) - f(x+iy))(f(x-y) - f(x-iy))}{(f(x+iy) - f(x-y))(f(x-iy) - f(x+y))}, \tag{10}$$

then, by (9) we have

$$|g(y)| = 1. \tag{11}$$

Since the numerator and the denominator of $g(y)$ in (10) are analytic for all y satisfying $0 < |y| \le r$ and since, by the fact that $w = f(z)$ is univalent in R, the denominator of $g(y)$ in (10) never vanishes in $0 < |y| \le r$, $g(y)$ is analytic in $0 < |y| \le r$. Next we will prove that $g(y)$ is also analytic at $y = 0$. As $y \to 0$, by L'Hopital's Rule (see [2]) and by (3) we have

$$\frac{f(x+y) - f(x+iy)}{f(x+iy) - f(x-y)} \to \frac{f'(x) - if'(x)}{if'(x) + f'(x)} = \frac{1-i}{1+i} \tag{12}$$

and

$$\frac{f(x-y) - f(x-iy)}{f(x-iy) - f(x+y)} \to \frac{-f'(x) + if'(x)}{-if'(x) - f'(x)} = \frac{1-i}{1+i}. \tag{13}$$

Hence, by (10), (12), (13), as $y \to 0$,

$$g(y) \to \left(\frac{1-i}{1+i}\right)^2 = -1. \tag{14}$$

If we define

$$g(0) = -1 \tag{15}$$

by (14), by Riemann's Theorem on removable singularities, the function $g(y)$ is analytic at $y = 0$. Furthermore, by (15) the equality (11) still holds at $y = 0$. Thus $g(y)$ is analytic in $|y| \le r$ and $|g(y)| = 1$ holds in $|y| \le r$. Therefore, by the Maximum Modulus Principle [8, p. 141] of analytic functions we obtain

$$g(y) = K \tag{16}$$

in $|y| \le r$, where K is a complex constant.

Setting $y = 0$ in (16) and using (15) it follows

$$K = -1. \tag{17}$$

By (10), (16), (17) we obtain

$$\begin{aligned}(f(x+y) - f(x+iy))(f(x-y) - f(x-iy))\\ + (f(x+iy) - f(x-y))(f(x-iy) - f(x+y)) = 0\end{aligned} \tag{18}$$

for all y satisfying $|y| \le r$.

Using Leibnitz's Rule for differentiation, differentiating both sides of (18) four times with respect to y, setting $y = 0$ and simplifying the resulting equality yields

$$f'''(x)f'(x) - \frac{3}{2}f''(x)^2 = 0. \tag{19}$$

Since $x \in R$ (the domain) was arbitrarily fixed, we can replace x by a variable z and thus by (19) we have

$$f'''(z)f'(z) - \frac{3}{2}f''(z)^2 = 0$$

in R. By the Identity Theorem (see [7, p. 106]) the above equality holds in $|z| < +\infty$. Hence

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 0$$

holds for all z satisfying $f'(z) \neq 0$.

Thus, the Schwarzian derivative of f vanishes for all z satisfying $f'(z) \neq 0$. Therefore, by a well-known fact $f(z)$ is a Möbius transformation of z.

Proof of the Corollary. By hypothesis $w = f(z)$ satisfies Property A. Hence, by the result of section 3, $w = f(z)$ satisfies Property B. Consequently, by the main theorem $w = f(z)$ is a Möbius transformation.

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